Goal

1. Define displacement and position vectors
2. Define the components of a vector and understand how to find them
3. Understand the properties of scalar multiplication and magnitude
4. Apply vectors to real world situations
5. Identify algebraic principles that govern the use of vectors

What is a vector? A vector is a description of movement from one location in a coordinate system to another. We signify this as an arrow with the base at the starting point and the tip at the destination. As such, vectors describe displacement in a coordinate system. If the starting location of an object is the origin, the vector is also called a position vector as the coordinates of the position will be the same as the amount of displacement.

We can add and subtract vectors just like integers or real numbers. To add two vectors together, we simply place them base to tip in the coordinate system and find the terminal location. To subtract vectors, we again place them base to tip, but the vector being subtracted is flipped in the opposite direction. This comes from the principle of scalar multiplication. Remember when we subtract an amount, it is the same as adding -1 times that amount. The principle applies to vectors as well. Vectors can be multiplied by a scalar multiple (usually an integer) and this is done by placing the given number of vector base to tip. If the number is negative, the base and tip are reversed, and the number of copies of the vector is the absolute value of the scalar multiple. We define the magnitude ($||\vec{v}||$) of a vector as its length, and this is always considered a positive value. If a vector is multiplied by a scalar, the magnitude of the resulting vector is the absolute value of the scalar time the magnitude of the vector. By definition the zero vector is a vector of length zero. Finally, two vectors are said to be parallel if they are scalar multiples of one another.

Now that we know what vectors look like graphically in a coordinate system, it is important that we can describe them with actual numbers. In any coordinate system, a vector displaces an object by some amount along the set of axis (x and y for two dimensions, x, y, and z for three dimensions), and as such we show displacement as a multiple of moving 1 in the positive x, y, and z. We use the symbols $\vec{i}$ for the x-axis, $\vec{j}$ for the y-axis, and $\vec{k}$ for the z-axis.

For example, the vector to show displacement for the origin to the point (2,4,-1) would look as follows:

$$\vec{v} = 2\vec{i} + 4\vec{j} - \vec{k}$$

These three terms for the vector are called the components of the vector. The number of components depends on the number of dimensions used in the coordinate system.

In general, to find the components of a displacement vector, subtract the corresponding values of each point and write in vector form. For example:

Displacement from (-4,3,2) to (1,4,-5)

$$\vec{v} = (1 - -4)\vec{i} + (4 - 3)\vec{j} + (-5 - 2)\vec{k} = 5\vec{i} + \vec{j} - 7\vec{k}$$

Given the components of a vector, finding a scalar multiple is as simple as multiplying the coefficients of $\vec{i}$, $\vec{j}$, and $\vec{k}$. The magnitude of a vector, given the components is found by taking the square root of the sum of the squares of the the coefficients of the components. For example:

Find the magnitude of the vector $3\vec{v}$, where $\vec{v} = 3\vec{i} - 2\vec{j} + \vec{k}$

$3\vec{v} = 9\vec{i} - 6\vec{j} + 3\vec{k}$, therefore $||3\vec{v}|| = \sqrt{9^2 + 6^2 + 3^2} = \sqrt{126} = 3\sqrt{14} = 3||v||$

We can use trigonometry to find the components of a 2-dimensional vector if we are given the length and the angle of the vector with respect to the x-axis. Given a vector, $\vec{v}$ with magnitude (or length), m, and the angle θ between the x-axis and $\vec{v}$, the components are as follows:

$$\vec{v} = m(\cos\theta)\vec{i} + m(\sin\theta)\vec{j}$$

A unit vector is a vector whose length is 1. Unit vectors have special properties which we will use in later sections, so it is beneficial to transform a given vector to a unit vector. This is accomplished by dividing each component coefficient by the magnitude of the vector. For example, find the unit vector that is parallel to the following vector:

$$\vec{v} = 3\vec{i} + 4\vec{j}$$

As we know the magnitude of this vector is 5, we determine that the unit vector is:

$$\vec{v}_u = \frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}$$

In applying the use of vectors, we can think of a vector as a velocity with the speed as the magnitude and the direction is determined by the angle with respect to the axes. We can also apply the concept of vector when describing acceleration as well as force.

When we combine vectors using addition, subtraction, and scalar multiplication, there are some very familiar rules which govern those operations.

Commutativity

1. $\vec{v} + \vec{w} = \vec{w} + \vec{v}$
2. As addition and subtraction are done component-wise, it is obvious that these operation are commutative. Also, two vectors must be of the same dimension to be added or subtracted.

Associativity

1. $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$
2. $a(b\vec{v}) = (ab)\vec{v}$
3. As addition, subtraction, and scalar multiplication are done component-wise, it is obvious that associativity holds for these operations on vectors.

Distributivity

1. $a(\vec{v} + \vec{w}) = a\vec{v} + a\vec{w}$
2. $(a + b)\vec{v}) = a\vec{v} + b\vec{v}$
3. As addition, subtraction, and scalar multiplication are done component-wise, it is obvious that distributivity holds for these operations on vectors.

Identity and inverse

1. $1\vec{v} = \vec{v}$
2. $\vec{v} + \vec{0} = \vec{v}$
3. $0\vec{v} = \vec{0}$
4. $\vec{v} + (-1)\vec{v} = \vec{v} - \vec{v} = \vec{0}$
5. As addition, subtraction, and scalar multiplication are done component-wise, it is obvious that these identity and inverse principles holds for these operations on vectors.

Problems to work on: 13-1; 32, 33, 34, 41, 43. 13-2; 8, 15, 22, 26