One way to analyze functions to to approximate them with polynomials. This allows us to estimate values of a function around a certain value of x with remarkable accuracy. These polynomials are called Taylor polynomials

Goal

1. Define Taylor polynomials
2. Identify a formula to find a Taylor polynomial about $x = 0$ and find it.
3. Identify a formula to find a Taylor polynomial about $x = a$ and find it.

What is a Taylor polynomial? A Taylor polynomial is a polynomial that approximates a function (usually exponential or trigonometric) around a certain value for x (usually 0). Why is this advantageous? It allows us to estimate with great accuracy values for the function around the point $(x, f(x))$.

How do we know if two functions are the same? It stands to reason that their first derivatives would be the same at the point, as well as their second, and third, and so on. So, we can algebraically calculate the derivatives of a function, evaluate them at the point in question, and compare them to the same derivatives of our desired degree polynomial. If the degree is 0, the approximation is a horizontal line; if it is 1, we are talking about the tangent line. If we take the location of $x = 0$, the first two Taylor polynomials are:

$$f(x) \approx P_0(x) = f(0) \qquad\qquad f(x) \approx P_1(x) = f(0) + f'(0)x$$

It is easy to see these approximations for 0 and 1 degree polynomials, but what about polynomials of larger degree, say five. How can we find a 5th degree polynomial that will have the same derivative values at $x = 0$ as our $f(x)$? Working through the algebra we get:

$$f(x) \approx P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5$$

Why does this work? If we take say the third derivative of this polynomial, the terms with degree less than 3 will differentiate to 0, while the terms with degree more than 3 will equal 0 when we plug in $x = 0$. Therefore, we can generalize this for any Taylor polynomial of degree n:

$$f(x) \approx P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + ... + \frac{f^{(n)}(0)}{n!}x^n$$

Now the question remains, "That is all fine and dandy when $x = 0$, but what about $x \neq 0$?" The same logic applies in this situation. Given that we want each derivative to be the same for the function and the polynomial at the point, we make the following changes to the polynomial:

$$f(x) \approx P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Again, this works because if we take say the 7th derivative (assume $n > 7$), the terms with degree less than 7 will differentiate to 0 while the terms with degree higher than 7 will equal 0 when evaluated at $x = a$

Find the Taylor polynomial around $x = 0$ for the following function with the corresponding degree:

$y = \cos x; 3 \qquad y = e^x; 4 \qquad y = \sin x; 8$