In the previous section, we approximated functions wit n-degree polynomials called Taylor polynomials. Now we will extend that concept to actual representations of a function with a polynomial of infinite degree

Goal

1. Define Taylor series
2. Apply Taylor series to find new representations of common functions
3. Apply Taylor series to find binomial expansions of non-positive integer exponents

What is a Taylor series? A Taylor series is an extension of a Taylor polynomial of degree $n \to \infty$.

From our previous study we know the following Taylor series (about $x = 0$):

1. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...$
2. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$
3. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$

Because these functions are defined for all $x \in R$, this series equals the function for all values of x. However, not all functions are like this. Take the natural log function about $x = 1$:

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + ...$$

The natural log function is not defined for $x \leq 0$, so the series is only good for estimating values of $x > 0$. Since we are centering the polynomial at $x = 1$, it makes sense that the function will approximate values of x the same distance to the right as to the left. As the series only works for a length of 1 away from the center to the left, it stands to reason that the series only works for a length of 1 to the right as well.

We know from the binomial expansion theorem how to find $(1+x)^p$ when p is a positive integer. However, what if p is not a positive integer, maybe negative or a positive non-integer? We can use the same algebraic calculation using the nth derivatives to generate the following series:

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \frac{p(p-1)(p-2)(p-3)}{4!}x^4 + ...$$

Again, because we don't know what p is, the series will only work for values of $|x| < 1$, because for some values of p, the function will not be defined for $x \leq -1$

Does this really work for a positive integer p? Absolutely, for each term with degree greater than p, the coefficient is 0 because of the subtraction in the numerator.

How does this change if we have the function $(1-x)^p$? Everything is the same except the terms will alternate positive and negative based on whether the degree of the term is even or odd.