The next step in the never-ending process of generalizing Francis's implicitly-shifted QR algorithm

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Washington State University

This is joint work . . .

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- . . . with Raf Vandebril.

This is joint work . . .

- . . . with Raf Vandebril.
- . . . mostly Raf’s work!

Francis's Algorithm

Francis's Algorithm

- requires Hessenberg matrix

Francis's Algorithm

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- we know how

Francis's Algorithm

- requires Hessenberg matrix
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$$\begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ & \times & \times & \times & \times \\ & & \times & \times & \times \\ & & & \times & \times \end{bmatrix}$$

Reduce to Triangular Form

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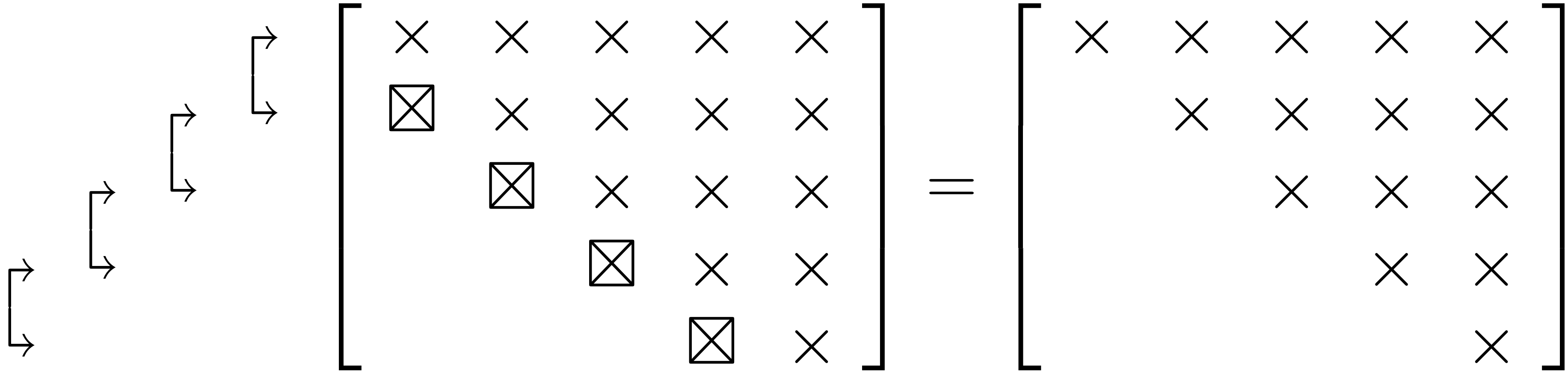
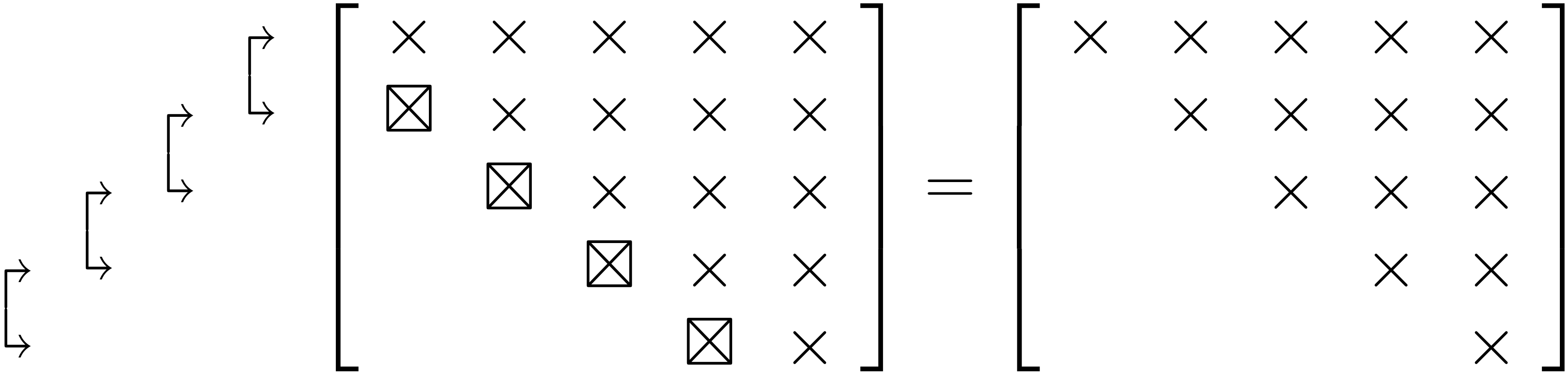
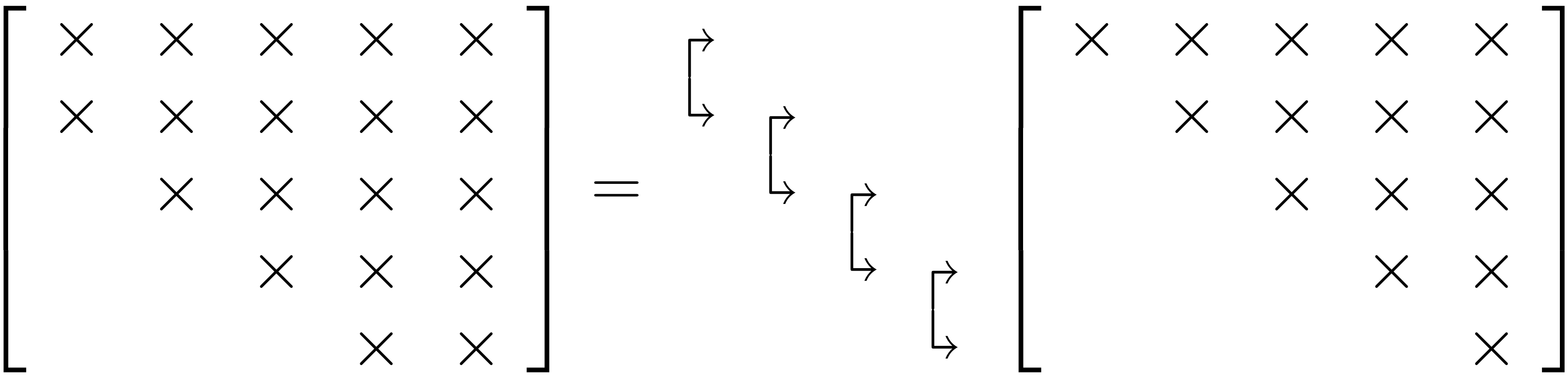


Reduce to Triangular Form



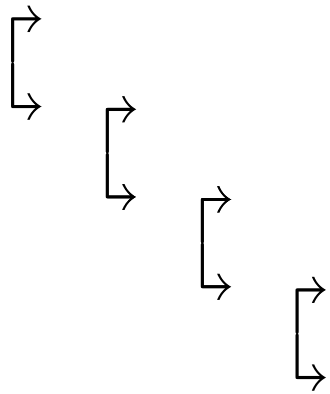
This yields a QR decomposition.

QR Decomposed Hessenberg matrix



QR Decomposed Hessenberg matrix

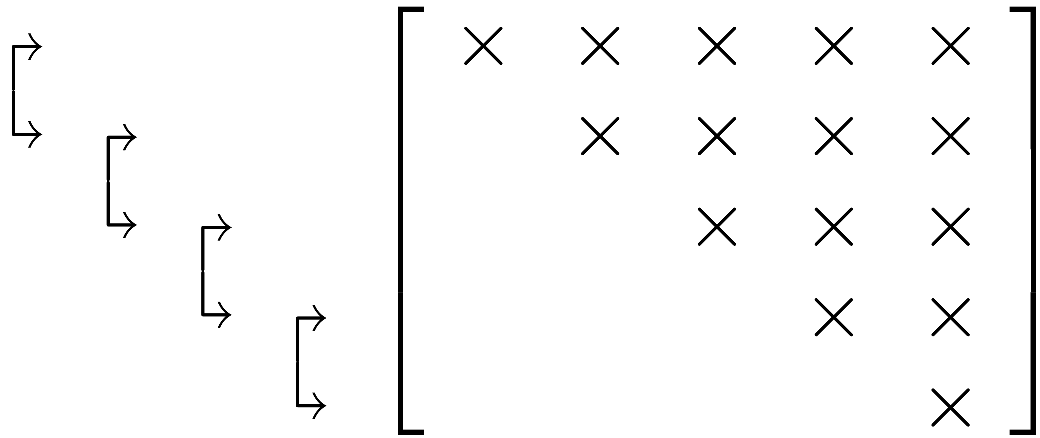
$$\begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ & \times & \times & \times & \times \\ & & \times & \times & \times \\ & & & \times & \times \end{bmatrix} =$$



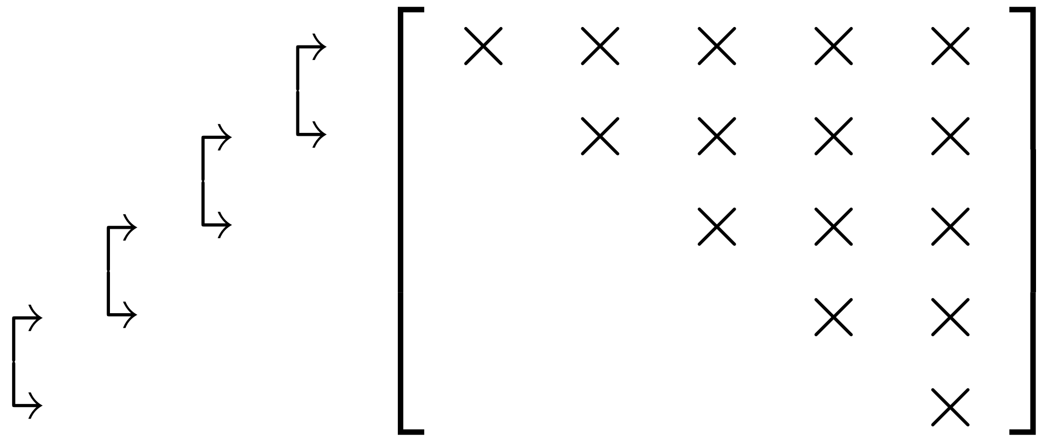
$$\begin{bmatrix} \times & \times & \times & \times & \times \\ & \times & \times & \times & \times \\ & & \times & \times & \times \\ & & & \times & \times \\ & & & & \times \end{bmatrix}$$

…a way to represent the matrix.

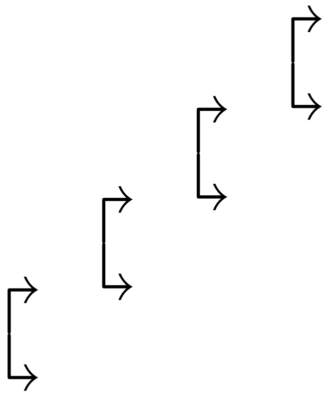
Hessenberg matrix (from now on)



Inverse of a Hessenberg matrix



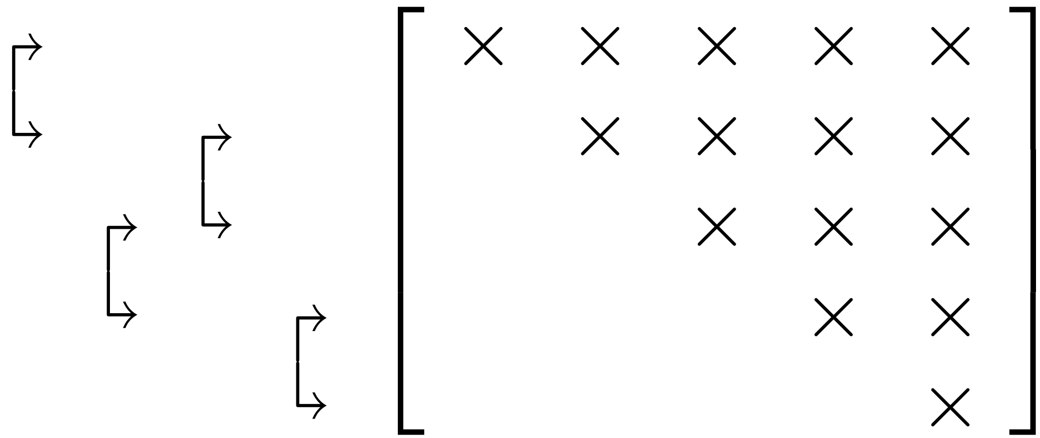
Inverse of a Hessenberg matrix



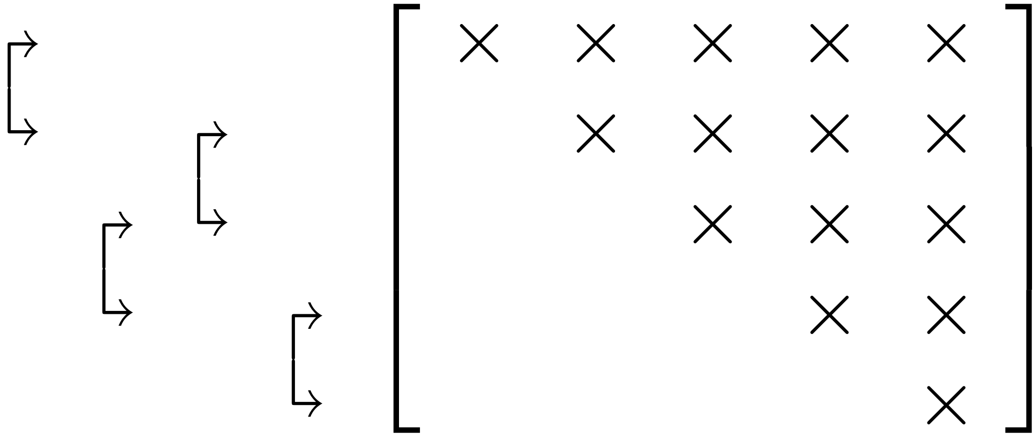
$$\begin{bmatrix} \times & \times & \times & \times & \times \\ & \times & \times & \times & \times \\ & & \times & \times & \times \\ & & & \times & \times \\ & & & & \times \end{bmatrix}$$

…an attainable form!

Another Possibility

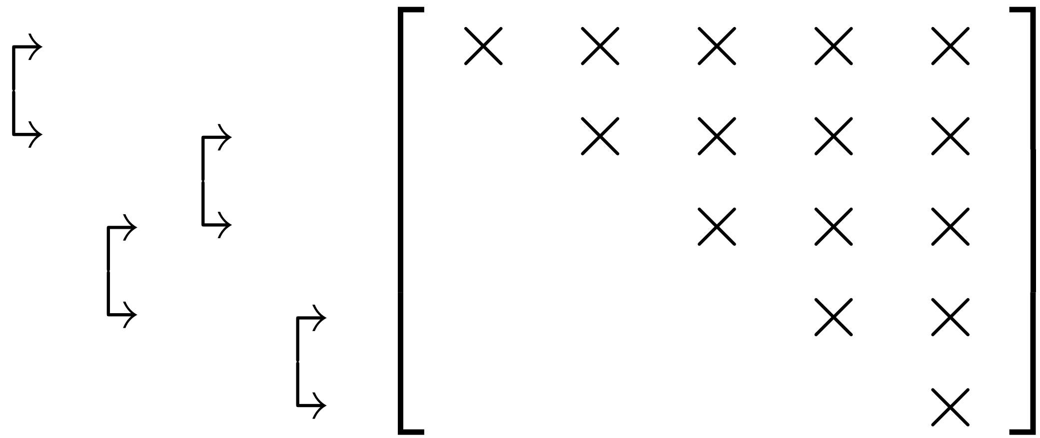


Another Possibility



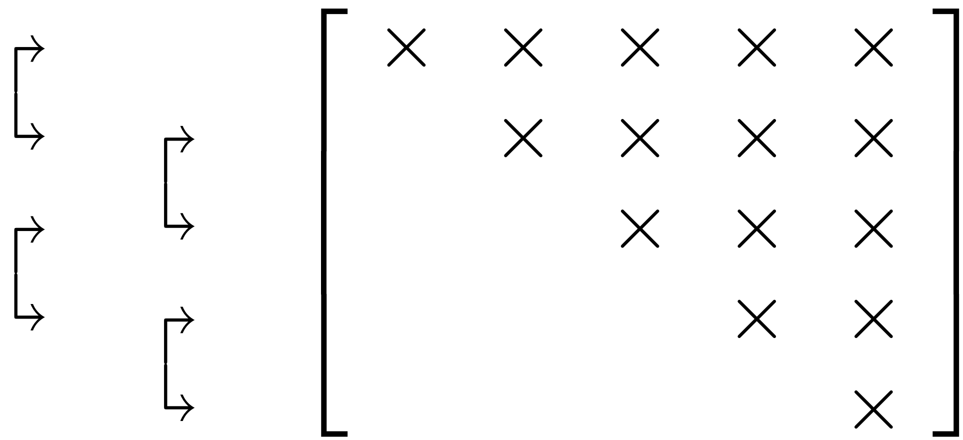
- CMV form

Another Possibility

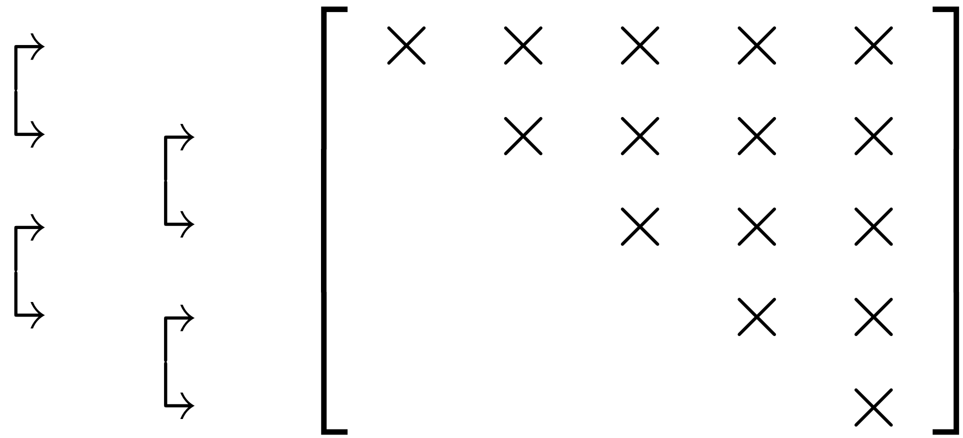


- CMV form
- Some rotations commute.

CMV Form

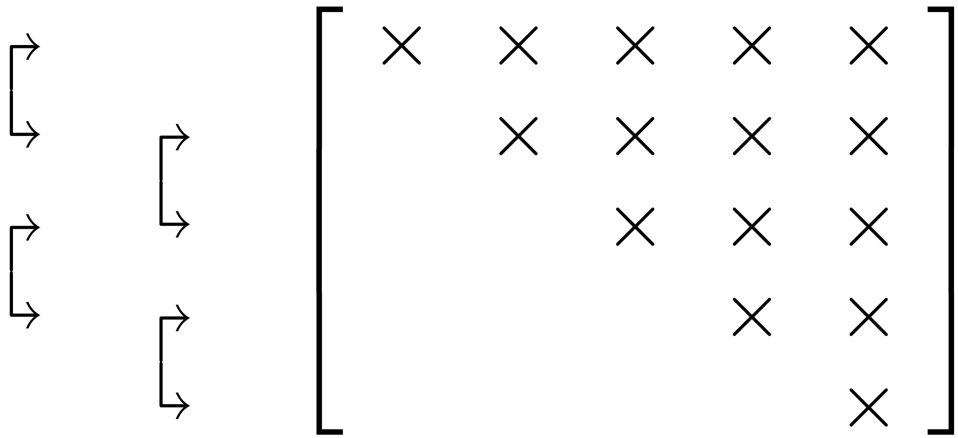


CMV Form



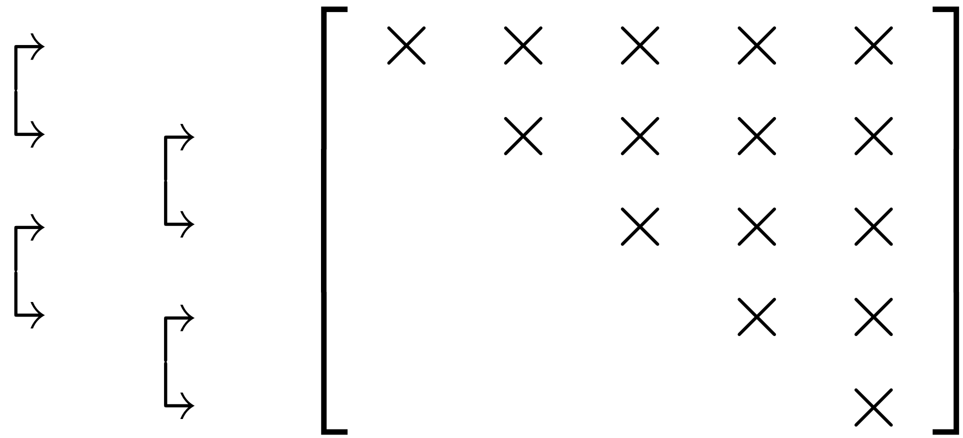
- also attainable

CMV Form



- also attainable
- rotators can appear in any order

CMV Form

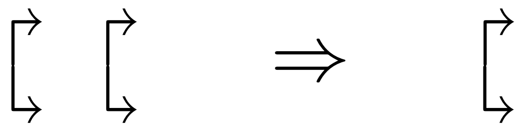


- also attainable
- rotators can appear in any order
- There are variants of Francis's algorithm for all of these forms.

Allowed Operations

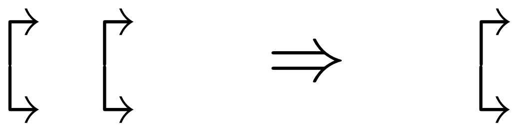
Allowed Operations

- fusion

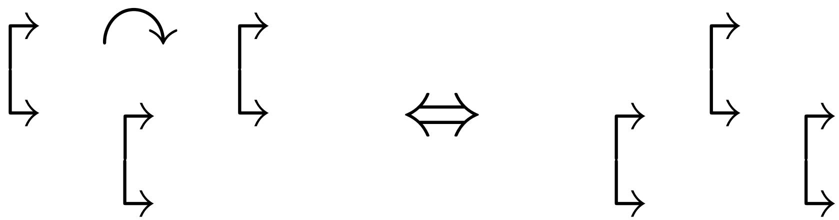


Allowed Operations

- fusion



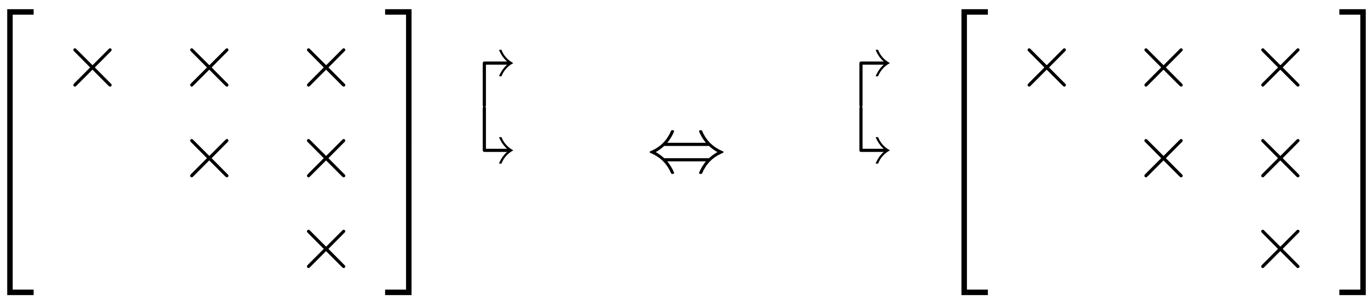
- shift through



Allowed Operations, continued

Allowed Operations, continued

- shift through triangular matrix



- structure commutes

Francis iteration on Hessenberg form

Francis iteration on Hessenberg form

- single shift for simplicity

Francis iteration on Hessenberg form

- single shift for simplicity (can do any number)

Francis iteration on Hessenberg form

- single shift for simplicity (can do any number)
- create a bulge

Francis iteration on Hessenberg form

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- create a bulge and chase it

Francis iteration on Hessenberg form

- single shift for simplicity (can do any number)
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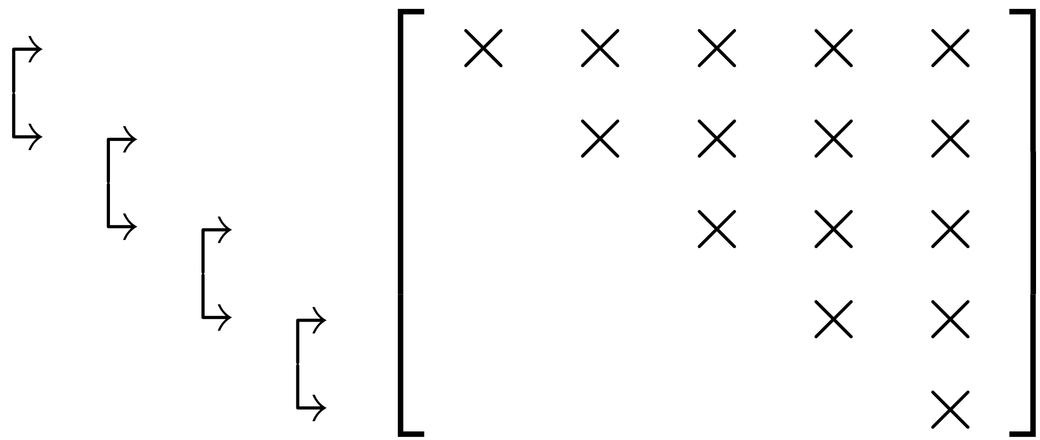
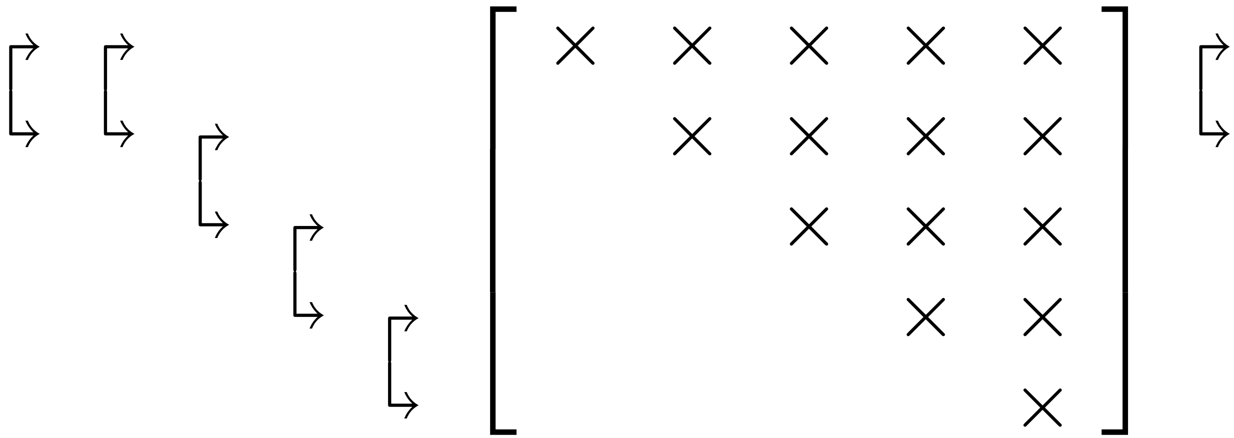
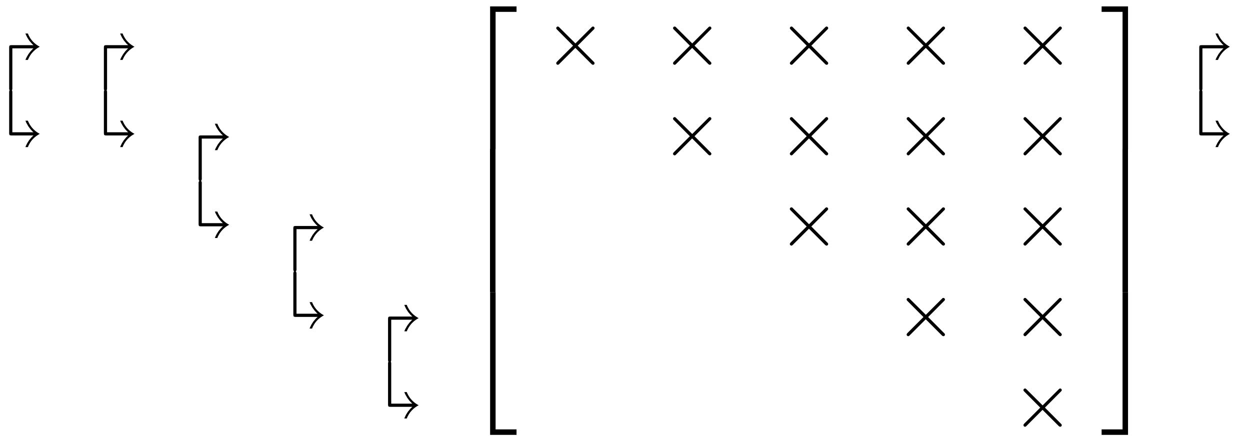
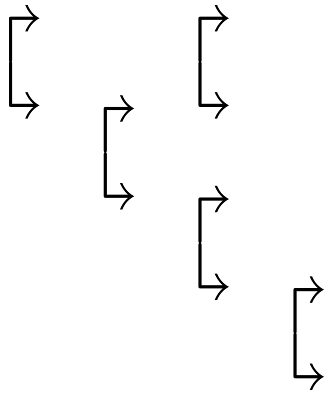


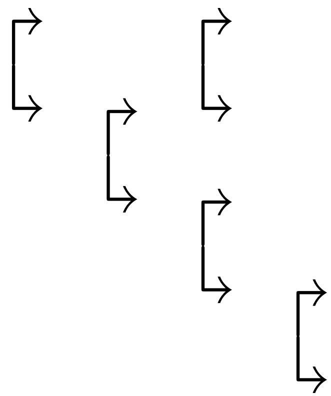
$$\begin{bmatrix} \times & \times & \times & \times & \times \\ & \times & \times & \times & \times \\ & & \times & \times & \times \\ & & & \times & \times \\ & & & & \times \end{bmatrix}$$



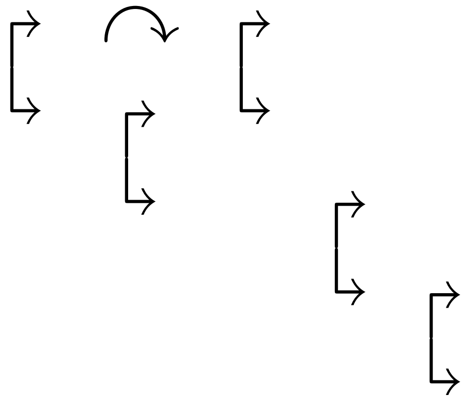


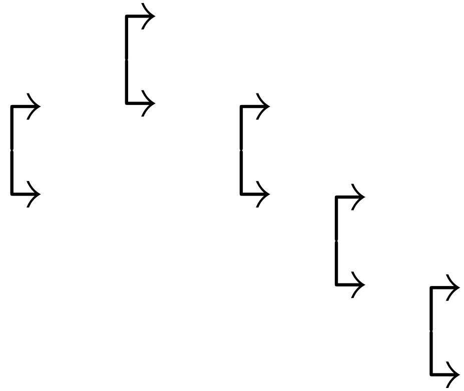
Suppress the triangular matrix.

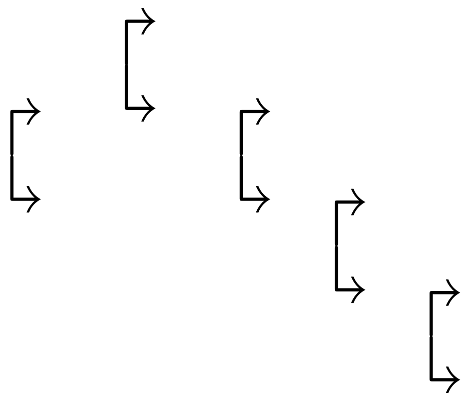




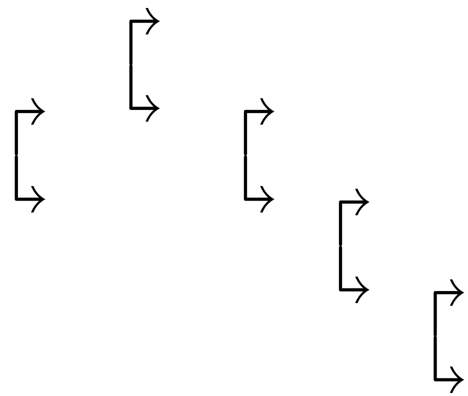
Think of the unitary case.



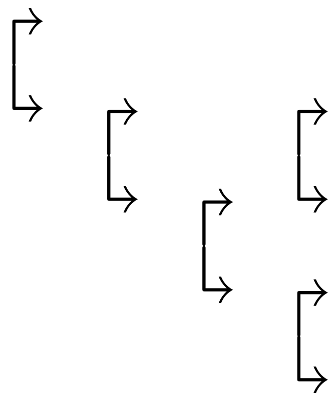


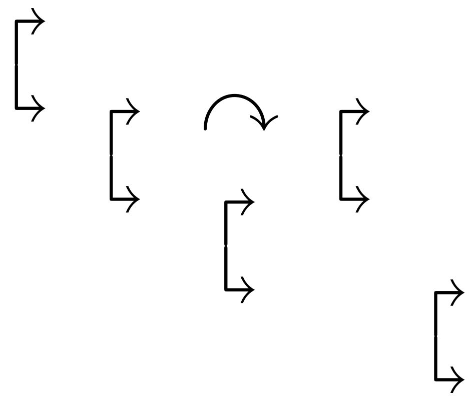


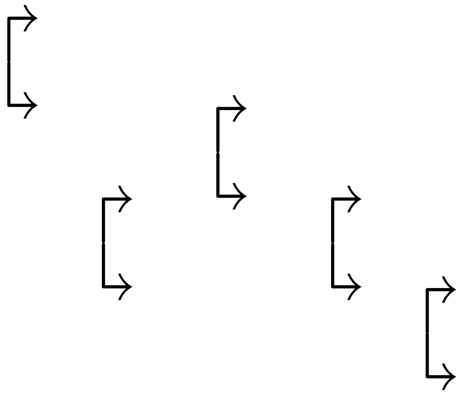
- Eliminate rotator in rows 2 and 3.

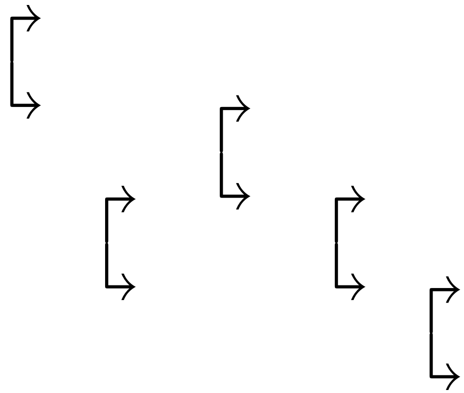


- Eliminate rotator in rows 2 and 3.
- Don’t touch first row.

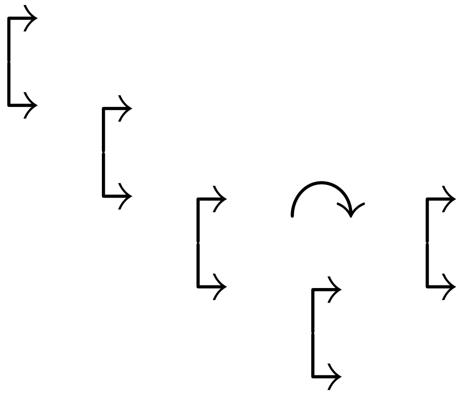


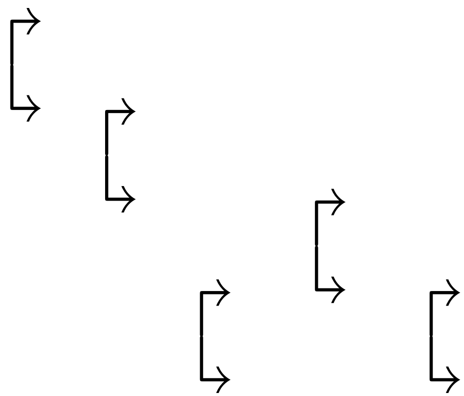


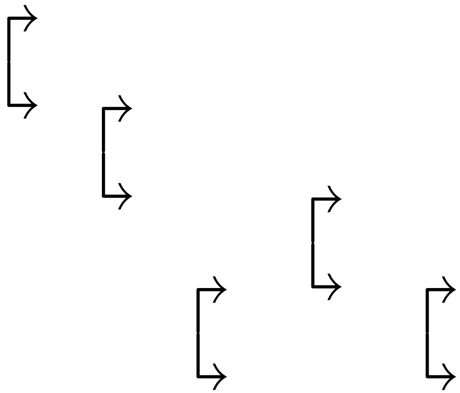




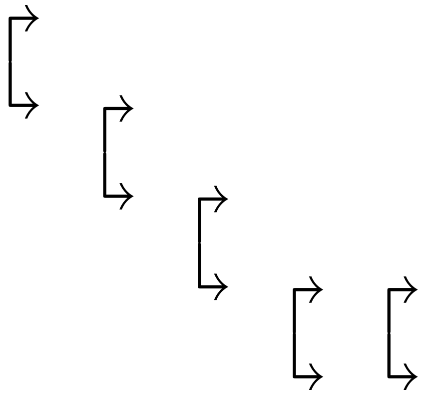
- Eliminate rotator in rows 3 and 4.

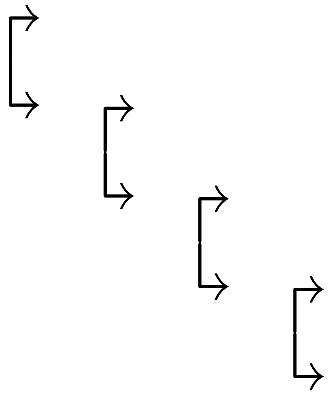


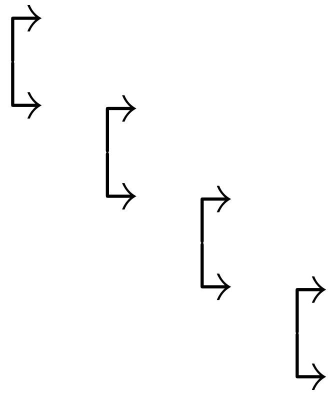




- Eliminate rotator in rows 4 and 5.



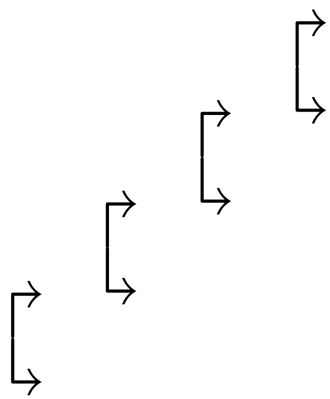




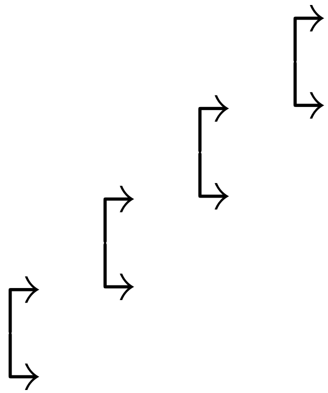
Done!

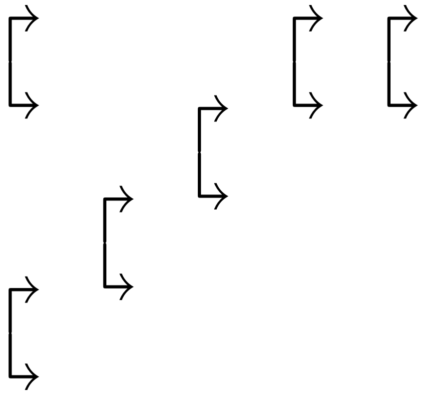
Francis iteration on inverse Hessenberg

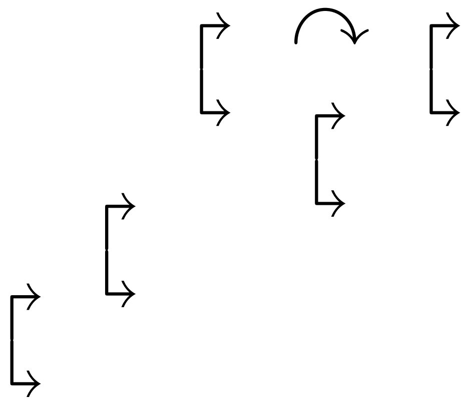
Francis iteration on inverse Hessenberg

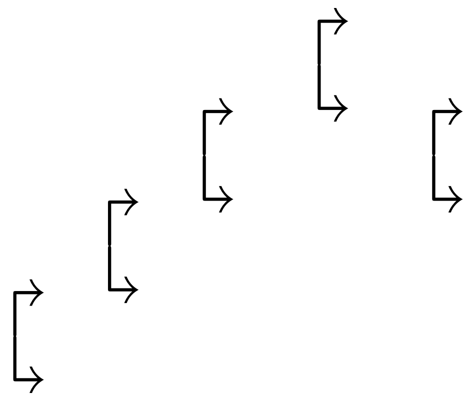


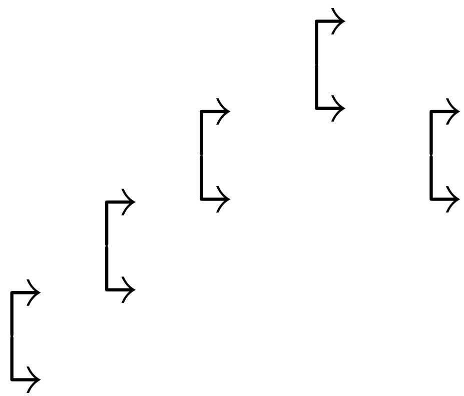
(triangular matrix suppressed)



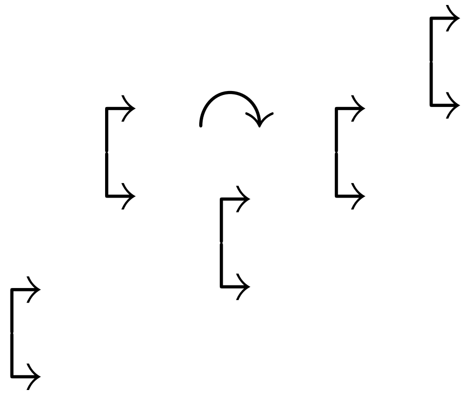


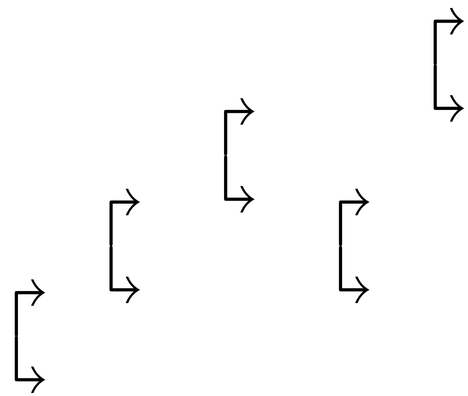


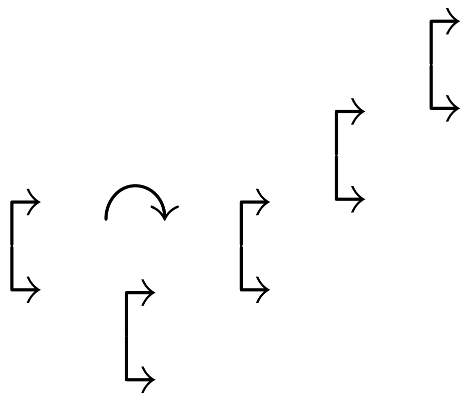


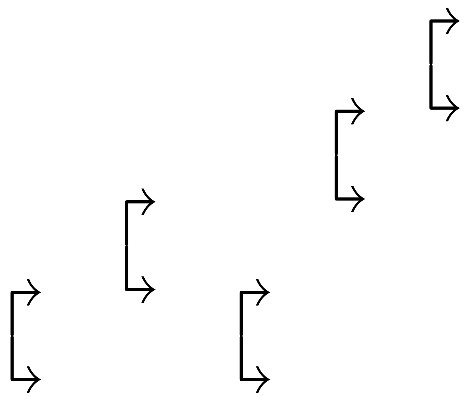


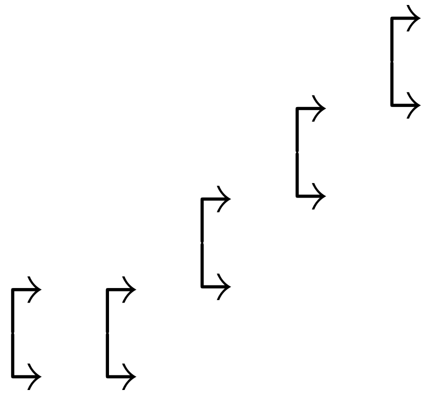
- Now eliminate the rotator on the right.

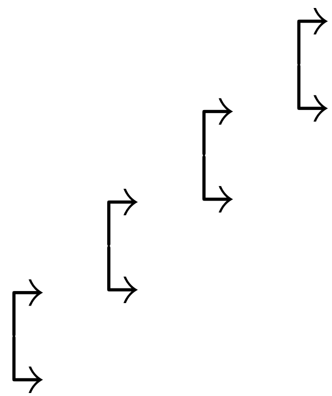


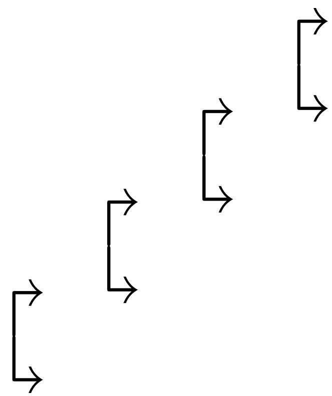






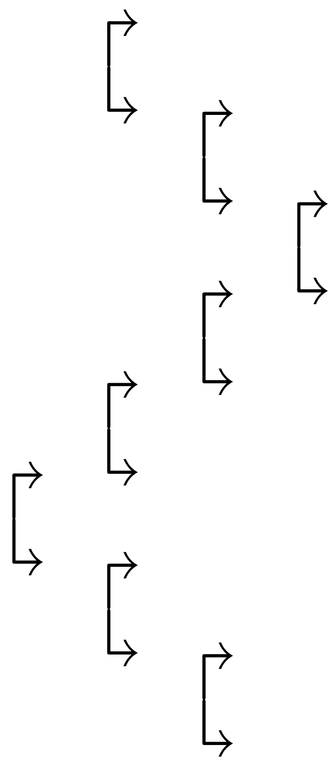




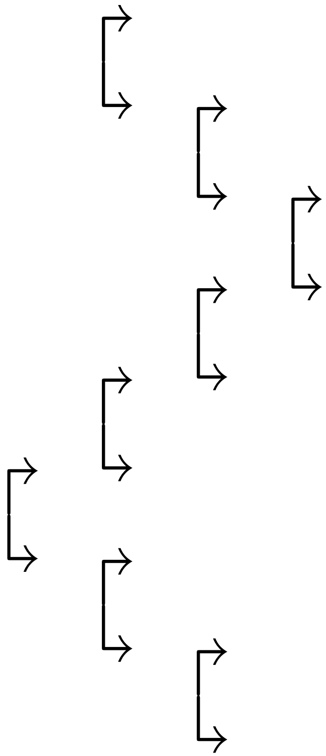


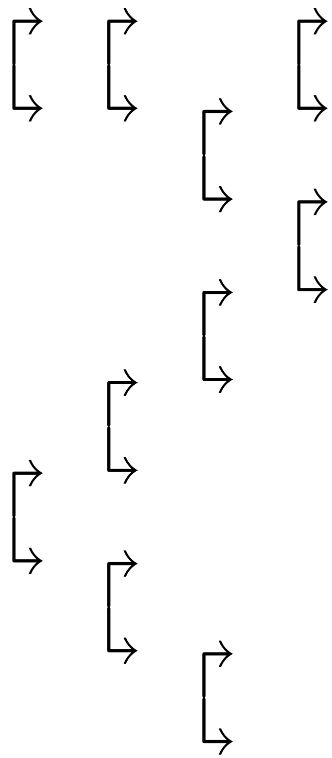
Done!

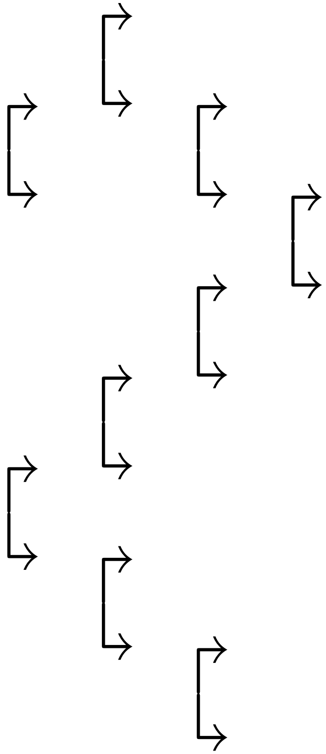
Francis iteration on an "arbitrary" pattern

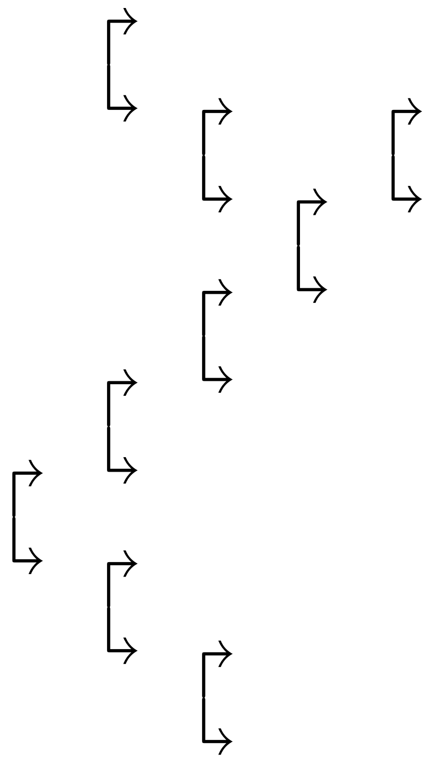


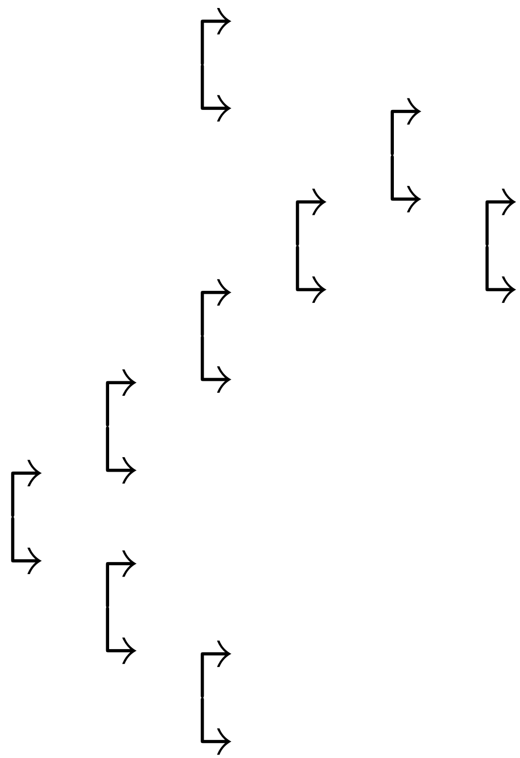
(triangular matrix suppressed)

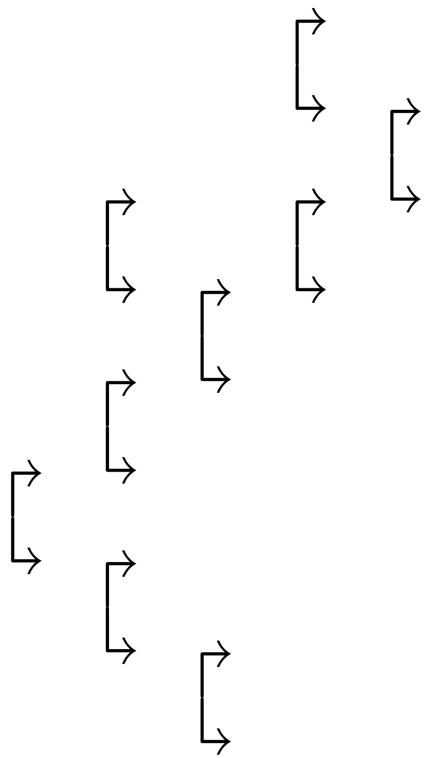


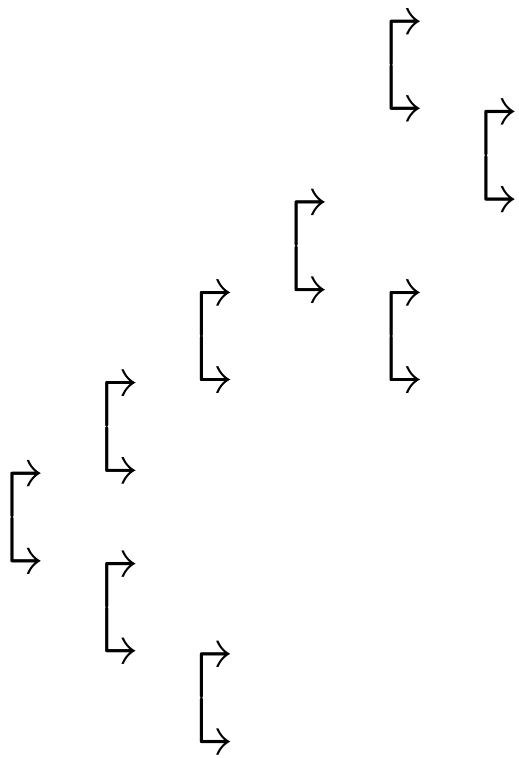


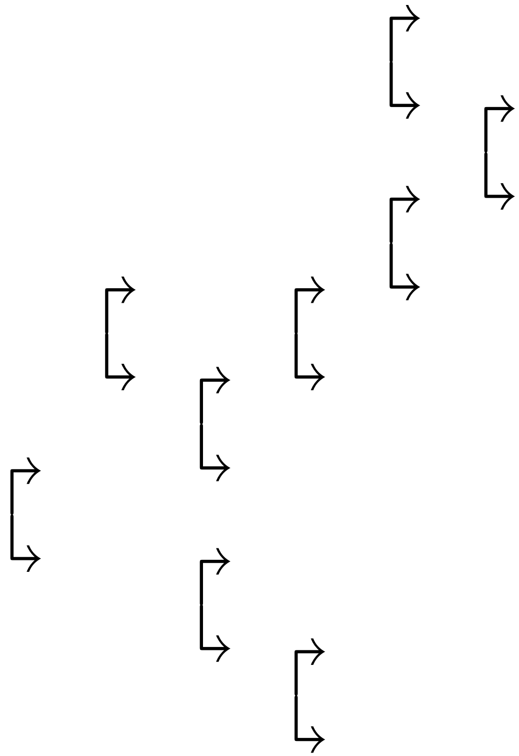


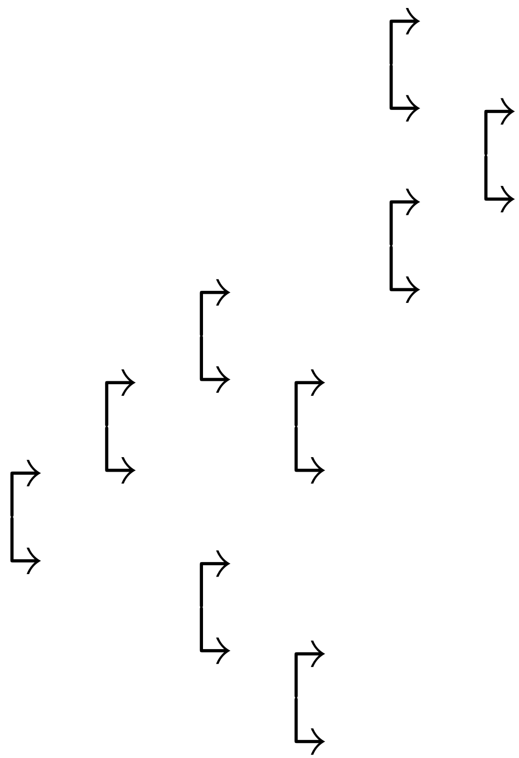


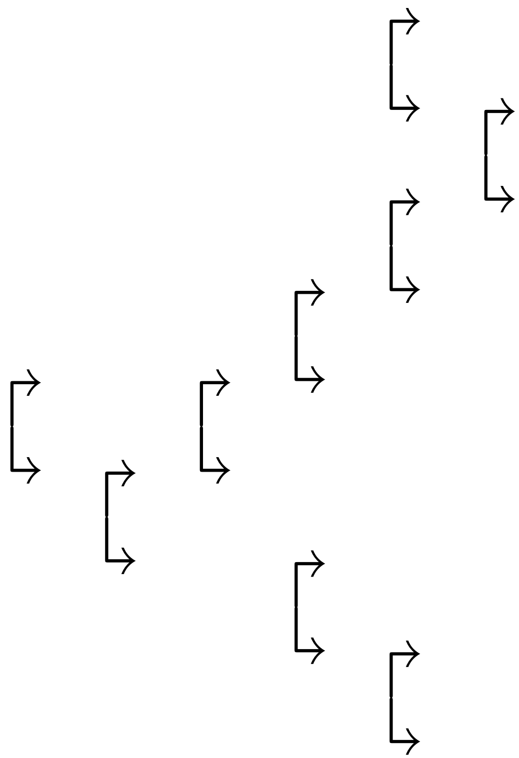


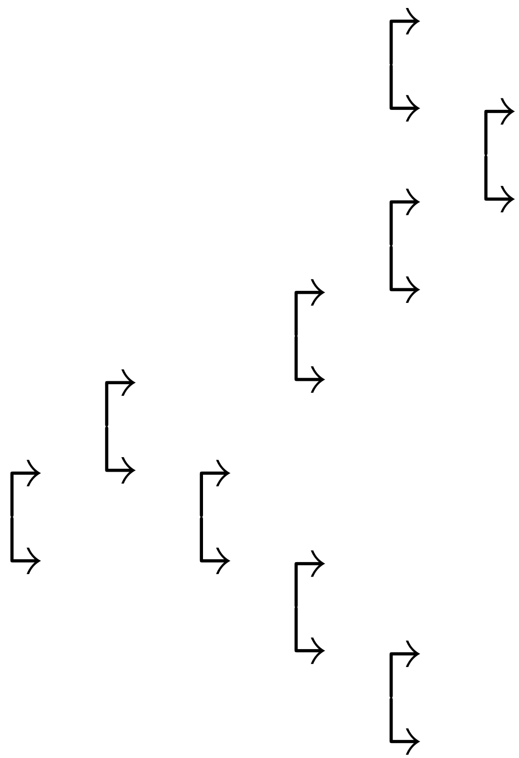


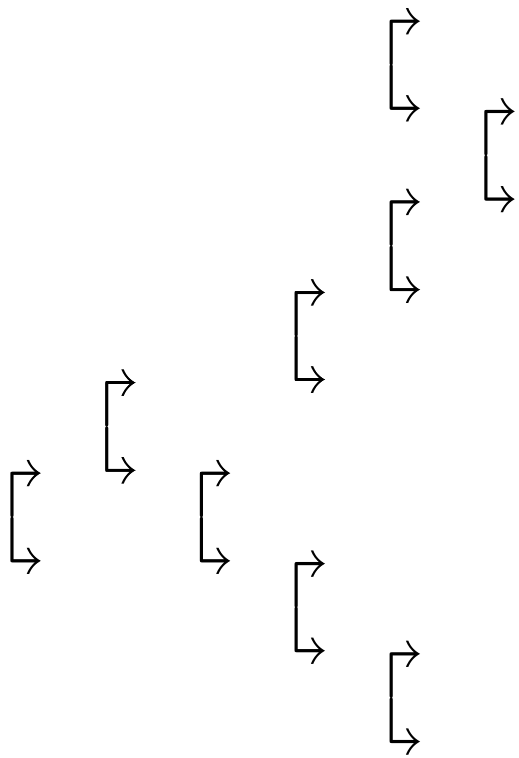






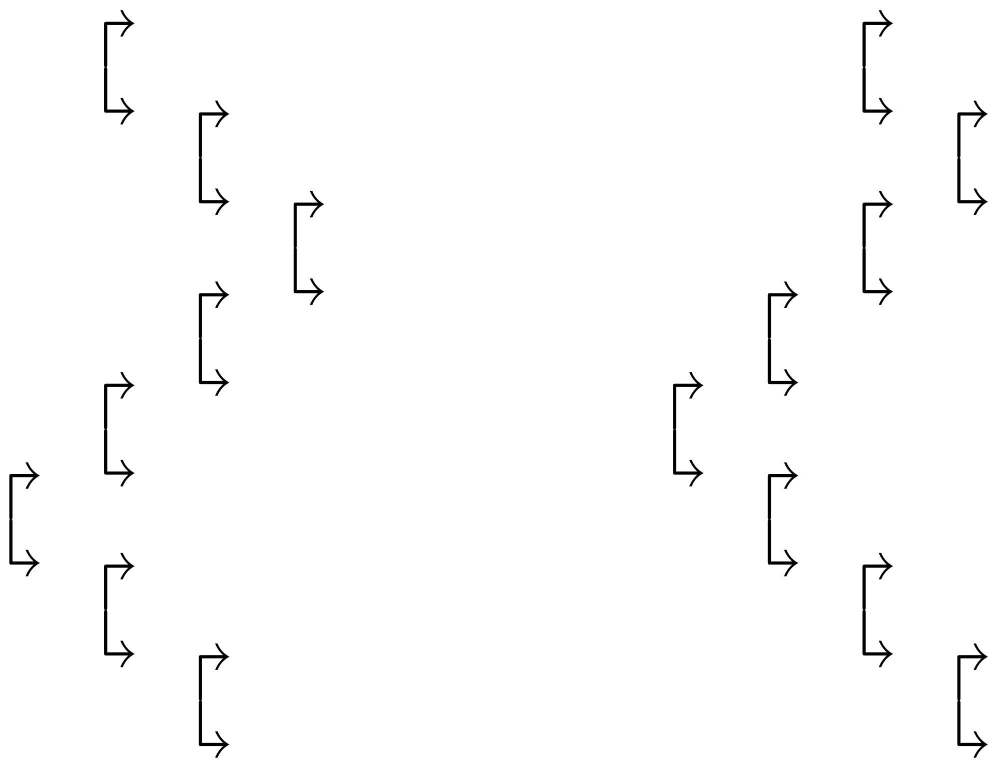






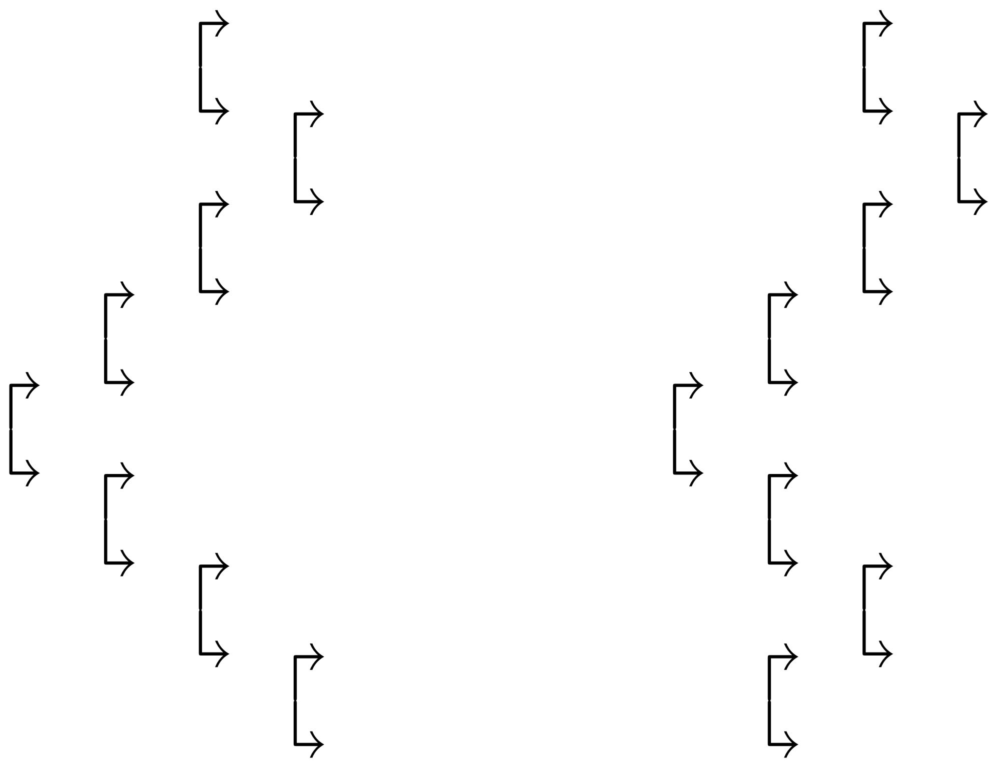
- Now go the other way.
- and so on . . .

Comparing start with finish



Pattern moves upward by one.

Two ways to finish



Bottom rotator can be on left or right.

Does it work?

Does it work?

- Raf tried it out.

Does it work?

- Raf tried it out.
- It works great!

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- Can we establish some convergence theory?

Does it work?

- Raf tried it out.
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- Yes, we can!

Does it work?

- Raf tried it out.
- It works great!
- Can we establish some convergence theory?
- Yes, we can!
- multishift iterations of any degree

What is Francis's algorithm?

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- It's nested subspace iteration . . .

What is Francis's algorithm?

- It's nested subspace iteration ...
  with changes of coordinate system.

What is Francis's algorithm?

- It's nested subspace iteration . . .
  with changes of coordinate system.
- No reliance on implicit-Q theorem.

What is Francis's algorithm?

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- DSW, A M Monthly (May 2011)

What is Francis's algorithm?

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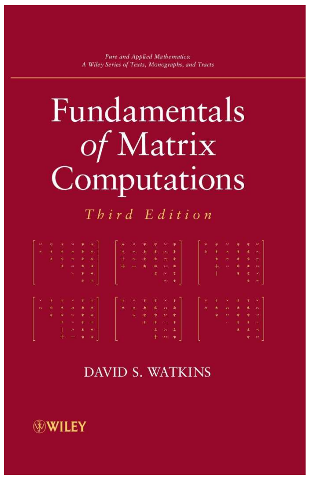

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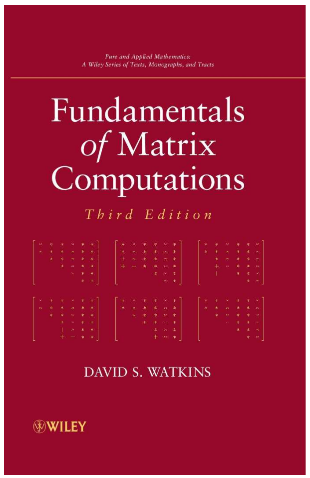

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- For other forms, adjust the Krylov subspaces

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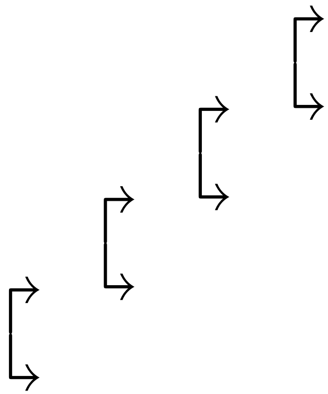

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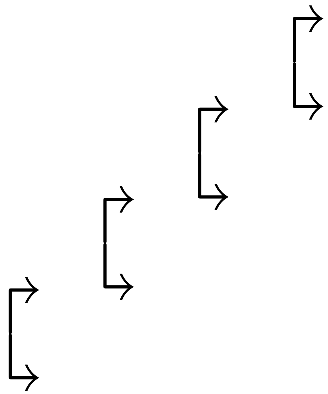

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- and in general …

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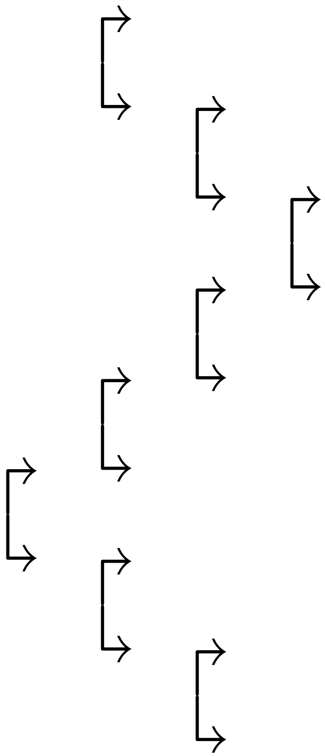

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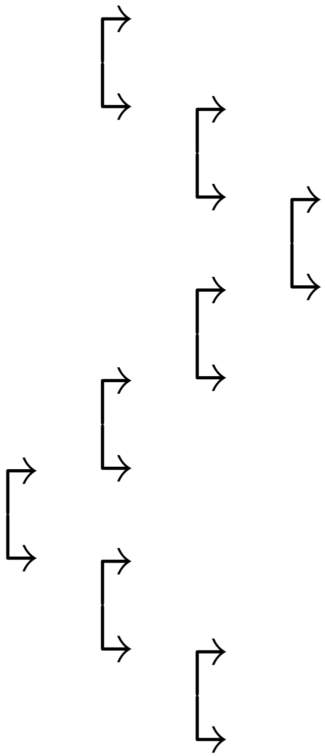

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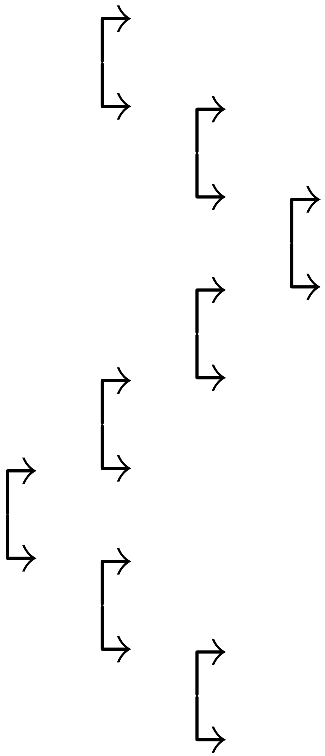

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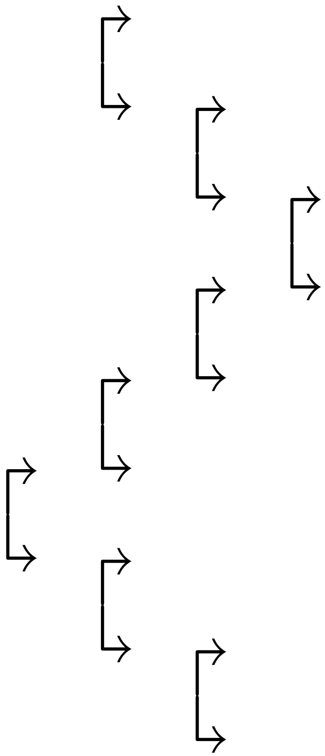

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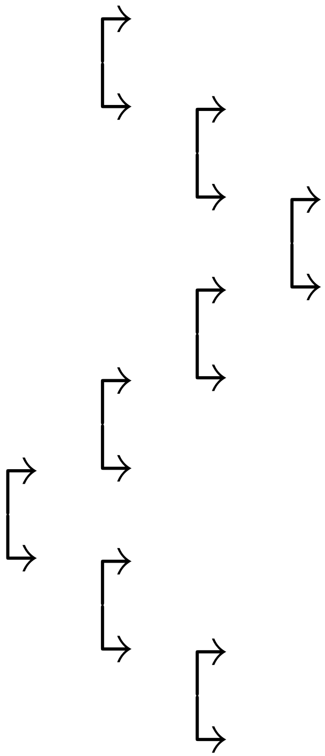

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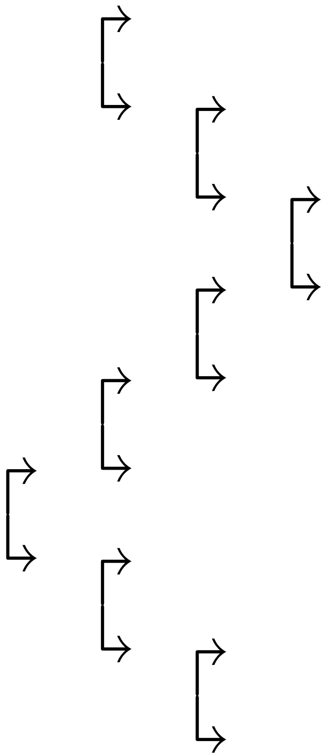

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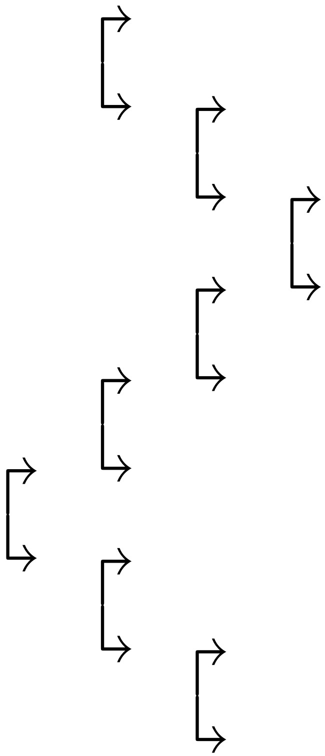

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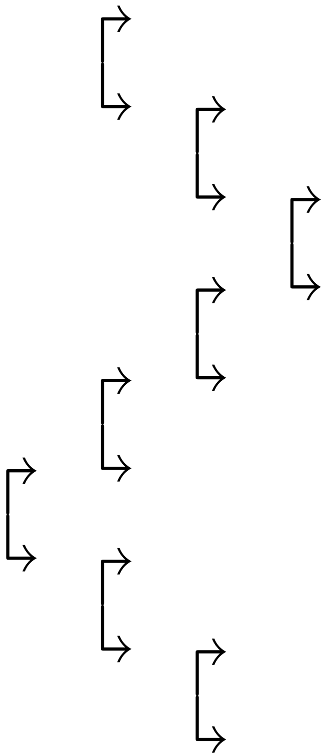

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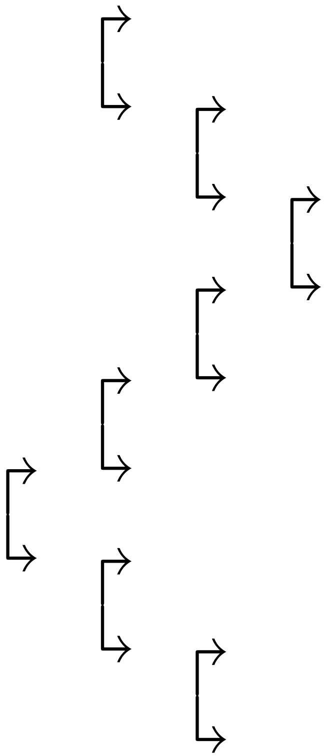

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- Thank you for your attention.