We know how to find definite integrals. However, what happens if our limits of integration contain ∞ or the function diverges within the interval? We call these integrals improper definite integrals.

Goal

1. Define an improper definite integral
2. Use limits to find an improper definite integral

What is an improper indefinite integral? Simply put, an improper indefinite integral exists is one of the limits of integration is ∞ or the function diverges within the interval. The following are examples of improper indefinite integrals:

$$\int_0^\infty e^{-x}\,dx \qquad \int_1^\infty \frac{1}{x}\,dx \qquad \int_2^4 \frac{1}{x-2}\,dx \qquad \int_{-1}^1 \frac{1}{\sqrt{x}}\,dx$$

So the question is, "How do I solve these problems?" The answer is first we must find out if the area converges or diverges. If the area diverges, there is no answer as the area is infinite, but if it converges, there is a finite area. Just as when we found the derivative using a limit, we use a limit to find the definite integral. Considering the first two examples with ∞ as a limit of integration, the solution is found this way:

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx = \lim_{b\to\infty} F(b) - F(a)$$

where a is a constant.

Working out these first two examples, we see one of them converges and has a value, while the other one diverges and does not.

In general, one only need to look at the power of the exponent in the denominator to determine whether the area converges or diverges. In considering $\int_a^\infty \frac{1}{u^p}\,dx$, where u is a linear function of x and a is a constant, if $p \leq 1$, then the area diverges. Otherwise, the area converges.

When solving improper integrals where the function diverges, you use the same process with the limit, except the limit goes to the x-value where the function diverges. You may need to split the integral if the point of divergence is in the interior of the interval. If you split the integral, and any portion of the split diverges, the entire integral diverges. Solving the last two examples, we see that one diverges and one converges.

In cases such as these, we can use symmetry to find the improper definite integral. For example:

$$\int_{-3}^{1} \frac{1}{x^2}\,dx = \int_{-3}^{-1} \frac{1}{x^2}\,dx + 2\int_{0}^{1} \frac{1}{x^2}\,dx$$

When we have such rational functions, we can use the principle of inverse to say that if a function converges either horizontally or vertically, it diverges the other way. The only exception is when the exponent is 1 and the inverse is the same as the function. In this case they both diverge.

Example: $\int_0^a \frac{1}{x^2}\,dx$ diverges while $\int_a^\infty \frac{1}{x^2}\,dx$ converges