## Goal

1. Define integration by substitution
2. Relate integration by substitution to the chain rule of differentiation
3. Change limits of integration using substitution

What is integration by substitution? Instead of finding $\int f(x)\,dx$ we find $\int f(u)\,du$ where $u$ is some $f(x)$. Why is this necessary? It is necessary if the original $f(x)$ does not have an apparent integral. For example:

$$\int x(x^2-5)^{-\frac{1}{2}}\,dx$$

Although it is possible not to multiply this expression out to polynomial form, we can substitute $u$ for the expression $(x^2-5)$ and $\frac{1}{2}du$ for $x\,dx$ giving us the following integral:

$$\int \frac{1}{2}u^{-\frac{1}{2}}\,du$$

This integral is easily solved to be $u^{\frac{1}{2}} = (x^2-5)^{\frac{1}{2}}$

You will notice that in effect we have reverse engineered the chain rule. We declared $x^2-5$ to be the inside function and $x^{-\frac{1}{2}}$ to be the outside function and used the principles of the chain rule in reverse to solve our integral. As we will see from our upcoming examples, not all the solutions will be so easily seen, but still can be achieved by using substitution.

Examples:

1. $\int \sin^6 x \cos x \, dx$

2. $\int t^2(t^3 - 3)^{10} \, dt$

3. $\int z e^{z^2} \, dz$

4. $\int \sqrt{1 + \sqrt{x}} \, dx$

5. $\int \frac{-\sin x}{\cos^2 x}, dx$

When dealing with definite integrals requiring the use of substitution, we have two methods to solve the problem. We can wait until the integration is done and we have substituted back to the original variable, or we can change the limits of integration to match the new variable. Take the first example, for instance.

$$\int_{\frac{\pi}{2}}^{\pi} \sin^6 x \cos x \, dx$$

Normally, we would take the integral with substitution and after substituting back we get:

$$\int_{\frac{\pi}{2}}^{\pi} \sin^6 x \cos x \, dx = \left. \frac{\sin^7 x}{7} \right|_{\frac{\pi}{2}}^{\pi} = 0 - \frac{1}{7} = -\frac{1}{7}$$

However, we can save the step of re-substituting by changing the limits of integration to match $u$ by replacing the limits $a$ and $b$ with the values $u(a)$ and $u(b)$.

$$\int_{\frac{\pi}{2}}^{\pi} \sin^6 x \cos x \, dx = \int_1^0 u^6 \, du = \left.\frac{u^7}{7}\right|_1^0 = 0 - \frac{1}{7} = -\frac{1}{7}$$

Write each of following definite integrals using new limits of integration based on substitution and then solve.

1. $\int_0^1 t^2(t^3 - 3)^{10} \, dt$

2. $\int_0^1 ze^{z^2} \, dz$

3. $\int_0^1 \sqrt{1 + \sqrt{x}} \, dx$

4. $\int_0^{\frac{\pi}{4}} \frac{-\sin x}{\cos^2 x}, dx$

Exercise Problems: 4, 17, 19, 23, 35, and 57