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ON THE CONDITIONING NUMBER AND THE SELECTION CRITERIA FOR p-VERSION APPROXIMATION FUNCTIONS

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Abstract—This paper discusses the full and reduced set one-, two- and three-dimensional p-version approximation functions based on Lagrange, Legendre and Chebyshev polynomials, and the criteria for their selection for linear as well as nonlinear problems. A study of the element conditioning number is presented for various aspect ratios and element distortions for various p-levels. Conditioning number studies are also presented for uniform as well as nonuniform meshes for various p-levels. The influence of the element and the mesh conditioning number on the accuracy of the computed solution as well as the convergence of the linear and nonlinear iterative solution procedures is discussed. A discussion of the ease of computation and recursive forms as well as the ease of describing boundary conditions is presented. Numerical examples are given to illustrate various aspects discussed and presented in the paper. Copyright © 1996 Elsevier Science Ltd.

INTRODUCTION

Among the various p-version approximations in current use, those based on Legendre, Lagrange and Chebyshev polynomials are most common. Babuska, Szabo and others have strictly employed Legendre polynomial-based approximations in their research work [1–4]. Surana and his co-workers have used Lagrange polynomial based p-version approximation functions [5–8]. The authors of Refs [9] and [10] and some other researchers have utilized Chebyshev polynomial based p-version approximation functions in their work.

Babuska *et al.* [11] are probably among the first ones to shed some light on the criteria for selecting p-version approximation functions. Their investigation focused on Legendre polynomial based p-version functions. The authors used the element conditioning number for a single element as a major criteria. Various studies of element conditioning numbers are presented for cases in which nodal, internal and side functions are selected in various combinations. The p-version functions, based on a reduced set with condensation, is advocated as a preferred strategy in conjunction with conjugate gradient and multilevel iteration methods as the solution procedures. Babuska and Elman [12] investigated the performance of the p-version square elements of type $Q(p)$ (full set) and $Q'(p)$ (reduced set) based on Legendre polynomials. The element performance was analyzed theoretically and numerical experiments were presented to confirm the theoretical conclusions. The paper investigated the approximation properties (in H^1-seminorm) of $Q(p)$ and $Q'(p)$ elements and their computational efficiency in achieving a prescribed accuracy for a specially designed problem over a square domain. The authors conclude that for a given p-level $Q(p)$ yields better accuracy than $Q'(p)$ for the test problem considered. Both $Q(p)$ and $Q'(p)$ elements exhibit just about the same convergence rates. The paper also investigates the performance of $Q(p)$ and $Q'(p)$ for distorted elements as well as triangular elements. Vijayakar *et al.* [9] advocated the use of Chebyshev polynomials in deriving p-version approximation functions which were used in finite elements designed specifically for quasi-prismatic bodies. They cited many properties of these polynomials which are well known and advocated these to be desirable. Yet we know that none of these properties are of any serious consequence for the elements or mesh conditioning numbers and their accuracy. Both of these important issues are not addressed in Ref. [9]. Devloo *et al.* [10] have used Chebyshev polynomials in h-p-adaptive finite element study of compressible flow. The properties cited by these authors (economical to compute, bounded and hierarchical) are of little consequence in terms of constituting a criteria for selecting the p-version approximation functions. To our knowledge there are no other publications containing discussions of the criteria for selecting the p-version approximation functions. The investigations presented by Babuska *et al.* [11] and Babuska and

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Elman [12] are too specific to either $Q(p)$ or $Q'(p)$ sets based on Legendre functions only, and are too narrowly focused for a specific numerical example.

SCOPE OF PRESENT RESEARCH

We ask the basic question: among the most commonly used p-version functions derived using Lagrange, Legendre and Chebyshev polynomials for full and reduced sets, which p-version functions are the base choice? The answer to this question requires a development of suitable criteria which can then be used to judge the performance of these approximation functions. At this point we make some remarks which will help us toward establishing such criteria.

Remarks

(1) In this study we shall consider one-, two- and three-dimensional p-version functions based on Lagrange, Legendre and Chebyshev polynomials.

(2) The approximation functions in set $Q(p)$ are complete polynomials, thus the approximation power and the accuracy of the computed solution is invariant of the choice of p-version functions as long as the numerical round off is not influencing the computations. For example, in linear solid mechanics we can write

$$[K]\{\delta\} = \{f\} + \{P\}. \tag{1}$$

The coefficients of $[K]$ and $\{f\}$ are different for different p-version functions and thus the computed solution $\{\delta\}$ will also be different for each of the three types of functions. However, when the computed $\{\delta^e\}$ for an element e is substituted in the element approximation

$$\{\phi\} = [N]\{\delta^e\}, \tag{2}$$

the solution $\{\phi\}$ will be the same for all three types of functions. This conclusion stems from the fact that all three types of p-version functions are complete algebraic polynomials and hence they must produce the same $\{\phi\}$, even though $[K]$, $\{f\}$ and $\{\delta\}$ in the intermediate computational steps are different. Thus from the point of view of the accuracy of solution $\{\phi\}$ (which is what we are interested in) all three types of $Q(p)$ functions behave the same, i.e. no one type is superior over the others.

(3) The argument presented in item (2) also applies to the p-version functions based on the reduced set $Q'(p)$ as long as identical order polynomial terms are omitted from the $Q(p)$ set in all three types of functions in generating the corresponding $Q'(p)$ sets. Thus from the point of view of accuracy of solution $\{\phi\}$, all three types of $Q'(p)$ functions behave the same, i.e. no one type is superior over the others.

(4) We have seen from item (2) that different p-version functions yield different element matrices $[K^e]$. Is this of any consequence? The ratio of the largest to smallest non zero eigenvalues of $[K^e]$ is known as the element conditioning number. The element conditioning number varies depending upon the type of functions, element aspect ratio (ratio of the largest to the smallest side) and the element distortion. The elements with bigger aspect ratios and more distortion generally result in a larger conditioning number. The conditioning numbers of the individual elements in a discretization influence the conditioning number of the assembled matrix $[K]$. The assembled matrix $[K]$ with a large conditioning number will generate more numerical round-off error and a slower convergence (in the case of iterative methods) in the process of solving for $\{\delta\}$. Thus the element conditioning number and perhaps the conditioning numbers of some representative meshes for the three types of functions may be one valid criterion for comparison. The functions resulting in a lower element conditioning number are superior and are preferred over those with a larger conditioning number.

(5) The discussion presented in item (4) necessitates an investigation of the element conditioning numbers of $Q(p)$ and $Q'(p)$ sets of different aspect ratios as well as distortions and also perhaps the conditioning numbers of some representative meshes.

(6) Since the conditioning number influences numerical round-off errors, investigations are needed for linear problems as well as for nonlinear problems. For linear problems we may utilize elimination methods or iterative methods to solve for $\{\delta\}$. On the other hand, for nonlinear problems iterative methods must be employed to solve for $\{\delta\}$. How crucial is the conditioning number in elimination methods? How does the conditioning number influence the convergence of the iterative solution procedure? What orders of magnitude of the conditioning numbers can be tolerated without severely affecting the efficiency of computations? Answers to these questions need to be quantified.

(7) The ease of computations, hierarchical structure and recursive forms are other desirable properties which to some extent all three types of functions possess.

(8) An examination of the hierarchical degrees of freedoms for the three types of approximation functions reveals that for p-version functions based on Lagrange polynomials the hierarchical degrees of freedom are tangential derivatives of various orders at the midside nodes. In the case of p-version functions based on Legendre and Chebyshev polynomials, the hierarchical degrees of freedom are not physically meaningful quantities. Thus the element approximations based on these three types of polynomials may differ in their ability to represent the boundary conditions, which in turn may influence the accuracy of the computed results, convergence of the iterative solution procedure, etc.

This paper addresses all of the questions and issues raised and discussed in the above remarks. First we present a brief overview of one-dimensional p-version functions based on Lagrange, Legendre and Chebyshev polynomials. This is followed by a section containing a brief discussion of $Q(p)$ and $Q'(p)$ two- and three-dimensional approximation functions. Numerical studies of the conditioning numbers are presented for a single element one-dimensional case, two-dimensional cases with various aspect ratios and distortions and three-dimensional elements for $Q(p)$ and $Q'(p)$ sets using Lagrange, Legendre and Chebyshev polynomial approximation functions. Numerical studies are also presented for representative two-dimensional meshes. These studies are followed by a discussion of the importance of the conditioning number for linear as well as nonlinear problems. The suitability of the various types of approximation functions in describing prescribed boundary conditions is discussed and illustrated. The last section of the paper contains summary and conclusions.

$Q(p)$ AND $Q'(p)$ SETS

The $Q(p)$ set (full set) represents p-version approximation functions where the order of the polynomials is complete up to degree "p". In the present study we consider three-node one-dimensional elements, nine-node two-dimensional quadrilateral elements and 27 node three-dimensional solid elements. The elements in the cartesian coordinate space can be distorted and their sides can be curved. Each element of the discretization is mapped into its own natural coordinate space ξ, η, ζ, in a two unit length, square or cube with the origin of the coordinate system located at the center of the element [Fig. 1(a–c)]. The p-version hierarchical approximation functions $Q(p)$ and $Q'(p)$ are derived for the element map in the natural coordinate system.

Consider the element of Fig. 1(b). For this element, the p-version approximation functions belonging to set $Q(p)$ are complete polynomials of degree p both in ξ and η. Nodes 1, 3, 5 and 7 are nonhierarchical nodes and involve the approximation functions N_1, N_3, N_5 and N_7 which are linear in ξ and η. Mid-side nodes 2, 4, 6 and 8 and the center node 9 are hierarchical nodes. The number of approximation functions belonging to these nodes depend upon the p-levels in ξ and η directions. For the sake of simplicity we consider $p_\xi = p_\eta = p$. The mid-side nodes have $4(p-1)$, $p \geqslant 2$ approximation functions whereas the center node 9 has $(p-1)(p-1)$; $p \geqslant 2$ approximation functions. We note that as p-level increases the number of approximation functions at the center node increase tremendously. For example, at $p = 10$, there are a total of 121 approximation functions out of which 81 belong to the center node 9. The obvious question is of course whether or not we can reduce the number of approximation functions at the center node without sacrificing accuracy significantly. A similar situation exists for three-dimensional elements.

The set $Q'(p)$ (reduced set, only for two- and three-dimensional elements) is similar to set $Q(p)$ in that the approximation functions in the $Q'(p)$ set for the corner and mid-side nodes are the same as those in $Q(p)$, but the approximation functions at the center node are reduced significantly. This of course leads to incomplete polynomials in ξ, η and ζ. In other words, the set $Q'(p)$ is derived from the set $Q(p)$ by deleting certain polynomial terms and hence the name "reduced set". In this study for two-dimensional elements (for example), we will consider a specific form of $Q'(p)$ which contains $(p-2)(p-3)/2$; $p \geqslant 4$ approximation functions at the center node as opposed to $(p-1)(p-1)$ in the $Q(p)$ set. This reduction in the number of approximation functions at the center node is probably at the expense of accuracy and possibly the p-convergence rate. The concepts described for the nine-node two-dimensional element is similarly extended for the 27 node p-version three-dimensional elements.

For one-dimensional elements there is no concept of the $Q'(p)$ set. For two- and three-dimensional

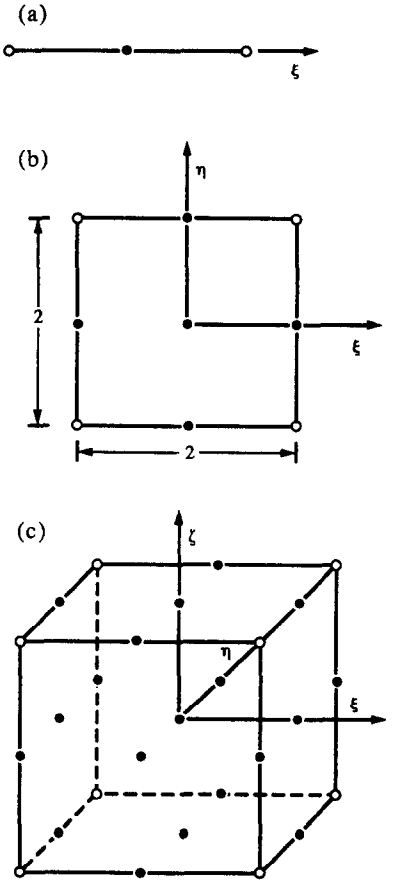

Fig. 1. Element configurations in ξ η space: (a) 3-node one-dimensional element; (b) 9-node two-dimensional element; (c) 27-node three-dimensional element.

elements the $Q(p)$ set functions are generated by taking tensor product of the one dimensional p-version functions in ξ, η and ξ, η, ζ directions. The $Q'(p)$ set functions are generated by deleting certain polynomial terms (as described above). Thus the one-dimensional p-version functions are the basis for both the $Q(p)$ and $Q'(p)$ sets. Details of the tensor product procedure and the procedure for generating $Q'(p)$ functions can be found in Ref. [13] and are omitted here for the sake of brevity.

ONE DIMENSIONAL p-VERSION FUNCTIONS

Consider a three-node configuration of length two units in the natural coordinate system ξ with the origin of the coordinate system located at the center of the element. In the following section we describe the p-version one dimensional approximation functions for this three-node configuration using Lagrange, Legendre and Chebyshev polynomials.

1. p-*Version Lagrange functions*

For $(p_\xi + 1)$ equally spaced nodes between $-1 \leqslant \xi \leqslant 1$, the Lagrange interpolation functions, defined for each node along with the corresponding nodal values of the dependent variables, uniquely define an element approximation of order p_ξ. These $(p_\xi + 1)$ equally spaced nodes with Lagrange interpolation functions and nodal variables can be reduced to a three-node configuration between $-1 \leqslant \xi \leqslant 1$ [Fig. 1(a)] for which we can write the following for a field variable ϕ:

$$\phi(\xi) = N_1^{1\xi}\phi_1 + N_2^{1\xi}\phi_2 + \sum_{i=2}^{p_\xi} N_3^{i\xi}\left(\frac{1}{i!}\left(\frac{\partial^i \phi}{\partial \xi^i}\right)_{\xi=0}\right), \quad (3)$$

where

$$N_1^{1\xi} = \left(\frac{1-\xi}{2}\right), \quad N_2^{1\xi} = \left(\frac{1+\xi}{2}\right),$$

$$N_3^{i\xi} = (\xi^i - a); \quad i = 2, \ldots, p_\xi;$$

$$\begin{cases} a = 1, & \text{if } i \text{ is even} \\ a = \xi & \text{if } i \text{ is odd} \end{cases}. \quad (4)$$

Equation (3) defines a p-version hierarchical approximation of order p_ξ for the one-dimensional three-node element in ξ space. Equation (4) defines the explicit form of the approximation functions. The nodal variable operators in the approximation defined by (1) are 1, 1 and $\partial^i/i!\partial\xi^i$; $i = 2, \ldots, p_\xi$ for nodes 1, 2 and 3, respectively.

2. p-*Version Legendre functions*

Legendre polynomials $P_n(\xi)$ are solutions of Legendre differential equation (Ref. [14]) for $n = 0, 1, 2, \ldots$, and are given by

$$P_0 = 1, \quad P_1(\xi) = \xi, \quad (5)$$

$$(n+1)P_{n+1}(\xi) = (2n+1)\xi P_n(\xi) - nP_{n-1}(\xi) \quad \text{for } n = 1, 2, \ldots. \quad (6)$$

The p-version one-dimensional approximation functions for the three node configuration of Fig. 1(a) can be easily constructed using eqns (5) and (6) and are given by

$$\bar{N}_1^{1\xi} = \left(\frac{1+\xi}{2}\right), \quad \bar{N}_2^{1\xi} = \left(\frac{1+\xi}{2}\right),$$

$$\bar{N}_3^{i\xi} = \frac{1}{\sqrt{2(2i-1)}}(P_i(\xi) - P_{i-2}(\xi)); \quad i = 2,3, \ldots \quad (7)$$

Note that $\bar{N}_3^{i\xi}$ are polynomials of degree i. We also note that if ϕ is the field variable to be interpolated then ϕ_1 and ϕ_2 are the degrees of freedoms at nodes 1 and 2. However, at node 3 (hierarchical node) the degrees of freedom are not physically meaningful quantities as they are in the case of Lagrange p-version functions.

3. p-*Version Chebyshev functions*

One-dimensional Chebyshev polynomials are defined by

$$C_0 = 1, \quad C_1 = \xi, \quad C_{n+1}(\xi) = 2\xi C_n(\xi) - C_{n-1}(\xi); \quad n \geqslant 2. \quad (8)$$

Using the Chebyshev polynomials defined by eqn (8), the p-version approximation functions for the three-node configuration of Fig. 1(a) can be defined as

$$\hat{N}_1^{1\xi} = \left(\frac{1+\xi}{2}\right), \quad \hat{N}_2^{1\xi} = \left(\frac{1+\xi}{2}\right)$$

$$\hat{N}_3^{i\xi} = C_i(\xi) - \hat{N}_2^{1\xi} - (-1)^i \hat{N}_1^{1\xi}; \quad i = 2, 3 \ldots \quad (9)$$

Here also we note that $\hat{N}_3^{i\xi}$ are polynomials of degree i. We also note that if ϕ is the field variable to be interpolated, then ϕ_1 and ϕ_2 are the degrees of freedom at nodes 1 and 2. However at node 3 the degrees of freedom are not physically meaningful quantities unlike for the case of Lagrange functions.

Remarks

(1) All three one-dimensional p-version functions are complete polynomials, are hierarchical in nature, possess a recursive form and are equally easy to compute.

(2) Functions $\bar{N}_1^{1\xi}$, $\bar{N}_2^{1\xi}$ and $\bar{N}_3^{i\xi}$ based on Legendre polynomials have some important properties:

$$\bar{N}_3^{i\xi}(-1) = \bar{N}_3^{i\xi}(+1) = 0; \quad i = 2, 3, \ldots, \quad (10)$$

$$\int_{-1}^{1} \frac{d\bar{N}_3^{i\xi}}{d\xi} \frac{d\bar{N}_3^{i\xi}}{d\xi} d\xi = \begin{cases} 1; & i = j \\ 0; & i \neq j \end{cases}. \quad (11)$$

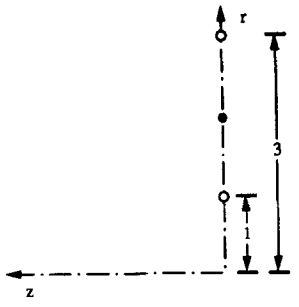

Fig. 2. One element model for p-version radial heat conduction.

(3) Hierarchical degrees of freedom for the Lagrange functions are physically meaningful quantities, whereas for Legendre and Chebyshev p-version functions this is not the case.

(4) p-Version Lagrange and other functions for two- and three-dimensional $Q(p)$ elements can be easily generated by taking the tensor product of the one-dimensional p-version functions in ξ, η, ζ directions. Corresponding degrees of freedom are generated by taking tensor products of the nodal variable operators.

(5) $Q'(p)$ two- and three-dimensional functions are generated by deleting selected polynomial product terms as described earlier [13].

(6) It is important to note that the lack of specific definitions of the hierarchical degrees of freedom for node 3 of the one-dimensional three-node configuration for Legendre and Chebyshev polynomials results in similar insignificant degrees of freedom for the midside and center hierarchical nodes for the two-dimensional elements and midside, or face and center nodes for the three-dimensional elements. This may have some serious consequences in terms of defining higher order, complex boundary conditions.

INVESTIGATION OF ELEMENT CONDITIONING NUMBER

In this section we present numerical studies for the element conditioning number for Lagrange, Legendre and Chebyshev polynomials based upon one-, two- and three-dimensional p-version approximation functions for regular and distorted elements with different aspect ratios.

1. *One-dimensional elements*

As an example, we consider one-dimensional radial heat conduction. Figure 2 shows a three-node p-version element with an inside radius of 1 and an outside radius of 3 (element length of 2 units). Figure 5 shows plots of the element conditioning number as a function of p-level for Lagrange, Legendre and Chebyshev polynomials computed using one-dimensional p-version approximation functions. Legendre p-version functions yield the lowest element conditioning number which almost remains constant

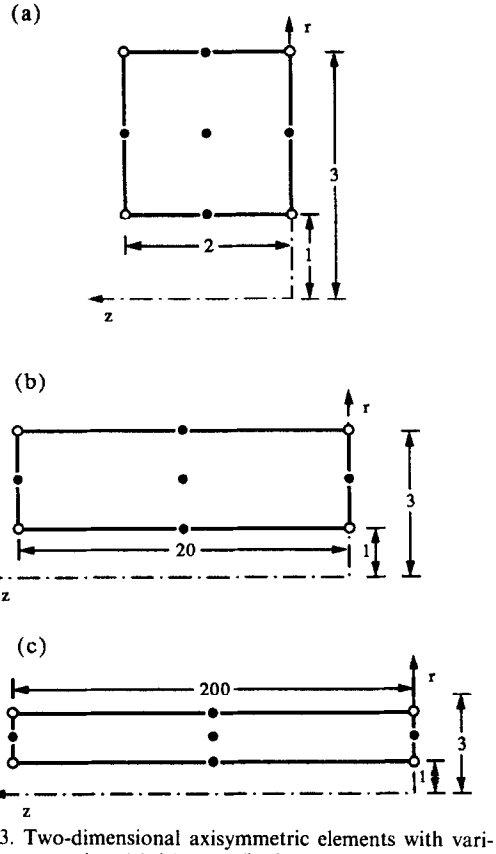

Fig. 3. Two-dimensional axisymmetric elements with various aspect ratios: (a) $b/a = 1$; (b) $b/a = 10$; (c) $b/a = 100$.

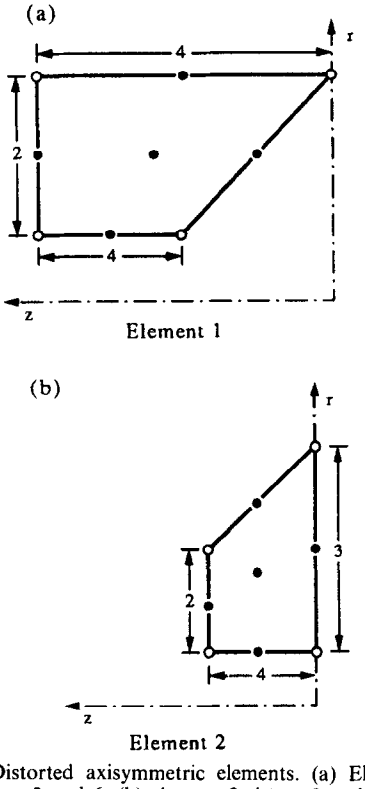

Fig. 4. Distorted axisymmetric elements. (a) Element 1: $b/a = 2$ and 6; (b) element 2: $b/a = 2$ and 6.

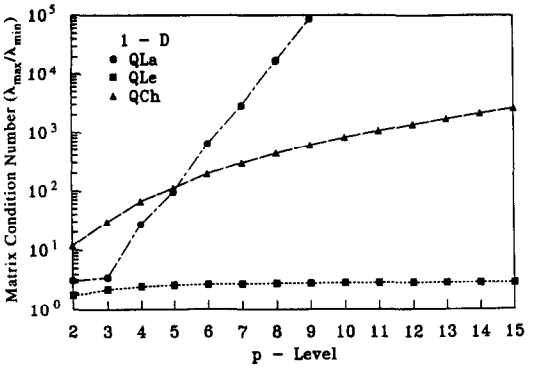

Fig. 5. Element conditioning number vs p-levels for the one-dimensional case (Fig. 2).

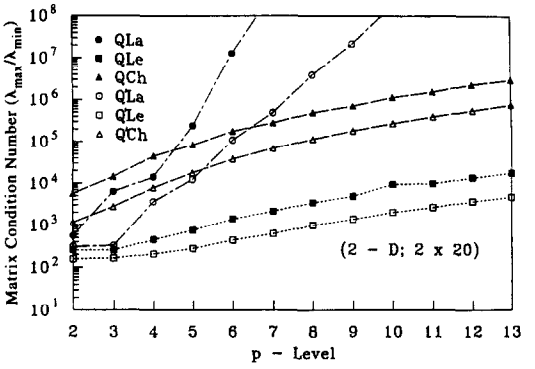

Fig. 7. Element conditioning number vs p-levels for the two-dimensional case (Fig. 3(b), $b/a = 10$).

as p-level increases. Chebyshev polynomial-based functions have the second best performance. For Lagrange p-version functions, the element conditioning number increases quite rapidly as the p-level increases. At $p = 9$, the element conditioning number climbs to 10^5.

2. *Two-dimensional element*

In this study we consider two-dimensional elements for axisymmetric heat conduction. Figure 3(a–c) shows p-version nine-node square and rectangular elements with aspect ratios of 1, 10 and 100, respectively. The element conditioning numbers for these elements are computed for $p = 2$–15 for $Q(p)$ and $Q'(p)$ sets using Lagrange, Legendre and Chebyshev polynomials. The results are presented in Figs 6–8. First we consider Fig. 6 showing the results for a two unit square element.

(a) Legendre functions yield the lowest element conditioning number.

(b) Chebyshev functions yield the second lowest conditioning number.

(c) Lagrange functions yield the highest element conditioning number.

(d) The element conditioning number increases as the p-level increases. Legendre and Chebyshev functions exhibit about the same rate of increase as p-level increases, however the Lagrange functions exhibit a much faster rate of increase compared to Legendre and Chebyshev functions.

(e) The functions based on the $Q'(p)$ set yield lower conditioning numbers for all the three types of functions considered here. Again the rate of increase is most significant for the Lagrange functions.

(f) From Fig. 6, we note that the $Q'(p)$ Legendre functions yield the lowest element conditioning number.

From the results presented in Figs 7 and 8 for aspect ratios (b/a) of 10 and 100, we can confirm the same observations as described above for Fig. 6. Some additional observations are:

(a) As the element aspect ratio increases the element conditioning number increases. The rate of increase of the element conditioning number as the p-level increases is approximately the same for $b/a = 1$ and 10 for Legendre and Chebyshev based functions.

(b) For $b/a = 100$, the Legendre functions yield almost constant element conditioning numbers as the p-level increases.

(c) Lagrange functions yield the most increase in the element conditioning number as element distortion increases. For $b/a = 100$, the element conditioning number for the Lagrange set reaches a value of 10^8 at $p = 6$–7. For Chebyshev polynomials a similar value of element conditioning number is reached at

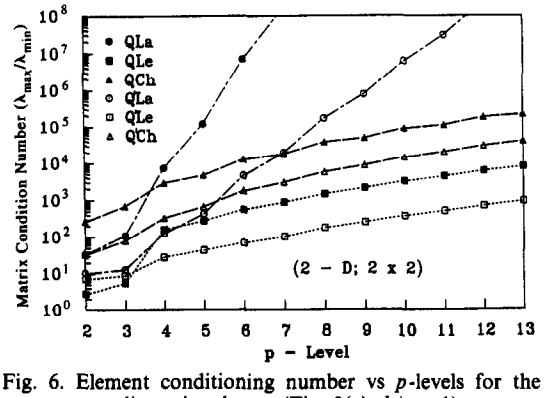

Fig. 6. Element conditioning number vs p-levels for the two-dimensional case (Fig. 3(a), $b/a = 1$).

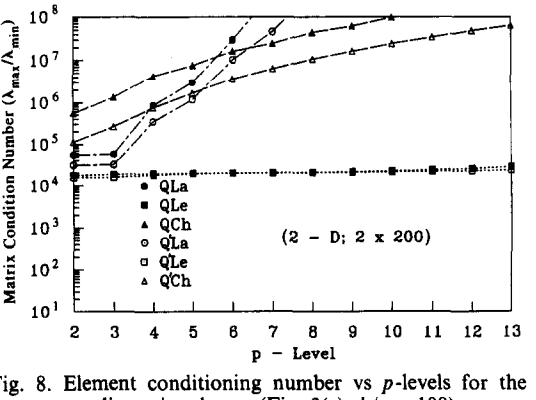

Fig. 8. Element conditioning number vs p-levels for the two-dimensional case (Fig. 3(c), $b/a = 100$).

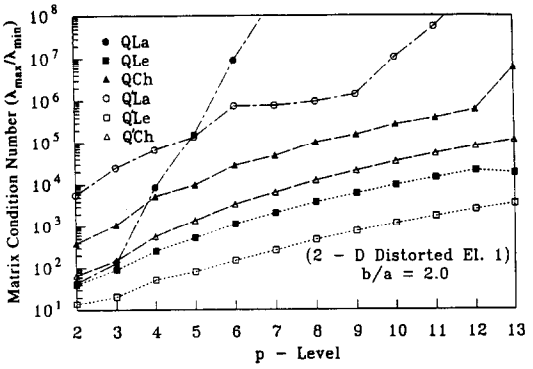

Fig. 9. Element conditioning number vs p-levels for the two-dimensional case (Fig. 4(a), element 1, $b/a = 2$).

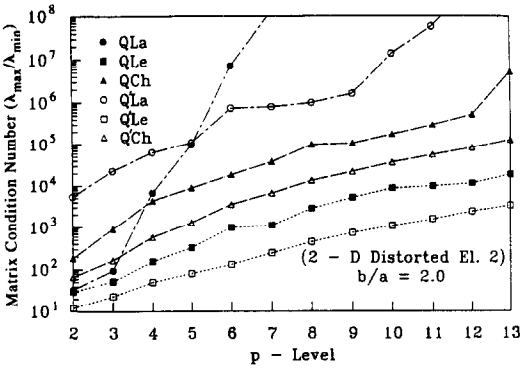

Fig. 11. Element conditioning number vs p-levels for the two-dimensional case (Fig. 4(b), element 2, $b/a = 2$).

$p = 10$ for the $Q(p)$ set and at $p = 13$ for the $Q'(p)$ set, whereas both $Q(p)$ and $Q'(p)$ sets based on Legendre functions yield almost a constant value of the element conditioning number, approximately 10^4, for the entire range of p-levels (2–13) considered.

Figure 4 shows two two-dimensional distorted axisymmetric elements. The elements of Fig. 4(a) and (b) are considered for $b/a = 2$ and 6. The element conditioning numbers as a function of p-level for the element of Fig. 4(a) for $b/a = 2$ and 6 for the three types of functions are shown in Figs 9 and 10. Similar results for the element of Fig. 4(b) for $b/a = 6$ are shown in Figs 11 and 12. We make the following observation from the results presented in Figs 9–12:

(a) The element conditioning number is the lowest for Legendre functions. Chebyshev and Lagrange functions rank second and third, respectively.

(b) The element conditioning numbers increase slightly as b/a increases for the distorted elements.

(c) Element orientation seems to have very little effect on the element conditioning number.

3. *Three-dimensional elements*

We now consider three-dimensional heat conduction. Figure 1(c) shows a twenty seven-node three-dimensional p-version element ($2 \times 2 \times 2$ cube) used for computing the conditioning number. Figure 13 shows plots of the element conditioning number for $Q(p)$ and $Q'(p)$ sets for $p_\xi = p_\eta = p_\zeta = p = 2$–10, for Lagrange, Legendre and Chebyshev functions.

(a) In general, the observation noted for the two-dimensional cases holds here as well. p-Version $Q'(p)$ Legendre functions yield the lowest element conditioning numbers.

(b) $Q(p)$ Legendre set yields a substantially higher conditioning number than the $Q'(p)$ set as p-levels are increased.

(c) We observe similar behavior as described in (b) for Lagrange and Chebyshev functions as well.

(d) For elements with larger aspect ratios and distortion, the element conditioning number increases just like it does for the two-dimensional elements (results not presented for the sake of brevity).

(e) The Lagrange $Q(p)$ set yields excessively large conditioning number even for p-levels as low as 4 or 5.

CONDITIONING NUMBERS OF REGULAR AND GRADED MESHES

Even though the element conditioning number is an important criterion for judging the performance of the various types of elements, in actual application it is the conditioning number of the discretization. The elements that exhibit the largest and smallest conditioning numbers often define the bounds of the

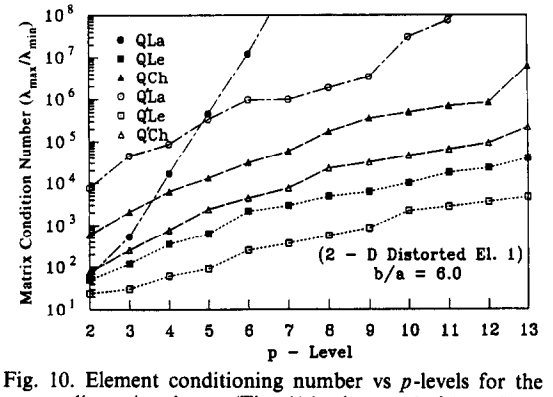

Fig. 10. Element conditioning number vs p-levels for the two-dimensional case (Fig. 4(a), element 1, $b/a = 6$).

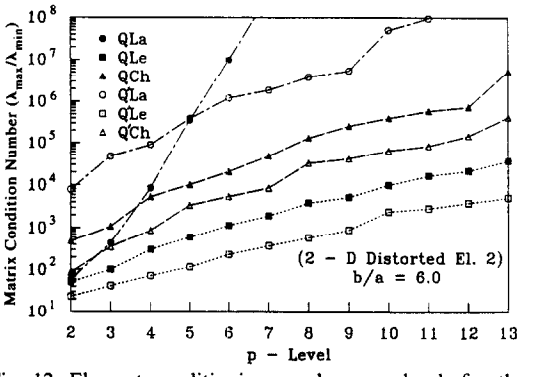

Fig. 12. Element conditioning number vs p-levels for the two-dimensional case (Fig. 4(b), element 2, $b/a = 6$).

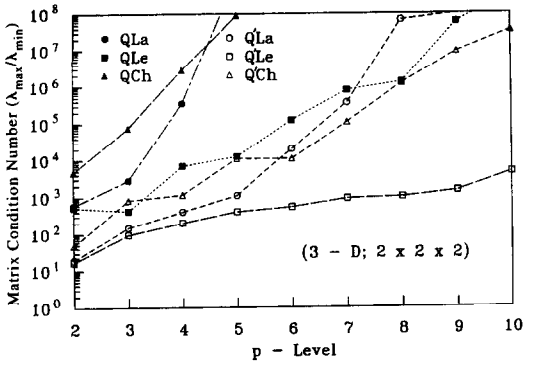

Fig. 13. Element conditioning number vs p-levels for the three-dimensional case (Fig. 1(c)).

conditioning number of the discretization. Uniform discretization yields a better conditioning number. More severely graded meshes result in progressively higher conditioning numbers for higher p-levels and thus are more likely to suffer from round-off errors. These findings have been confirmed with specific numerical studies (not presented here for the sake of brevity). Legendre functions yield the lowest conditioning number. Chebyshev and Lagrange functions rank second and third, respectively.

ACCURACY OF THE COMPUTED SOLUTION FOR LINEAR PROBLEMS

A large number of numerical studies have been conducted for linear problems in solid mechanics and linear heat conduction for one-, two- and three-dimensional problems using Lagrange, Legendre and Chebyshev functions with uniform as well as severely graded meshes. In each case the assembled equations were solved using the wavefront solution method. In all studies p-levels were varied from 2 to 15. The results are summarized in the following:

(1) The computed values of the primary variables from all three functions agree almost to the last decimal place in all cases.

(2) The sum of the secondary variables at the nodes without specified essential boundary conditions was always of the order of 10^{-7} or lower, indicating that the equilibrium of forces, moments, etc. is satisfied within 10^{-7} or better. The computations were performed on a 32 bit computer in double precision.

(3) Thus we note that even though Lagrange elements have excessively high conditioning number compared to Legendre elements, the numerical values of the solution are not affected at all when elimination methods are employed to solve the resulting assembled algebraic equations.

(4) Some other researchers [15] have reported slightly different findings when iterative methods are employed in solving the system of linear algebraic equation:

(a) as the conditioning number of the mesh increases the number of iterations increase significantly;

(b) for excessively large mesh conditioning number the iterative solution procedure may even fail to converge;

(c) failure of the convergence of the iterative solution procedure is directly related to the round-off errors.

CONDITIONING NUMBER AND ACCURACY OF THE SOLUTION FOR NONLINEAR PROBLEMS

The situation for nonlinear problems is more complex than for linear problems. First of all the assembled matrix coefficients are nonlinear functions of the unknown solution and thus iterative methods must be employed to find the solution. Since the solution is continuously changing during the iterative solution procedure so are the coefficients of the assembled matrix and the conditioning number. The mesh conditioning number for a null starting solution ($\{\delta\} = \{0\}$) is perhaps only a vague indicator of what really happens during the actual iterative solution procedure.

As for linear problems, a significant number of studies using all three types of functions for a number of nonlinear problems in fluid dynamics (steady state as well as transient) have been conducted. In all numerical studies, Newton's method with line search was utilized to find the solution vector. The resulting linear equations during each iteration were solved using the Gaussian elimination method (wavefront solution method). In all numerical studies, p-levels ranged from 2 to 15. We make the following specific remarks based on our findings:

(a) The numerical results from the three types of functions showed similar agreement as for the linear problems.

(b) The number of iterations required by Newton's method with line search was also the same when the three types of functions were used.

(c) Studies reported by other researchers [15] indicate that when the preconditioned conjugate gradient methods are employed to find the solution vector, the convergence slows down significantly as the mesh conditioning number increases. Thus when using such solution methods Legendre p-version functions are preferable over the others.

SPECIFYING BOUNDARY CONDITIONS

As discussed in an earlier section of the paper, the hierarchical degrees of freedoms for the p-version Lagrange elements are second and higher order tangential derivatives at the midside and face nodes, whereas for Legendre and Chebyshev functions these degrees of freedom are not meaningful quantities. This fact may have some consequences in prescribing complex and varying boundary conditions. This in

turn may influence the convergence of the iterative processes and the computed results especially when the meshes are coarse, which usually is the case when using p-version elements. This occurs even though in general, any set of complex boundary conditions can also be accurately represented by both Legendre and Chebyshev functions by using a sufficient number of conditions which would permit the computations of the corresponding hierarchical degrees of freedom. This procedure will require the inverse of a matrix, the size of which will depend upon the p-levels. A failure to do so will only permit linear boundary conditions which may influence the computed results. We present the following example to illustrate this point.

We consider a two-dimensional lid driven cavity as shown in Fig. 14(a). We assume Newtonian fluid and laminar flow. The two upper corners of the cavity represent points of singularity. Figure 14(b) shows a sixteen element graded mesh ($a = 0.1$, $b = 0.4$, $c = 0.2$). The velocity field at the upper two corners is assumed to be zero. Thus the velocity must increase from zero at the upper two corners to -1 (the velocity of the moving lid). This is accomplished within the small lengths a of the two elements at the upper corners. Figure 14(c) shows a cubic distribution of the specified velocity field at the two top corner elements. Figure 14(d) shows the lincar velocity distribution.

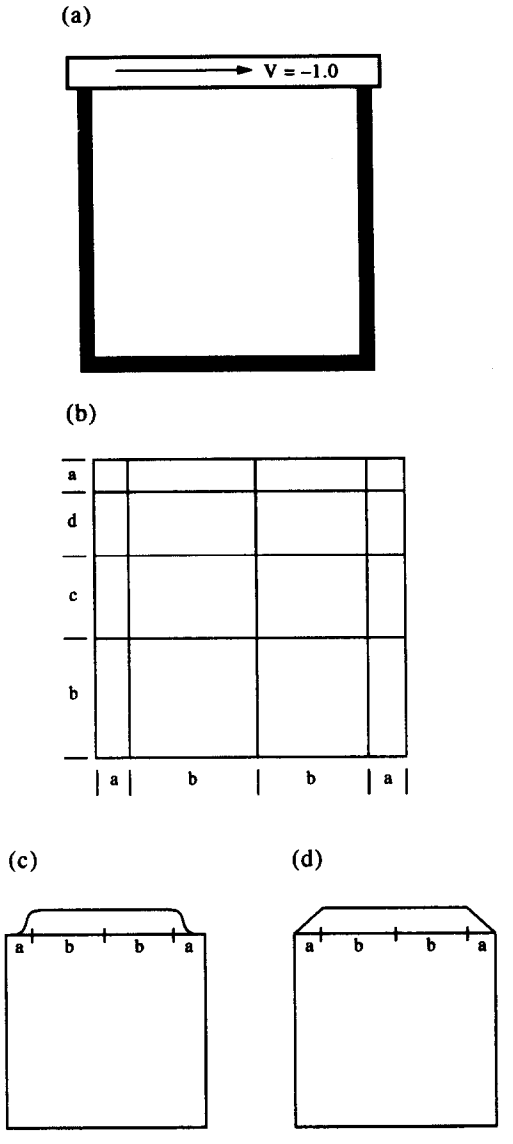

Fig. 14. Lid driven cavity: (a) schematic; (b) 16 element graded mesh; (c) cubic boundary conditions at the top corners; (d) linear boundary conditions at the top corners.

Numerical computations were performed (with the $Q(p)$ set) using the velocity distribution of Fig. 14(c) for Lagrange and that of Fig. 14(d) for Legendre and Chebyshev p-version functions. The maximum of the absolute value of stream function and the recirculation zones in the bottom left and right corners were monitored. $|\psi|_{max} = 0.1364$ was obtained using Lagrange functions for boundary conditions of Fig. 14(c) at $p = 5$. Legendre and Chebyshev functions predicted $|\psi|_{max} = 0.1020$ and 0.1009 for the boundary conditions of Fig. 14(d) for $p = 5$. In the case of Legendre and Chebyshev functions p-levels were increased up to 8 resulting in $|\psi|_{max} = 0.1237$ and 0.1122. In all cases, the computed results predicted recirculation zones in the bottom corners.

The studies with Legendre and Chebyshev functions do not predict a correct value of $|\psi|_{max}$. This is due to the linear boundary conditions at the upper two corners. This situation can be corrected either by performing local mesh refinement in the vicinity of the upper two corners, such that element lengths containing linear distribution become very small or by computing correct values of the hierarchical variables though the inverse process. We use a 36 element mesh with the upper two corner element lengths of 0.05. Numerical computations with this mesh yield $|\psi|_{max} = 0.1361$ and 0.1359 for Legendre and Chebyshev polynomials at $p = 6$.

SUMMARY AND CONCLUSIONS

p-Versions of full and reduced sets with one-, two- and three-dimensional approximation functions based on Lagrange, Legendre and Chebyshev polynomials have been presented, and the criteria for their selections for linear as well as nonlinear problems have been discussed. Element conditioning number studies have been presented for different aspect ratios and element distortions. The results of the conditioning number studies for uniform as well as nonuniform meshes have also been discussed. The influence of the element and the mesh conditioning number on the accuracy of the solution and the convergence of the linear and nonlinear iterative solution procedure have been discussed. Based on the numerical studies and the results presented in the paper, we can draw the following specific conclusions:

(1) Legendre functions yield the lowest conditioning number. Chebyshev functions yield the second lowest conditioning number and the Lagrange functions yield the highest conditioning number.

(2) The element conditioning number increases as the p-levels increase. For the cases studied, Legendre and Chebyshev functions exhibit just about the same rate of increase as the p-levels increase. However, the Lagrange functions exhibit a much faster rate of increase compared to Legendre and Chebyshev functions.

(3) The functions based on the $Q'(p)$ set yields a lower conditioning number than those of the $Q(p)$ set. The conditioning number of the $Q'(p)$ Legendre set is the lowest out of all the studies conducted during this investigation.

(4) In general, as the element aspect ratio increases, the element conditioning number also increases. The increase is most significant for Lagrange functions.

(5) The element distortion also increases the conditioning number. Thus the distorted element with a high aspect ratio would yield the highest conditioning number.

(6) From our numerical studies we confirm the fact that all three p-version functions produce identical results and no one type exhibits superiority over the others in terms of accuracy as long as numerical round-off does not influence the computations. In other words, for a given mesh and p-level all three functions will produce an identical solution (dependent on variables and secondary variables) as long as the results are not influenced by round-off.

(7) The elements exhibiting the highest and the lowest conditioning numbers usually defines the bounds of the conditioning number of the discretization. Uniform meshes yield the best conditioning number. Severely graded meshes result in progressively higher conditioning numbers and are more likely to suffer from round-off errors.

(8) Numerical studies with linear problems using elimination methods (wavefront) for p-levels up to 15 show that the solution from all three functions agree up to the last decimal place even though the conditioning number of the Lagrange p-version elements is $0(10^8)$ compared to $0(10^3$–$10^4)$ for Legendre.

(9) In non-linear problems, the solution accuracy, number of iterations, etc. also remains unaffected by the choice of functions when Newton's method with line search is utilized to find the solution iteratively.

(10) Lagrange functions have some advantage over the others in terms of defining the boundary conditions.

In case of Lagrange functions, instead of using high p-levels ($p > 8$) one should undertake local mesh refinement and utilize lower p-levels. This will yield reasonable conditioning number without unduly diminishing the accuracy of the local approximation. There are, however, exceptions to this, such as problems with strong singularities.

In summary, Legendre $Q'(p)$ sets yield the lowest conditioning numbers for the elements and the discretizations. Computed numerical results are not influenced by the conditioning number (at least up to p-levels of 15) when using elimination methods for linear problems and Newton's method with line search and elimination methods for nonlinear problems. If iterative solution methods are employed, such as GMRES or any preconditioned gradient based solvers, it is critical that the conditioning number of the discretization remains as low as possible. Therefore, Legendre based functions would be most appealing in this situation.

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REFERENCES

1. B. A. Szabo, Some recent developments in finite element analysis. *Comput. Math. Applic.* **5,** 99–115 (1995).
2. I. Babuska, B. A. Szabo and I. N. Katz, The p-version of the finite element method. *SIAM J. numer. Anal.* **18,** 515–545 (1981).
3. B. A. Szabo and G. J. Sahrmann, Hierarchical plate and shell models based on p-extension. *Int. J. numer. Meth. Engng* **26,** 1855–1881 (1988).
4. I. Babuska, B. A. Szabo and R. L. Actis, Hierarchic models for laminated composites. *Int. J. numer. Meth. Engng* **33,** 503–535 (1992).
5. K. S. Surana and R. M. Sorem, p-Version hierarchical three-dimensional curved shell element for elastostatics. *Int. J. numer. Meth. Engng* **31,** 649–676 (1991).
6. K. S. Surana and N. J. Orth, p-Version hierarchical axisymmetric shell element for heat conduction. *Numer. Heat Trans. 20,* 80–103 (1991).
7. K. S. Surana and N. J. Orth, p-Version hierarchical axisymmetric shell element. *Comput. Struct.* **39,** 257–268 (1991).
8. K. S. Surana and N. J. Orth, p-Version hierarchical three-dimensional curved shell element for heat conduction. *Comput. Mech.* **7,** 341–353 (1991).
9. S. M. Vijayakar, H. R. Busby and D. R. Houser, Finite element analysis of Quasi-prismatic bodies using Chebyshev polynomials. *Int. J. numer. Meth. Engng* **24,** 1461–1477 (1987).
10. P. Devloo, J. T. Oden and P. Pattani, An h-p adaptive finite element method for the numerical simulation of compressible flow. *Comput Meth. Appl Mech. Engng* **70,** 203–235 (1988).
11. Babuska, M. Griebel and J. Pitkaranta, The problem of selecting the shape functions for a p-type finite element. *Int. J. numer. Meth. Engng* **28,** 1891–1908 (1989).
12. I. Babuska and H. C. Elman, Performance of the h-p version of the finite element method with various elements. *Int. J. numer. Meth. Engng* **36,** 2503–2523 (1993).
13. K. S. Surana and G. Y. Gan, Full and reduced set p-version approximations and their role in smooth and singular problems. In *Material, Design and Analysis Symp. ETCE.* Houston, Texas (1995).
14. B. Szabo and I. Babuska, *Finite Element Analysis.* Wiley, New York (1991).
15. W. H. Press, S. A. Teukolsky, W. T. Vetterling and B. P. Flannery. *Numerical Recipes in Fortran.* Cambridge University Press, Cambridge (1992).