Now that we can construct Taylor polynomials, how can we analyze how close to the actual function is the polynomial?

Goal

1. Find a method to determine the error of the Taylor polynomial in relation to its corresponding function.
2. Apply this method to find a large enough degree polynomial to ensure accuracy to a given degree.
3. Apply this method to find a range of values to which a given degree polynomial is accurate to a given tolerance.

If we define the error function to be the difference between the Taylor polynomial of degree n and the function, it is easy to see that all derivatives of this error function from the first to the nth must be 0. If we take the $(n+1)$th derivative, we get our first opportunity for a non-zero value. Therefore, if we can find a bound on the $(n+1)$th derivative, we can find a function which will bound our error for us (LaGrange Error Bound).

Given a Taylor polynomial of degree n about $x = a$ and M equal to a bound of $f^{(n+1)}(x)$ on a given interval of x about a, the function for absolute value of the error is as follows:

$$|E_n(x)| \leq \frac{M}{(n+1)!}(|x-a|)^{n+1}$$

Therefore, the bound on the error is based on the next term of the Taylor series of the Taylor polynomial. Of course, in some instances we can calculate the error directly.

Examples:

1. What is the error on the Taylor polynomial of degree 6 centered at $x = 0$ for e^1?

2. What degree Taylor polynomial centered at $x = 0$ is necessary to guarantee e^1 to the nearest .0000001?

3. At what interval $[-x, 0]$ is the degree 6 Taylor polynomial for $f(x) = e^x$ centered at $x = 0$ guaranteed to be accurate to the nearest .0001?

4. What is the error on the Taylor polynomial of degree 6 centered at $x = 0$ for $\ln 2$? (Note: We use the function $\ln(1 + x) = f(x)$)

5. What degree Taylor polynomial centered at $x = 0$ is necessary to guarantee $\ln 2$ to the nearest .000001?

6. At what interval $[0, x]$ is the degree 6 Taylor polynomial for $f(x) = \ln(x + 1)$ centered at $x = 0$ guaranteed to be accurate to the nearest .0001?