Francis's Algorithm

David S. Watkins

watkins@math.wsu.edu

Department of Mathematics

Washington State University

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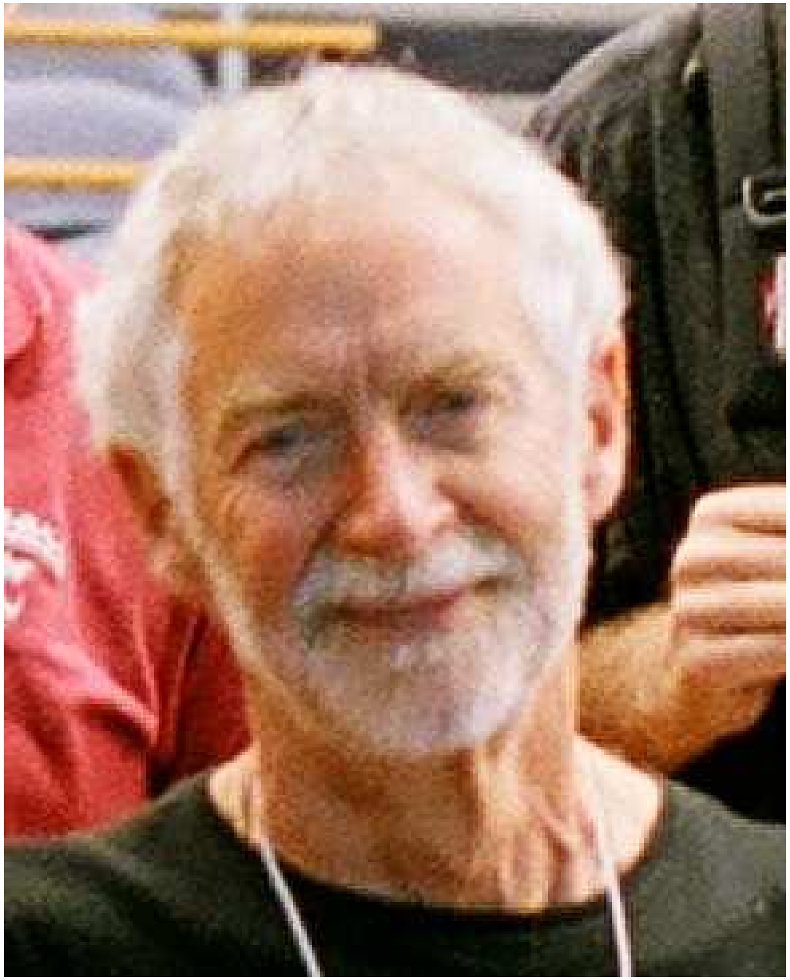
- 50 years!
- Top Ten of the century (Dongarra and Sullivan)

John Francis



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- moved on to other things

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- . . . but this is not "Francis's Algorithm"

Francis's Algorithm

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- radically different from basic QR
- Usual justification: Francis's implicit-Q theorem

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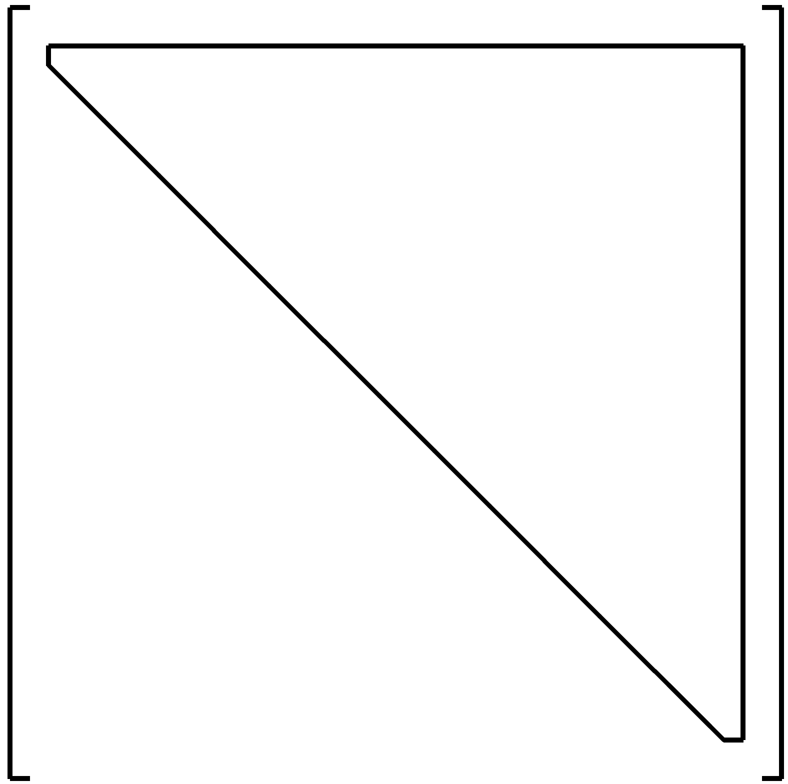
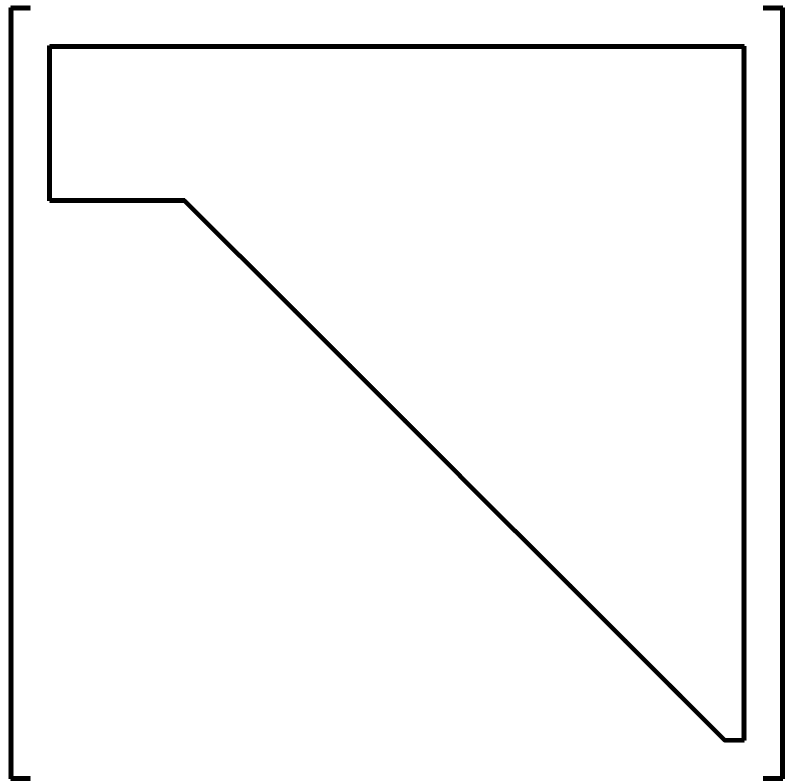
- Hessenberg form is disturbed.

An Upper Hessenberg Matrix



After the Transformation ($Q_0^{-1}AQ_0$)



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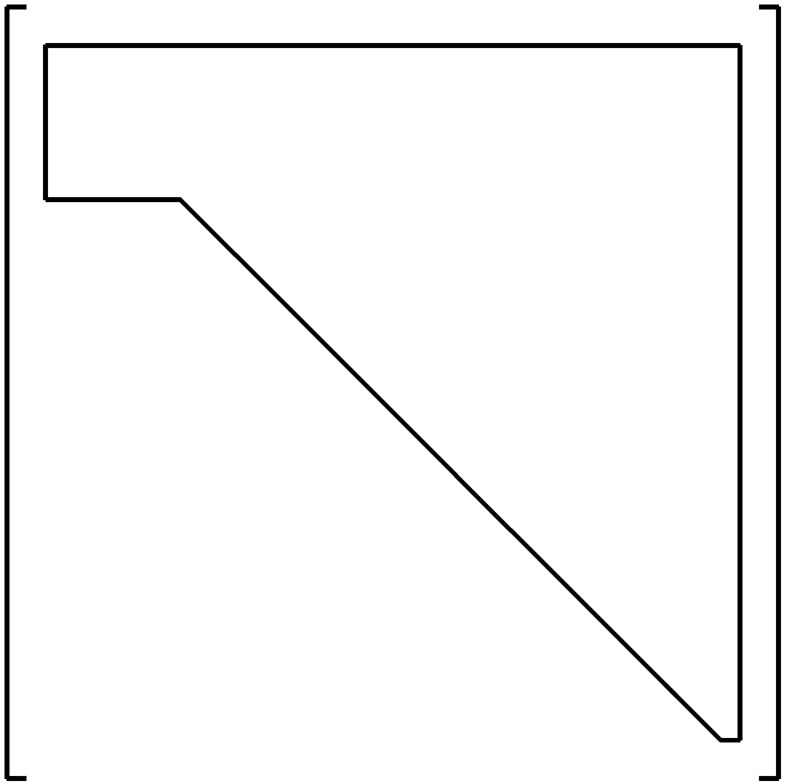
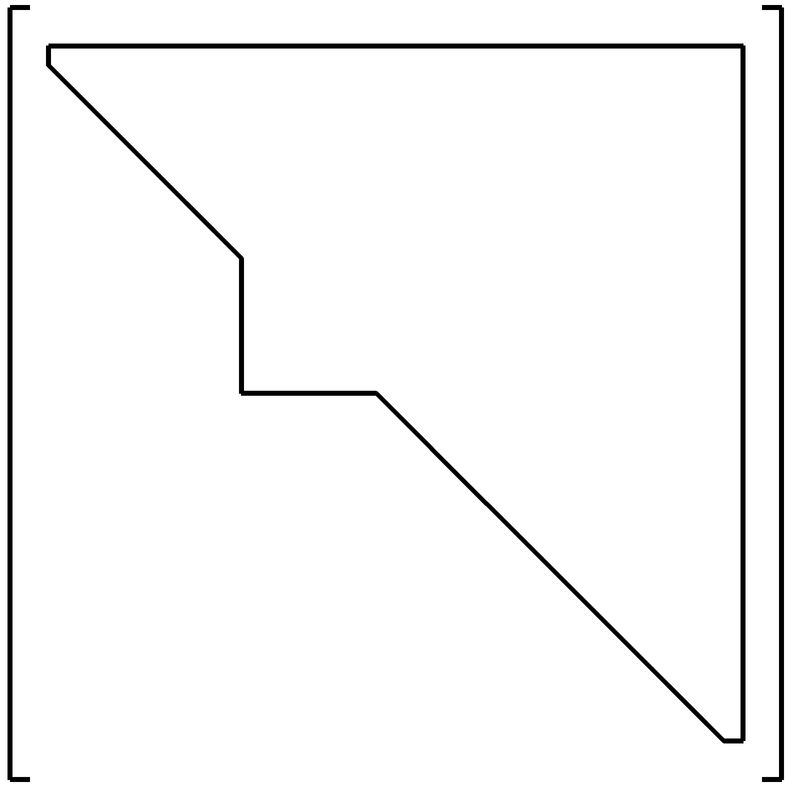


Now return the matrix to Hessenberg form.

Chasing the Bulge



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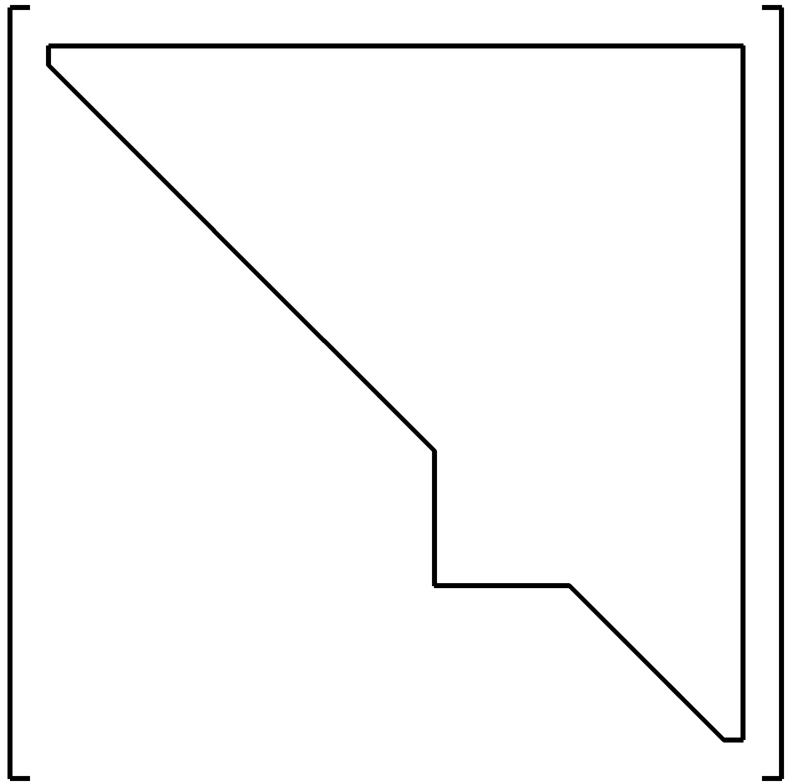
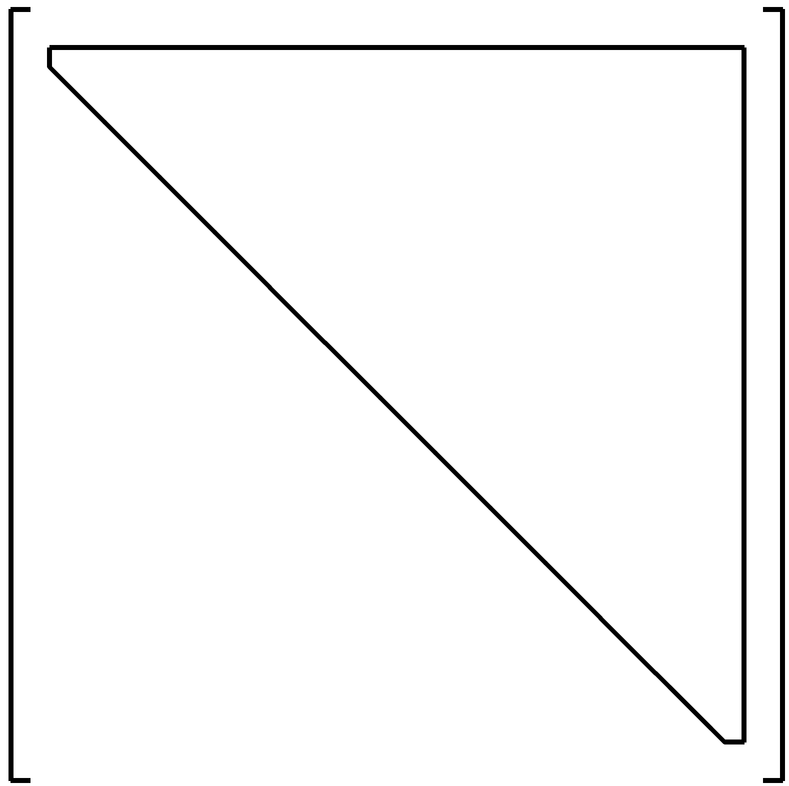


Done



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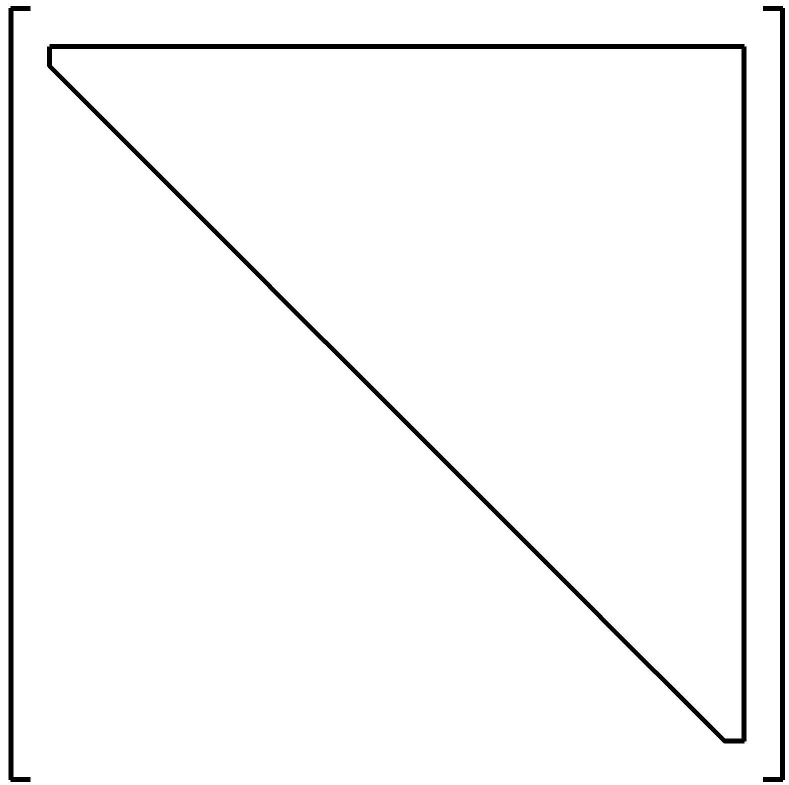

The Francis iteration is complete!

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- This is not a radical move.

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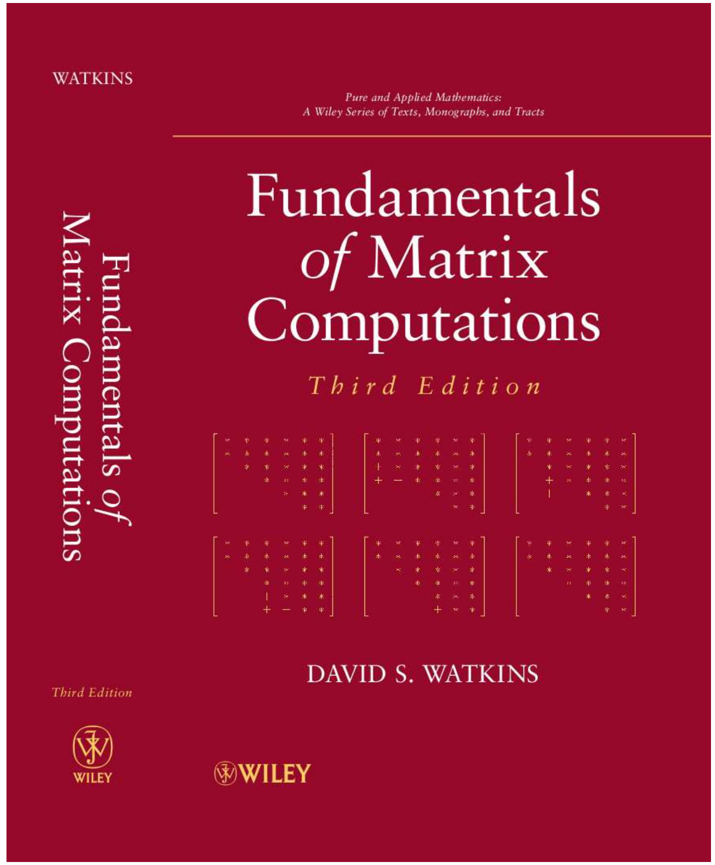
- I'm putting my money where my mouth is.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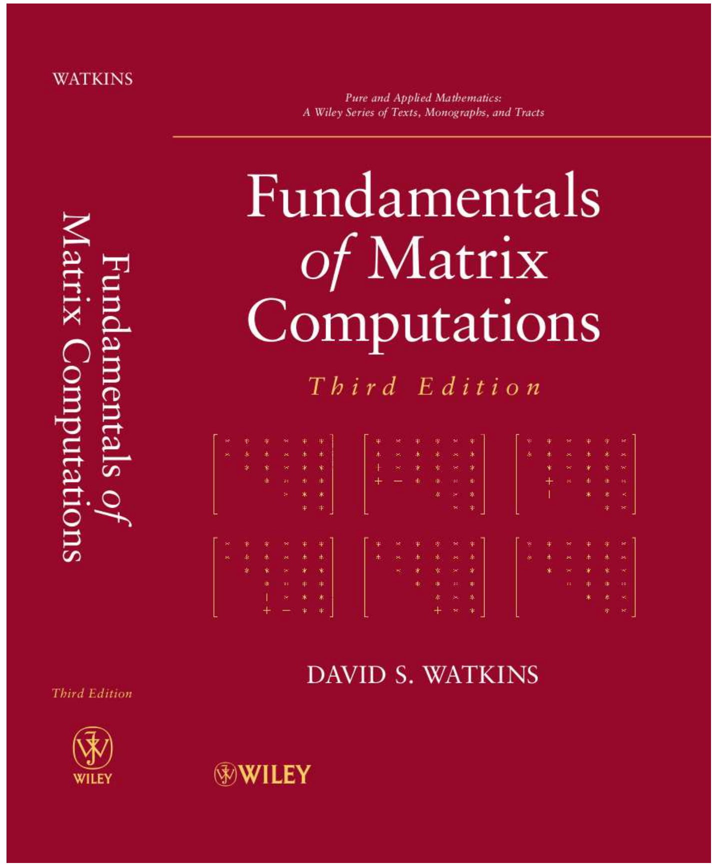

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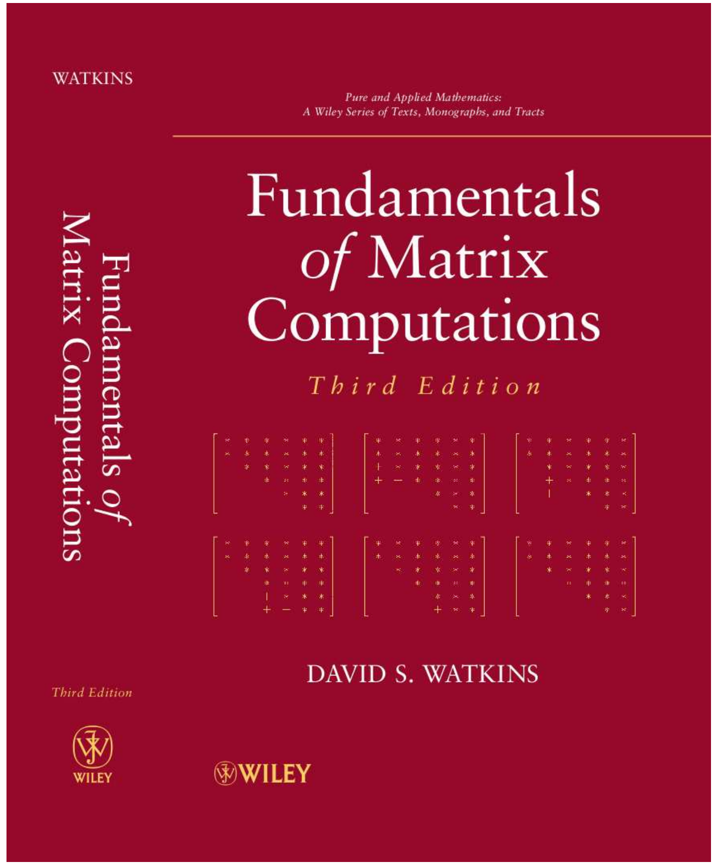

- I'm putting my money where my mouth is . . .
- . . . and saving one entire section!

Pedagogical Pathway

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- reduction to Hessenberg form

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- Why does it work?

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- $\text{span}\{q_1, \ldots, q_j\} \to \text{span}\{e_1, \ldots, e_j\}$
- ready for next iteration

This version of subspace iteration . . .

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- …holds the subspace fixed

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- …holds the subspace fixed
- while the matrix changes.

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- $A \rightarrow \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$ (A_{11} is $j \times j$.)

Application to Francis's Iteration (first pass)

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- power method $+$ change of coordinates
- $q_1 \rightarrow Q^{-1}q_1 = e_1$
- case $j = 1$ of subspace iteration with a change of coordinate system
- $\ldots$ but this is just a small part of the story.

Krylov Subspaces . . .

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. . . and Subspace Iteration

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- $\mathcal{K}_j(A, q)$ are "determined by q".

Krylov Subspaces . . .
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- Conclusion: Power method induces nested subspace iterations on Krylov subspaces.

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- $|p(\lambda_{j+1})/p(\lambda_j)| \qquad j = 1, 2, 3, \ldots, n-1$

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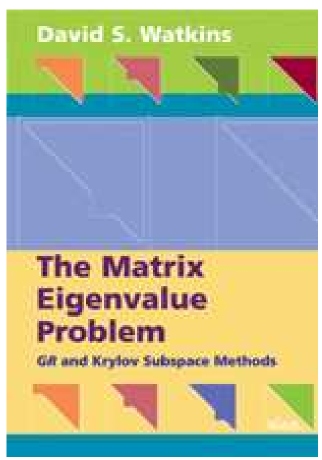

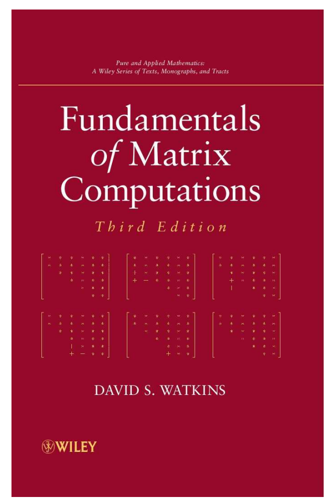

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John Francis speaking in Glasgow

A Portion of the Audience

Afterwards

Photos courtesy of Frank Uhlig