So far, we have found how to find integrals using substitution and integration by parts. Today, will use the technique called integration by partial fractions.

Goal

1. Define integration by partial fractions
2. Solve problems using integration by partial fractions

When we integrate a rationale function, the degrees of the polynomials tell us what we can do to find the integral. If the degree of the denominator is one more than the degree of the numerator, we might be able to use a version of the natural log function integral. But if this is not the case, we need another strategy. If the degree of the numerator is greater than or equal to the degree of the denominator, we may try long division first to see if the denominator is a factor of the numerator. If not, we can use the technique of partial fractions to find the derivative of the fractional part of the quotient. We also use this technique if the degree of the denominator is greater than the degree of the numerator and the numerator is not a multiple of the derivative of the denominator. Given that the denominator factors into the product of polynomials of smaller degree, we assume the fraction with the large polynomial denominator is the sum of fractions whose denominators are factors of the larger denominator. What does this like in general?

$$\int \frac{y}{(x-a_1)(x-a_2)...(x-a_n)}\, dx = \int \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + ... + \frac{A_n}{x-a_n}\, dx$$

where y is a polynomial in x and a_i, A_i are real-valued constants.

For example:

$$\int \frac{3x+1}{x^2-1}\,dx = \int \frac{A_1}{x-1} + \frac{A_2}{x+1}\,dx$$

Using principles of algebra we can determine that the values of A_1 and A_2 are 2 and 1, respectively. Once we have determined each of the fractions, we can integrate each one separately.

However, things don't always work out quite so nice. If there are multiplicities within the factors of the denominator's polynomial, then there is a fraction for each of the multiplicities. Also, if a factor is an irreducible polynomial of degree greater the one, the numerator is considered a general polynomial of one degree less than the factor.

For example:

$$\int \frac{4x^3+9x^2+4x+3}{x^4+2x^3+2x^2+2x+1}\,dx = \frac{A_1}{x+1} + \frac{A_2}{(x+1)^2} + \frac{A_3x+B_3}{x^2+1}\,dx$$

Solving for our constants we get $A_1 = 1$, $A_2 = 2$, $A_3 = 3$, and $B_3 = 0$. Now we can integrate.

Exercise problems: 5,6,17,38,39,44,60,61