We can apply these integration techniques to the principles of mass, density, and center of mass

Goal

1. Identify an integral formulas relating density and mass
2. Identify an integral formula relating mass, density, and center of mass
3. Apply these formulas to solve problems

We know from physics that density times area (or volume) equals mass (or total amount). So it only makes sense that if the density (whether constant or not) can be modeled by a function based on the position from a reference point, the mass of the object (or total amount of a substance) is the integral based on the Riemann sum of the function for the density times some small change in x. Therefore is the function for the density is $\delta(x)$, the mass (or total amount) will be as follows:

$$\texttt{Mass} = \int_a^b \delta(x)\,dx$$

For example, say the population density along U.S. 41 from here to Chassell is modeled by the function $\delta(x)$. Since it is 10 miles from Houghton to Chassell, we can find the population living between Houghton and Chassell by finding:

$$\int_0^{10} \delta(x)\,dx$$

However, our calculation need not be linear. Consider the following example. The population density of a region around a city is a function $\delta(x)$ of the distance x miles from the downtown of the city. Find the total population living within a radius of 10 miles from downtown. This case is a little unusual as our "slice" is now a concentric circle (much like the cylindrical shell and a vertical axis of rotation). Finding the population of each slice is $2\pi x\delta(x)$. So the total population that lives within 10 miles is:

$$2\pi \int_0^{10} x\,\delta(x)\,dx$$

How do we calculate center of mass? We know that the center of mass is the balance point for the position of a fulcrum. Imagine a see-saw board $20ft$ long with four children of identical weight on it, one on end and three on the other? On what point on the board should the fulcrum be to balance the see-saw? We know that the force equals the mass times the distance from the fulcrum, and doing the algebra, we can calculate that the fulcrum (center of mass) should be $5ft$ from the end with the three kids.

Suppose now that we have the same see-saw, but now we have 3 kids, with one sitting on one side of the fulcrum, and the other two sitting on the other side on two different places on the see-saw to create a balance. We can use the moment formula to find the center of mass. Given that $\bar{x}$ is the position of the center of mass, and x_i is the position and m_i is the mass for each kid, we get the formula:

$$m_1(x_1 - \bar{x}) + m_2(x_2 - \bar{x}) + m_3(x_3 - \bar{x}) = 0$$

Solving for $\bar{x}$, we get:

$$\bar{x} = \frac{\Sigma_{i=1}^{3} m_i x_i}{\Sigma_{i=1}^{3} m_i}$$

In effect, we have created a Riemann sum. Generalizing this for any object laying on the x-axis, we get the following formula:

$$\bar{x} = \frac{\int_a^b x\,\delta x\,dx}{\int_a^b \delta x\,dx}$$

Problems to work on: Exercises 3, 4, 7, 8, 12, 19

Up to this point, we have considered only one dimension This formula works for two- or three-dimensional figures as well. Therefore, we apply the same formula to each dimension taking into account the formula for each slice in that dimension. Given the appropriate dimension, the center of mass $(\bar{x}, \bar{y}, \bar{z})$ is:

$$\bar{x} = \frac{\int x\,\delta\,A_x(x)}{\texttt{Mass}},\ \bar{y} = \frac{\int y\,\delta\,A_y(y)}{\texttt{Mass}},\ \bar{z} = \frac{\int z\,\delta\,A_z(z)}{\texttt{Mass}}$$

where the function, A is the formula for the area (or volume) of a slice perpendicular to the given axis and the density, δ is constant.

Remember when dealing with center of mass, we are looking to split the figure into two equal portions if you will. Therefore, we can use principles of symmetry to aid in our calculations.

Problems to work on: Exercises 25, 26, 28