General solution for the Couette flow profile

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A general solution for the Couette velocity profile is reported. Taylor's classical one-dimensional profile is shown to be a special case of this solution for configurations whose aspect ratio is large. Numerical evaluation indicates that error between the two profiles is a logarithmic function of the aspect ratio and provides data to estimate when Taylor's profile should be replaced with the present solution. [S1063-651X(99)15111-0]

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I. INTRODUCTION

The Taylor-Couette cell has served as an important shear flow model since Taylor's seminal work in 1923 [1]. Investigators have studied its flow spectrum extensively, helping to elucidate many phenomena in fluid physics, especially laminar transition. The cell is typically run in its fundamental mode, the so-called "Couette" mode, and the system's response to prescribed disturbances is examined. Numerically, this is accomplished by employing a Couette velocity profile as an initial condition $\mathbf{u}(\mathbf{x},0)$, perturbing it, and computing the nonstationary flow evolution $\mathbf{u}(\mathbf{x},t)$. However, an exact solution for the general case has never been published. In most instances [2–4], the span dimension of the cell is assumed to be large enough relative to the gap dimension such that the problem can be modeled one-dimensionally. This results in the well-known solution attributed to Taylor [1], which has long been accepted as the standard theoretical model [5]. Variation vanishes along the axial span of the cell and end effects from the stator walls are neglected.

However, there exist important classes of problems where the one-dimensional assumption is not justified, for example when the span and gap dimensions are comparable or when local effects of the endwalls are of interest. While profiles can be computed numerically, this is generally not favored since it introduces an additional component of truncation error, may exceed reasonable computational effort for high resolution grids, and requires recomputation whenever parameters or grids are altered. Alternatively, approximate Couette solutions are sometimes used [6], but resulting numerical simulations for $t>0$ cannot, in the strictest sense, be considered true Navier-Stokes solutions [7]. A general exact solution for the profile is required for such cases, i.e., one that is independent of restrictions upon the geometric parameters. Such results are derived in this Brief Report.

II. SOLUTION PROCEDURE

An equation describing the Couette profile $u(r,z)$ can be obtained by simplifying the azimuthal component of the cylindrical Navier-Stokes equations via the unidirectional Couette flow model. In dimensionless form, this yields

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}-\frac{u}{r^2}=0. \tag{1}$$

Cell dimensions (Fig. 1) are the gap width a, cell span b, and rotor radius r_0, and the rotor is assumed to turn at a constant rate ω. The corresponding dimensionless geometric parameters are the aspect ratio $\phi=b/a$ and the radius ratio $R=r_0/a$. Using a and $a\omega$ as length and velocity scales, the boundary conditions can be stated nondimensionally as

$$u=0 \quad \text{at} \quad z=0, \quad z=\phi, \quad r=R+1 \tag{2a}$$

and

$$u=R \quad \text{at} \quad r=R. \tag{2b}$$

This system can be solved using an integral transform method [8]. An appropriate integral transform pair is given by

$$\overline{u}(r,\beta_m)=\int_0^{\phi} Z(\beta_m,z')u(r,z')dz' \tag{3a}$$

and

$$u(r,z)=\sum_{m=1}^{\infty}\frac{Z(\beta_m,z)\overline{u}(r,\beta_m)}{\int_0^{\phi} Z^2(\beta_m,z')dz'}. \tag{3b}$$

The overbar notation represents a transform in the z coordinate direction, β_m are corresponding eigenvalues, and

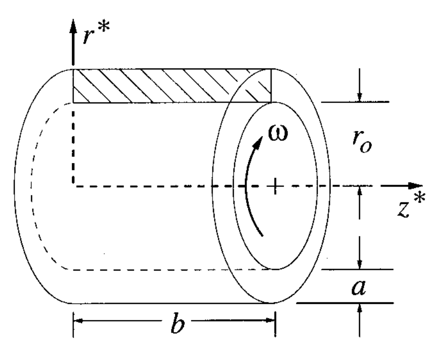

FIG. 1. Finite span Taylor-Couette cell shown in dimensional coordinates. Flow occurs in the cavity represented by cross-hatching.

$Z(\beta_m, z)$ are corresponding eigenfunctions. All eigen-related quantities can be derived from standard tables [9].

Application of transform (3a) to Eq. (1) yields an ordinary differential equation of the Bessel type,

$$\frac{d^2\overline{u}}{dr^2}+\frac{1}{r}\frac{d\overline{u}}{dr}-\left(\beta_m+\frac{1}{r^2}\right)\overline{u}=0. \tag{4}$$

An auxiliary Sturm-Liouville equation and integration by-parts have been used to evaluate various terms. The general solution of Eq. (4) has the form $\overline{u}=c_1I_1(\beta_m r)+c_2K_1(\beta_m r)$, where c_1 and c_2 are constants of integration and I_1 and K_1 are the first order modified (hyperbolic) Bessel functions of the first and second kind, respectively. Using the transformed boundary conditions $\overline{u}=0$ at $r=R+1$ and $\overline{u}=\phi R[1-(-1)^m]/m\pi$ at $r=R$, the transformed solution can be expressed as

$$\overline{u}(r,\beta_m)=-\frac{R[(-1)^m-1]\{I_1[\beta_m(R+1)]K_1(\beta_m r)-K_1[\beta_m(R+1)]I_1(\beta_m r)\}}{\beta_m\{I_1[\beta_m(R+1)]K_1(\beta_m R)-K_1[\beta_m(R+1)]I_1(\beta_m R)\}}. \tag{5}$$

Inverse transform (3b) is applied to Eq. (5) to obtain the physical solution in (r,z) space. Even-order eigenmodes, $m=2,4,6,...$, are not relevant to the solution; therefore, retaining only the participating odd-order modes and applying appropriate simplification, the final solution is found to be

$$u(r,z)=\frac{4R}{\pi}\sum_{m=1}^{\infty}\frac{\{I_1[\beta_m(R+1)]K_1(\beta_m r)-K_1[\beta_m(R+1)]I_1(\beta_m r)\}\sin(\beta_m z)}{\{I_1[\beta_m(R+1)]K_1(\beta_m R)-K_1[\beta_m(R+1)]I_1(\beta_m R)\}(2m-1)}, \tag{6}$$

where $\beta_m=(2m-1)\pi/\phi$.

III. TAYLOR'S SOLUTION: A SPECIAL CASE

Taylor's classical solution [1] can be shown to be a special case of Eq. (6) when the aspect ratio is large. Because ϕ^{-1} is a parameter in the set of all eigenvalues β_m, small argument approximations for the Bessel functions can be employed when ϕ is suitably large. That is, $I_1(r)$ and $K_1(r)$ behave respectively as $r/2$ and $1/r$ for small values of r. Substituting these expressions into Eq. (6) yields

$$u(r,z)=\frac{4R^2(R+1)[(R+1)/r-r/(R+1)]}{(2R+1)\pi}\times\sum_{m=1}^{\infty}\frac{\sin[(2m-1)\pi z/\phi]}{2m-1}. \tag{7}$$

Making use of a summation identity [10], the dependence upon z vanishes and Eq. (7) can be simplified to

$$u(r)=\frac{R^2}{2R+1}\left(\frac{(R+1)^2}{r}-r\right), \tag{8}$$

which is Taylor's classical one-dimensional solution.

This of course invites the question of how small ϕ can become before the one-dimensional assumption is violated. Previous work [5] has emphasized the view that one-dimensional modeling of the Taylor-Couette cell is not physically valid. Nevertheless, for the purpose of representing initial conditions in a numerical simulation, a limit for ϕ can be established for which the two profiles differ by less than an acceptable value. Figure 2 shows the rms difference between Eq. (6) and Eq. (8) as a function of ϕ and R, where the profile representing Eq. (6) is computed at $\phi/2$. (By definition, this is the location whose distance to the nearest end wall is a maximum.) Results show that the error is almost logarithmically related to the aspect ratio over a large spectrum of radius ratios. One-dimensionality becomes more difficult to maintain as R is increased. For example, assuming an acceptable rms error of 10^{-4}, Taylor's solution accurately models the Couette profile for cells of 3:1 aspect ratio and higher at $R=0.1$. However, for large values of R such as 1000, cells must have ϕ slightly greater than 10:1 in order to be modeled one-dimensionally. Trends in Fig. 2 suggest that the difference between the two solutions is not bounded in R. However, the issue is essentially irrelevant for $R\gg1000$ since simpler results based upon the "thin gap" approximation are valid and are usually used instead [11].

Cell aspect ratios are often much greater than 10:1 and for these configurations the Taylor solution appears to be a valid Couette profile. The one exception is if local effects near the

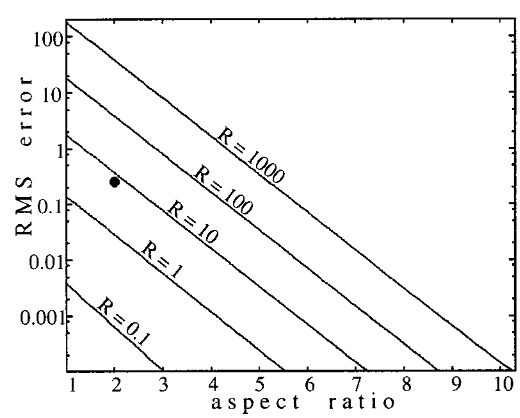

FIG. 2. RMS error between one-dimensional and two-dimensional profiles. Datum ● represents the configuration studied by Hua *et al.* [3].

end walls are to be studied, in which case Eq. (6) should be employed. For smaller aspect ratios the two profiles should first be compared to confirm the validity of one-dimensional modeling, if this simplification is being considered. In many instances it may not be justified; for example, Hua *et al.* [3], repeating the work of Jones [12], used Taylor's profile for a cell having a 2:1 aspect ratio and $R=7.039$. However, according to Fig. 2 (see datum) this has an associated rms error on the order of 0.26. The aspect ratio would have to be increased to about 7:1 to realize an rms error of 10^{-4}.

IV. CONCLUSION

A general solution for the Couette flow profile is reported. Our result is independent of geometric restrictions. Taylor's one-dimensional solution is shown to be a special case of this profile for large aspect ratio cells. Error between the two profiles is approximately a logarithmic function of the aspect ratio for a wide range of radius ratios. Corresponding data may be used to estimate which profile is appropriate for a given case.

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