Today we turn our attention to differential equations

Goal

1. Define a differential equation
2. Test solutions to differential equations
3. Use initial conditions to find constants of integration.
4. Define differential equations of higher orders.

What is a differential equation? A differential equation is an equation which defines y' or $\frac{dy}{dx}$ as a function in terms of x and y which assumes that y is a function of x. If the equation for y' is in terms of x only, we can integrate both sides and get the solution quite easily. If it is a combination of x and y the process is a little more involved and we will discuss how to solve this analytically in a later section. Whether or not we we can solve analytically, we can find a solution for y numerically by calculating values of y after choosing an arbitrary starting point.

Let's take the example of $y' = y$. If we assume a starting position of (1,2), we know the slope at the point is 2, and we can approximate the next integer point to be (2,6) and the next to be (3,18) and so on.

We know intuitively that one solution to this problem is $y = e^x$. We can test this by taking the derivative of both sides finding that $y' = e^x$, therefore $y' = y$. However, as we know when we take integrals (which we will have to do to solve the problem analytically), we have this thing called the constant of integration which allows for multiple solutions. In this case, how do we take into account the constant of integration? Well, how can you change the function e^x so that its derivative is the same of the original?

That's right, you can multiply it by a constant. Any function, $y = Ce^x$, has the property $y' = y$, where C is any constant. This collection of functions (using any value of C) is called the family of solutions. How can we find the value of C? We have to be given an initial condition. So given the differential equation, $y' = y$, and the initial condition that the point $(2, 5e^2)$ is on the graph of y, we can conclude that the value of C is 5.

Do differential equations have to limited to the first derivative? The answer is no. The number of the derivative tells us the order of the equations. The presence of a second derivative makes the equation of second order; a third derivative, third order, and so on.

This leads us to a natural question. How many constants can I expect to find in the solution of a differential equation? Just as you would expect, the answer is the number of times you would have to integrate. Therefore, the number of constants is equal to the order of the differential equation. Take for example the illustration of acceleration, velocity, and displacement. We know that the acceleration due to gravity is a constant 32 ft/sec^2. We also know that the acceleration is the second derivative of the displacement, so given the differential equation:

$$\frac{d^2s}{dt^2} = -32$$

We can integrate twice to get the equation for the displacement:

$$s = -16t^2 + C_1 t + C_2$$

where C_1 and C_2 turn out to be the initial velocity and displacement respectively.

Examples to work on:

1. Which of the following is a solution of the problem $y' + y = 0$?
   (a) e^x
   (b) e^{-x}
   (c) e^{2x}
   (d) e^{-2x}
2. Given that the original velocity of an object is 500 ft/sec and its original position is 100 ft above the ground, find the equation for its vertical position with respect to time.
3. Which is NOT an solution to the differential equation $y^{(4)} = y$?
   (a) $4e^x$
   (b) $\sin x$
   (c) $-5\cos x$
   (d) $3e^{-2x}$