Goal

1. Define the dot product of two vectors both geometrically and algebraically
2. Learn algebraic properties of dot product operations
3. Discover what information the dot product tells us about two vectors
4. Apply the dot product to geometric and real-world scenarios.

What is the dot product? The dot product a number generated by one or two vectors either geometrically or algebraically. In a geometric sense, the dot product between $\vec{v}$ and $\vec{w}$ is:

$$\vec{v} * \vec{w} = ||\vec{v}||\,||\vec{w}||\,\cos\theta$$

where θ is the angle between the two vectors.

Algebraically, the dot product of two vectors is the sum of the products of the corresponding coefficients of the components. For example, given the 2 2-dimensional vectors, $\vec{v} = v_1\vec{i} + v_2\vec{j}$ and $\vec{w} = w_1\vec{i} + w_2\vec{j}$, then the dot product is:

$$\vec{v} * \vec{w} = v_1 w_1 + v_2 w_2$$

Although the number of dimensions does not change the formula for the geometric definition, as the number of dimensions increase, the number of terms summed will increase to be n, the number of dimensions contained in the vector.

Using the Law of Cosines from geometry we can prove that these two definitions of the dot product yield the same value.

Now that we have defined the dot product, what are the rules that govern operations regarding the dot product. They are:

1. $\vec{v} * \vec{w} = \vec{w} * \vec{v}$ – Commutativity
2. $\vec{v} * (\lambda \vec{w}) = \lambda(\vec{v} * \vec{w}) = (\lambda \vec{v}) * \vec{w}$ – Associativity
3. $(\vec{v} + \vec{w}) * \vec{u} = \vec{v} * \vec{u} + \vec{w} * \vec{u}$ – Distributivity

All of these properties are easily shown using the algebraic definition of dot product as the computation of the dot product is done component-wise.

Two interesting facts are easily shown using the definition of dot product; the geometric for the first, and the algebraic for the second.

1. If $\vec{v} * \vec{w} = 0$, then $\vec{v}$, and $\vec{w}$ are perpendicular. Proof: $\cos \frac{\pi}{2} = 0$
2. $\vec{v} * \vec{v} = ||\vec{v}||^2$. Take any vector and move its base to the origin. By the distance formula, and length of the vector is the square root of the sum of the square of its component coefficients. This sum is the dot product, so the dot product is the square of the length or magnitude of the vector.

Given this fact that the dot product of vectors equals 0 if they are perpendicular, it is possible, given any 3-dimensional vector (written in component form) and a point on the vector, to find the equation of the plane through the point normal (or perpendicular) to the vector. Given $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$ normal to the plane containing the point (x_0, y_0, z_0) and any vector in the plane $\vec{w} = (x - x_0)\vec{i} + (y - y_0)\vec{j} + (z - z_0)\vec{k}$, the formula for the plane is:

$$ax + by + cz = ax_0 + by_0 + cz_0$$

Another algebraic application of the dot product is the use of projections to use one vector as a component of another. Most often, a vector is described in terms of it relationship to the axes of a coordinate system (which are perpendicular to each other. When calculating a projection, we write a vector in terms of a scalar multiple of a second plus a vector perpendicular to the second vector. The projection a vector $\vec{y}$ onto another vector $\vec{u}$ is:

$$\hat{y} = \frac{\vec{y} * \vec{u}}{\vec{u} * \vec{u}}\vec{u}$$

By definition, the projection, $\hat{y}$, is parallel to $\vec{u}$. Subtracting $\hat{y}$ from $\vec{y}$, we get the vector we call $y_{\perp}$, because $\hat{y} * y_{\perp} = 0$

A real world application of the dot product is the finding of work. The regular equation for work is $W = Fd$, but this assumes that the force and displacement are in the same direction. However if the force vector and the displacement vector are not parallel, the formula is no longer a simple multiplication, but the dot product of the force and displacement vectors.

Problems to work on: 29, 35, 43, 44, 47, 50,