Goal

1. Define the cross product of two vectors both geometrically and algebraically
2. Relate the cross product to the concept of determinants of matrices
3. Learn algebraic properties of cross product operations
4. Apply the cross product to geometric scenarios.

What is the cross product? The cross product of two vectors is a vector perpendicular to both vectors with a length equal to the area of the parallelogram generated by the two vectors pointing the direction as determined by the orientation of the two vectors. Note that the cross product is only defined for 3-dimensional vectors. In a geometric sense, the cross product between $\vec{v}$ and $\vec{w}$ is:

$$\vec{v} \,\mathtt{x}\, \vec{w} = (||\vec{v}||\,||\vec{w}||\,\sin\theta)\vec{n}$$

where $\vec{n}$ is the correctly pointing unit vector perpendicular to $\vec{v}$ and $\vec{w}$

Algebraically, the cross product is found by finding the determinant of the 3x3 matrix as defined by the two vectors in question and a vector of $\vec{i}$, $\vec{j}$, and $\vec{k}$. Writing the vectors as rows of the matrix (row or columns does not matter), we get:

$$\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

and we define the cross product to be the determinant of the matrix:

$$\vec{v} \,\mathtt{x}\, \vec{w} = (v_2w_3 - v_3w_2)\vec{i} + (v_3w_1 - v_1w_3)\vec{j} + (v_1w_2 - v_2w_1)\vec{k}$$

How do you determine the correct direction for this perpendicular vector to point. Imagine the vectors on the xy-plane with the bases at the origin. Consider the direction of the smallest angle from the first vector to the second. If this angle is in a counter-clockwise direction, the vector points in the $+z$ direction. If the angle is in a clockwise direction, the vector points in a -z direction. This is called the right-hand rule. In this respect, the algebraic definition is preferable, as the correct orientation for the perpendicular vector is determined.

What properties exist in cross produce operations? There is a slight variation with our normal properties of associativity, commutativity, and distributivity.

1. Associative (with scalars): $(\lambda\vec{v})\,\mathtt{x}\,\vec{w} = \lambda(\vec{v}\,\mathtt{x}\,\vec{w}) = \vec{v}\,\mathtt{x}\,(\lambda\vec{w})$

   There is no associativity with 3 or more vectors with the cross product operation

2. Commutative: $\vec{w}\,\mathtt{x}\,\vec{v} = -(\vec{v}\,\mathtt{x}\,\vec{w})$

   This is called anti-commutative.

3. Distributive: $\vec{u}\,\mathtt{x}\,(\vec{v} + \vec{w}) = (\vec{u}\,\mathtt{x}\,\vec{v}) + (\vec{u}\,\mathtt{x}\,\vec{w})$

   There is no difference here.

There are numerous uses for the cross product in solving algebraic and geometric problems. It can be used to find the equation of a plane given three points, the area of a parallelogram, or even the volume of a parallelepiped.

To find the equation of the plane containing three non-collinear points, choose two vectors determined by the three points. Using the cross product, we can determine the vector normal to the unknown plane, and then determine the equation of the plane.

Since the definition of the cross product includes the area of the contained vectors, one simply needs to find the length of the cross product vector to find the area of the parallelogram determined by two vectors.

To find the volume of a parallelepiped, one uses the formula $(\vec{b}\,\mathtt{x}\,\vec{c}) * \vec{a}$. Computing this value is the same as the absolute value of the following determinant:

$$\left| \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \right| = |a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)|$$

Problems to work on: 14, 16, 24, 26, 44