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CONTRACTIVITY OF WAVEFORM RELAXATION RUNGE–KUTTA ITERATIONS AND RELATED LIMIT METHODS FOR DISSIPATIVE SYSTEMS IN THE MAXIMUM NORM*

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Abstract. Contractivity properties of Runge–Kutta methods are analyzed, with suitable interpolation implemented using waveform relaxation strategy for systems of ordinary differential equations that are dissipative in the maximum norm. In general, this type of implementation, which is quite appropriate in a parallel computing environment, improves the stability properties of Runge–Kutta methods. As a result of this analysis, a new class of methods is determined, which is different from Runge–Kutta methods but closely related to them, and which combines its high order of accuracy and unconditional contractivity in the maximum norm. This is not possible for classical Runge–Kutta methods.

Key words. waveform relaxation, Runge–Kutta method, stability, contractivity, differential system

AMS subject classifications. 65L05, 65L07

CR classification. G1.7

1. Introduction.

Consider the initial-value problem for systems of ordinary differential equations (ODEs)

$$\begin{cases} y'(t) = f(t, y(t)), & t \geq t_0, \\ y(t_0) = y_0, \end{cases} \tag{1.1}$$

$f : [t_0, \infty) \times R^m \to R^m$, which are dissipative in the maximum norm. It is the purpose of this paper to study contractivity properties of various numerical methods obtained by applying continuous Runge–Kutta (CRK) methods to waveform relaxation (WR) iterations of the problem (1.1). We discuss the methods corresponding to a fixed number of WR iterations as well as the methods obtained in the limit as the iteration index goes to infinity. It is proved that if the underlying CRK method has some desirable stability properties with respect to the test equations with forcing terms, then the methods corresponding to a fixed number of WR iterations are unconditionally contractive in the maximum norm. This is the context of Theorems 4.2 and 4.3 in §4. Iterating the WR iterations to convergence leads to the new classes of one-step methods, which give the different treatment to the diagonal part of the ODE system (1.1). For this reason, we will call them diagonally split Runge–Kutta (DSRK) methods. These methods differ from RK methods, and they are not even included in the class of general linear methods introduced by Butcher. It turns out that if the underlying CRK methods have appropriate stability properties, then the DSRK methods

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are also unconditionally contractive in the maximum norm. This is the context of Theorem 4.4 in §4. This is remarkable since the unconditionally contractive methods are very difficult to construct. For example, it was demonstrated by Spijker [19] (see also Kraaijevanger [9]) that the requirement of unconditional contractivity leads in the case of RK methods to the very restrictive order barrier $p \leq 1$. These new DSRK methods overcome this barrier. In summary, the approach of this paper leads to the new numerical methods for systems of ODEs, which are at the same time of high order and unconditionally contractive.

Dissipative systems arise in practice from semidiscretization of parabolic partial differential equations and are discussed by Verwer [8], Kraaijevanger [9], [10], and Renaut–Williamson [18]. Usually in real life applications the dimension m of the system is large and the application of traditional numerical techniques to such systems can be quite time consuming. This is most evident when the system (1.1) is stiff and, as a consequence, the use of implicit schemes is required. This involves the solution of large systems of nonlinear equations at each step of the integration. In the quest for improving the efficiency of numerical simulation, it was proposed to first use some continuous-time iterations (Picard-Lindelöf iterations) to "decouple" the system (1.1) and then discretize the resulting subsystems. The resulting WR methods were first introduced by Lelarasmee [11] and Lelarasmee, Ruehli, and Sangiovanni–Vincentelli [12] for time domain analysis of large scale nonlinear dynamical systems. A survey of this technique with emphasis on simulation of large electrical circuits was written by White, Sangiovanni–Vincentelli, Odeh, and Ruehli [20]. The convergence theory of WR methods has been brought on a firm mathematical basis by Nevanlinna and his coworkers [14]–[17]. We also refer to Lie and Skålin [13] for related results.

To date, most of the work on waveform relaxation methods was concerned with convergence, error estimation and control, acceleration of convergence of Picard–Lindelöf iterations, and implementation advantages in a parallel computing environment. Recently, the authors [4], [5] attempted the investigation of stability properties of numerical algorithms based on waveform relaxation techniques by considering a special two-dimensional test system with two real parameters. In [4] the time-point relaxation Heun method with linear interpolation was analyzed, implemented with fixed number of Picard–Lindelöf iterations, as well as the method obtained in the limit as the iteration index goes to infinity. It was found that the region of absolute stability is increased as compared to the usual implementation. The time-point relaxation method is a variant of waveform relaxation in which each window is equal to the stepsize used in numerical integration. In [5] they observed similar phenomena for Runge–Kutta methods of higher orders with interpolants given by natural continuous extensions (NCEs) defined by Zennaro [21]. A message one can get from such studies is that, in general, the new DSRK methods have better stability properties than the CRK methods from which they were derived. In this paper we consider much more general systems of ODEs and find out that this trend is confirmed. To be more specific, we study contractivity properties of numerical methods obtained by applying first to the system (1.1) the Gauss–Jacobi (WRGJ), the Gauss–Seidel (WRGS), and successive-over-relaxation (WRSOR) continuous-time iterations, and then applying CRK methods to the resulting systems of ODEs. The discretized iterations obtained in this way will be denoted by WRGJRK, WRGSRK, and WRSORRK methods, respectively. It is demonstrated that under some technical assumptions these iterations are unconditionally contractive for dissipative systems (1.1) for any mesh $\{t_0, t_1, \ldots\}$ if the underlying CRK method is semi $AN_f(0)$-stable or semi $A_f(0)$-stable. These stability properties were introduced recently by Bellen and Zennaro [2] and Zennaro [23] to investigate contractivity properties of RK methods with respect to test equations with forcing terms. Similar results are also obtained for DSRK methods, i.e., methods obtained in the limit from WRGJRK, WRGSRK, and WRSORRK methods as the iteration index goes to infinity.

We also investigate contractivity properties of numerical iterations based on explicit CRK methods. It is demonstrated that under some assumptions the regions of contractivity of such iterations and the corresponding DSRK method (which is implicit) contains the regions of $AN_f(0)$- or $A_f(0)$-stability of the underlying CRK formula. This again shows the potential of numerical WR iterations although the improvement is not as dramatic as in the implicit case (in particular, the resulting implicit DSRK method cannot be unconditionally contractive). However, it may happen that the region of contractivity of WR iterations and DSRK methods is quite large even if the region of contractivity of the underlying explicit RK method is empty (compare the example corresponding to explicit three-stage RK method of order 3 in §6).

Although the analysis of this paper is restricted only to the systems of ODEs that are dissipative in the maximum norm, we would like to stress that similar ideas are also applicable in more general situations. This approach may lead to new classes of one-step methods that treat different parts of the system (1.1) in different ways depending on particular characteristics of the problem. The resulting methods could be called split Runge–Kutta (SRK) methods. In the case of dissipative systems the diagonal elements of the Jacobian matrix are special and are given special treatment (this leads to DSRK methods), but in general it may be useful to consider different types of splitting if the properties of the underlying system of ODEs suggest to do so.

The organization of this paper is as follows. In the next section we define continuous-time WRGJ, WRGS, and WRSOR iterations and the class of CRK methods with interpolation given by NCEs introduced by Zennaro [21]. We also define discretized iterations WRGJRK, WRGSRK, and WRSORRK resulting from application of CRK methods to the corresponding systems of differential equations for continuous-time iterations. These discretized iterations are well defined for sufficiently small stepsize and for all finite windows and converge to the same limit denoted by DSRK method as the iteration index goes to infinity.

In §3 we prove that the DSRK method has the same order of convergence as the underlying RK method and that the interpolants associated with it satisfy the properties of NCEs.

In §4 we formulate sufficient conditions for contractivity of WRGJRK, WRGSRK, and WRSORRK iterations and in §5 we investigate the conditions for the unique solvability of the systems of algebraic equations that define DSRK method for arbitrary stepsize h, and the conditions for the convergence of discretized WR iterations. These results are independent of the length of the window (which may even be unbounded) and the number of steps.

In §6 we present some examples of unconditionally contractive methods of orders 2 and 3 and discuss explicit methods up to the order 4. Finally, in §7 some concluding remarks are given.

2. Continuous-time and discretized waveform relaxation iterations.

In this section we will discuss the iterative methods (continuous-time and discretized) for the solution of the system (1.1). Assume that the kth iterative solution $y^k(t)$ is given on some window $[t_0, T]$, where in some cases T may be equal to infinity. Then the continuous-time iterations WRGJ, WRGS, and WRSOR are defined recursively as follows:

WRGJ iteration:

$$(2.1)\quad \begin{cases} \dfrac{dy_i^{k+1}(t)}{dt} = f_i(t, y_1^k(t), \dots, y_{i-1}^k(t), y_i^{k+1}(t), y_{i+1}^k(t), \dots, y_m^k(t)), \\ y_i^{k+1}(t_0) = y_{0,i}, \end{cases}$$

$$i = 1, 2, \dots, m, t \in [t_0, T], k = 0, 1, \dots;$$

WRGS iteration:

$$(2.2)\quad \begin{cases} \dfrac{dy_i^{k+1}(t)}{dt} = f_i(t, y_1^{k+1}(t), \dots, y_{i-1}^{k+1}(t), y_i^{k+1}(t), y_{i+1}^k(t), \dots, y_m^k(t)), \\ y_i^{k+1}(t_0) = y_{0,i}, \end{cases}$$

$$i = 1, 2, \dots, m, t \in [t_0, T], k = 0, 1, \dots;$$

WRSOR iteration:

$$(2.3)\quad \begin{cases} \dfrac{d\bar{y}_i^{k+1}(t)}{dt} = f_i(t, y_1^{k+1}(t), \dots, y_{i-1}^{k+1}(t), \bar{y}_i^{k+1}(t), y_{i+1}^k(t), \dots, y_m^k(t)), \\ \bar{y}_i^{k+1}(t_0) = y_{0,i}, \\ y_i^{k+1}(t) = (1 - \omega) y_i^k(t) + \omega \bar{y}_i^{k+1}(t), \end{cases}$$

$$i = 1, 2, \dots, m, t \in [t_0, T], k = 0, 1, \dots.$$

Here ω is a real parameter. Observe that for $\omega = 1$ WRSOR method reduces to WRGS iteration.

It is evident that WRGJ method is quite appropriate for the implementation of numerical algorithms on parallel computers, since (2.1) is the decoupled system of ODEs and all the components can be handled simultaneously by different processors. As for the WRGS and WRSOR iterations, although at first sight one does not see any opportunities of exploiting some parallelism across the system due to the dependence of each component on the previous ones, it is still possible to take advantage of another type of parallelism across the time (see Bellen and Tagliaferro [1]).

Let the grid $t_0 < t_1 < \cdots < t_n < t_{n+1} < \dots$ be given and define the stepsize $h_{n+1} = t_{n+1} - t_n$. To solve (2.1)–(2.3) numerically consider the class of ν-stage CRK methods with interpolation given by NCEs. This class, applied to (1.1), takes the form

$$(2.4a)\quad Y_r^{n+1} = \eta(t_n) + h_{n+1} \sum_{s=1}^{\nu} a_{rs} f(t_n + c_s h_{n+1}, Y_s^{n+1}), \qquad r = 1, 2, \dots, \nu,$$

$$(2.4b)\quad \eta(t_{n+1}) = \eta(t_n) + h_{n+1} \sum_{s=1}^{\nu} b_s f(t_n + c_s h_{n+1}, Y_s^{n+1}),$$

$$(2.4c)\quad \eta(t_n + \theta h_{n+1}) = \eta(t_n) + h_{n+1} \sum_{s=1}^{\nu} b_s(\theta) f(t_n + c_s h_{n+1}, Y_s^{n+1}), \qquad \theta \in [0, 1],$$

$n = 0, 1, \dots,$ where $c_r = \sum_{s=1}^{\nu} a_{rs}$. We restrict attention to the methods for which $c_r \in [0, 1]$. We recall from Zennaro [21] that the functions $b_s(\theta)$ in (2.4c) are polynomials of degree d, $[(p+1)/2] \le d \le p$, where p is the nodal order of the method (2.4) and $[(p+1)/2]$ stands for the integer part of $(p+1)/2$. These functions satisfy the conditions

$$b_s(0) = 0 \quad \text{and} \quad b_s(1) = b_s, \qquad s = 1, 2, \dots, \nu,$$

so that the piecewise polynomial function $\eta(t)$ is continuous on the whole interval of integration and (2.4c) includes (2.4b) as a special case. This function $\eta(t)$ will be referred to as an NCE of the numerical solution $\{\eta(t_0), \eta(t_1), \dots\}$ defined on the grid $\{t_0, t_1, \dots\}$.

Applying the CRK method (2.4) to the systems (2.1)–(2.3) along the mesh $\{t_0, t_1, \dots, t_N\}$ of the window $[t_0, t_N := T]$ we obtain WRGJRK, WRGSRK, and WRSORRK discretized iterations defined by the following.

WRGJRK method:

(2.5a)
$$Y_{r,i}^{k+1} = \eta_i^{k+1}(t_n) + h \sum_{s=1}^{\nu} a_{rs} f_i(t_n + c_s h, \eta_1^k(t_n + c_s h), \dots,$$
$$\eta_{i-1}^k(t_n + c_s h), Y_{s,i}^{k+1}, \eta_{i+1}^k(t_n + c_s h), \dots, \eta_m^k(t_n + c_s h)),$$

(2.5b)
$$\eta_i^{k+1} = (t_n + \theta h) = \eta_i^{k+1}\,(t_n) + h \sum_{s=1}^{\nu} b_s(\theta) f_i(t_n + c_s h, \eta_1^k(t_n + c_s h), \dots,$$
$$\eta_{i-1}^k(t_n + c_s h), Y_{s,i}^{k+1}, \eta_{i+1}^k(t_n + c_s h), \dots, \eta_m^k(t_n + c_s h)),$$

$$r = 1, 2, \dots, \nu, i = 1, 2, \dots, m, n = 0, 1, \dots, N-1, \theta \in [0,1], k = 0, 1, \dots;$$

WRGSRK method:

(2.6a)
$$Y_{r,i}^{k+1} = \eta_i^{k+1}(t_n) + h \sum_{s=1}^{\nu} a_{rs}\; f_i(t_n + c_s h, \eta_1^{k+1}(t_n + c_s h), \dots,$$
$$\eta_{i-1}^{k+1}(t_n + c_s h), Y_{s,i}^{k+1}, \eta_{i+1}^k(t_n + c_s h), \dots, \eta_m^k(t_n + c_s h)),$$

(2.6b)
$$\eta_i^{k+1} = (t_n + \theta h) = \eta_i^{k+1}(t_n) + h \sum_{s=1}^{\nu} b_s(\theta) f_i(t_n + c_s h, \eta_1^{k+1}(t_n + c_s h), \dots,$$
$$\eta_{i-1}^{k+1}(t_n + c_s h), Y_{s,i}^{k+1}, \eta_{i+1}^k(t_n + c_s h), \dots, \eta_m^k(t_n + c_s h)),$$

$$r = 1, 2, \dots, \nu, i = 1, 2, \dots, m, n = 0, 1, \dots, N-1, \theta \in [0,1], k = 0, 1, \dots;$$

WRSORRK method:

$$Y_{r,i}^{k+1} = \eta_i^{k+1}(t_n) + h \sum_{s=1}^{\nu} a_{rs} f_i(t_n + c_s h, \eta_1^{k+1}(t_n + c_s h), \dots, \tag{2.7a}$$
$$\eta_{i-1}^{k+1}(t_n + c_s h), Y_{s,i}^{k+1}, \eta_{i+1}^k(t_n + c_s h), \dots, \eta_m^k(t_n + c_s h)),$$

(2.7b)
$$\eta_i^{k+1}\,(t_n + \theta h) = (1 - \omega)\eta_i^k(t_n + \theta h)$$
$$+ \omega \left(\eta_i^{k+1}(t_n) + h \sum_{s=1}^{\nu} b_s(\theta) f_i(t_n + c_s h, \eta_1^{k+1}(t_n + c_s h), \dots, \right.$$
$$\left. \eta_{i-1}^{k+1}(t_n + c_s h), Y_{s,i}^{k+1}, \eta_{i+1}^k(t_n + c_s h), \dots, \eta_m^k(t_n + c_s h)) \right),$$

$r = 1, 2, \ldots, \nu, i = 1, 2, \ldots, m, n = 0, 1, \ldots, N-1, \theta \in [0, 1], k = 0, 1, \ldots.$

Here, $\eta^0(t)$ is a given initial guess (often chosen as the constant function y_0). In the above formulas $h = h_{n+1}$ and for notational convenience we suppressed the dependence of $Y_{r,i}^{k+1}$ on the index $n+1$.

Using standard contraction principle arguments we can show that for sufficiently small h the methods (2.5)–(2.7) are well defined and have the same limit as $k \to \infty$. This limit depends on the dimension m of the system (1.1) and is given by

(2.8a)

$$\overline{Y}_{r,i} = \overline{\eta}_i(t_n) + h \sum_{s=1}^{\nu} a_{rs} f_i \, (t_n + c_s h, \overline{\eta}_1(t_n + c_s h), \ldots, \overline{\eta}_{i-1}(t_n + c_s h), \overline{Y}_{s,i}, \overline{\eta}_{i+1}(t_n + c_s h), \ldots, \overline{\eta}_m(t_n + c_s h)),$$

(2.8b)

$$\overline{\eta}_i(t_n + \theta h) = \overline{\eta}_i(t_n) + h \sum_{s=1}^{\nu} b_s(\theta) f_i(t_n + c_s h, \overline{\eta}_1(t_n + c_s h), \ldots, \overline{\eta}_{i-1}(t_n + c_s h), \overline{Y}_{s,i}, \overline{\eta}_{i+1}(t_n + c_s h), \ldots, \overline{\eta}_m(t_n + c_s h)),$$

$r = 1, 2, \ldots, \nu, i = 1, 2, \ldots, m, n = 0, 1, \ldots, N-1, \theta \in [0, 1]$. This method gives a special treatment to the diagonal part of the system (1.1) and will be called diagonally split Runge–Kutta (DSRK) method. This is a new class of one-step methods, different from CRK methods but strictly related to them, which as will be demonstrated in §4 is more promising with respect to stability properties. Observe that the method (2.8) is not even included in the class of general linear methods introduced by Butcher [7] (see also Burrage [6]). Although (2.8) was obtained as the limit of WRGJRK, WRGSRK, or WRSORRK iterations, this method is of interest in its own right.

We summarize the above discussion in the following theorem (see also Theorem 2.2 in Bellen and Zennaro [3]).

THEOREM 2.1. *Assume that the function f in* (1.1) *is continuous and satisfies the Lipschitz condition with respect to the second argument. Then there exists $\overline{h}_0 > 0$ such that, for all grids $\{t_0, t_1, \ldots, t_N\}$ such that* $\max\{h_{n+1} : 0 \le n \le N-1\} < \overline{h}_0$ *and all finite windows $[t_0, t_N := T]$, the following statements hold.*

(i) *The numerical* WRGJRK, WRGSRK, *and* WRSORRK *iterations are well defined, i.e., the $\nu \times \nu$-systems* (2.5a), (2.6a), *and* (2.7a) *have unique solutions $\{Y_{r,i}^{k+1}\}_{r=1}^{\nu}$ for $i = 1, 2, \ldots, m, n = 0, 1, \ldots, N-1$, and $k = 0, 1, \ldots$;*

(ii) *The limit* DSRK *method is well defined, i.e., the $2m\nu \times 2m\nu$ system* (2.8) *has a unique solution $\{\overline{Y}_{r,i}\}_{r=1,i=1}^{\nu \;\; m}$ and $\{\overline{\eta}_i(t_n + c_r h)\}_{r=1,i=1}^{\nu \;\; m}$ for all $n = 0, 1, \ldots, N-1$;*

(iii) *The numerical* WRGJRK, WRGSRK, *and* WRSORRK *iterations converge to the limit* DSRK *method* (2.8) *as $k \to \infty$, i.e.,*

$$\lim_{k \to \infty} \sup_{t_0 \le t \le t_N} \|\eta^k(t) - \overline{\eta}(t)\|_\infty = 0.$$

A common strategy to implement WRGJRK, WRGSRK, or WRSORRK methods is to fix a tolerance TOL and to iterate until

$$\max\{\|\eta^{k+1}(t_n) - \eta^k(t_n)\|_\infty : 0 \le n \le N\} \le \text{ TOL}.$$

Of course, one should match the tolerance TOL with the accuracy of the method, i.e., with the choice of the grid $\{t_0, t_1, \ldots, t_N\}$ on appropriately chosen window $[t_0, T]$.

These important aspects are not addressed in this paper and will be discussed elsewhere.

We conclude this section with a few general remarks about numerical WR iterations.

First of all, observe that $\eta^{k+1}(t)$ given by (2.5b), (2.6b), or (2.7b), or $\overline{\eta}(t)$ given by (2.8b) are defined on the whole window $[t_0, T]$ and not only on the grid $\{t_0, t_1, \dots, t_N\}$. It is necessary to use these interpolants since integrating the ith component of (2.1), (2.2), or (2.3) by CRK method (2.4) we have to approximate the other components using the information from the previous iteration, and/or from the already computed components in the present iteration, at nongrid points $t_n + c_s h, s = 1, 2, \dots, \nu$. Similarly, integrating (1.1) by the DSRK method we have to compute the approximations $\overline{\eta}(t)$ at $t = t_n + c_s h, s = 1, 2, \dots, \nu$. It is evident that these interpolants must be sufficiently accurate in order to maintain the nodal order p of the underlying RK method (2.4).

Our second remark is that with the approach we have adopted the same stepsize $h = h_{n+1}$ is used for all the components of the systems (2.1)–(2.3) or (1.1). Hence, we could also choose to approximate the other components of y^k by means of the quantities $Y_{s,j}^k$ and/or $Y_{s,j}^{k+1}$ instead of the quantities $\eta_j^k(t_n + c_s h)$ and/or $\eta_j^{k+1}(t_n + c_s h)$, respectively. This is exactly what Lie and Skålin [13] did. With this choice the limit method (2.8) degenerates into the CRK method (2.4). We have adopted a different strategy than Lie and Skålin, searching for new classes of one-step methods with better stability properties than the underlying RK methods. The results presented in §4 (compare in particular Theorems 4.1–4.3) as well as the examples presented in §6 give the evidence that this strategy has proven to be successful.

The next observation is that if the underlying CRK method is a projection method (in particular, collocation), then we have $a_{rs} = b_s(c_r), r, s = 1, 2, \dots, \nu$ (see Zennaro [22]) and, as a consequence, $Y_{s,j}^k = \eta_j^k(t_n + c_s h)$. Therefore, also in this case, the DSRK method (2.8) degenerates to the CRK formula (2.4).

We could also define WR iterations applying the CRK method (2.4) to the "pure" Picard iterations

$$\begin{aligned} \frac{dy_i^{k+1}(t)}{dt} &= f_i(t, y_1^{(k)}, \dots, y_m^{(k)}), \\ y_i^{k+1}(t_0) &= y_{0,i}, \end{aligned}$$

$i = 1, 2, \dots, m, t \in [t_0, T], k = 0, 1, \dots$. Then it is easy to see that the limit method would degenerate into the (classical) CRK method with the coefficient matrix $[b_s(c_r)]_{r=1,s=1}^{\nu\ \ \nu}$, which, in general, differs from the coefficient matrix A of the underlying method (2.4) (in fact, $[b_s(c_r)] = A$ if and only if (2.4) is a projection method). Hence, in view of the order barrier by Spijker [19] and Kraaijevanger [9] (compare §4) the resulting method cannot have good contractivity properties.

3. Order of accuracy of the DSRK method. To investigate the order of convergence of WR methods as $k \to \infty$, observe that the limit of WRGJRK, WRGSRK, or WRSORRK iterations on the window $[t_0, T]$ is the same as the result of applying the limit DSRK method (2.8) in a step-by-step fashion over the mesh $\{t_0, t_1, \dots, t_N\}$. Therefore, without loss of generality, we can consider only the local discretization error of (2.8) over the step $[t_n, t_{n+1}]$. The WR iterations restricted to one step only are called time-point relaxation methods.

We have the following theorem.

THEOREM 3.1. *Assume that the function f in* (1.1) *is sufficiently smooth; the ν-stage* RK *method* (2.4a–b) *has nodal order p and that the interpolant* (2.4c) *is an* NCE *of degree d. Then the* DSRK *method* (2.8) *also has nodal order p and the polynomial $\overline{\eta}$ defined by* (2.8b) *satisfies the properties of the* NCEs, *i.e.*,

$$\sup_{t_n \le t \le t_{n+1}} \|y'(t) - \overline{\eta}'(t)\|_\infty = O(h^d) \tag{3.1}$$

and

$$\left\| \int_{t_n}^{t_{n+1}} G(x)(y'(x) - \overline{\eta}'(x))dx \right\|_\infty = O(h^{p+1}), \tag{3.2}$$

as $h \to 0$, for every sufficiently smooth matrix-valued function $G(x)$. Here, y is the local solution, i.e., the solution to $y'(t) = f(t, y(t)), t \in [t_n, t_{n+1}], y(t_n) = \eta(t_n)$.

Proof. By Theorem 2.1 the method (2.8) is well defined for sufficiently small h, and using standard arguments it follows that $\overline{\eta}(t)$, which is a polynomial of degree d on $[t_n, t_{n+1}]$, is uniformly bounded together with all its derivatives as $h \to 0$.

Consider the initial-value problem

$$\begin{cases} w_i'(t) = f_i(t, \overline{\eta}_1(t), \ldots, \overline{\eta}_{i-1}(t), w_i(t), \overline{\eta}_{i+1}(t), \ldots, \overline{\eta}_m(t)), \\ w_i(t_n) = \overline{\eta}_i(t_n), \end{cases} \tag{3.3}$$

$t \in [t_n, t_{n+1}]$. It is clear that the application of the CRK method (2.4) yields exactly (2.8). Hence, $\overline{\eta}$ defined by (2.8b) is an NCE. Since $\overline{\eta}$ is bounded together with all of its derivatives, this property is inherited by the functions $w_i(t)$ defined by (3.3). Using the properties of the NCEs (see Zennaro [21]) we obtain

$$\sup_{t_n \le t \le t_{n+1}} |w_i'(t) - \overline{\eta}_i'(t)| = O(h^d), \tag{3.4}$$

$$\left| \int_{t_n}^{t_{n+1}} g(x)(w_i'(x) - \overline{\eta}_i'(x))dx \right| = O(h^{p+1}), \tag{3.5}$$

as $h \to 0$, for every sufficiently smooth (scalar) function $g(x)$. Here, p is the nodal order of the underlying RK method (2.4a–b). We also have (see again [21])

$$\sup_{t_n \le t \le t_{n+1}} |w_i(t) - \overline{\eta}_i(t)| = O(h^{d+1}) \tag{3.6}$$

and

$$\left| \int_{t_n}^{t_{n+1}} g(x)(w_i(x) - \overline{\eta}_i(x))dx \right| = O(h^{p+1}), \tag{3.7}$$

as $h \to 0$, for every sufficiently smooth (scalar) function $g(x)$.

Using the nonlinear variation-of-constants formula of Gröbner and Alekseev we get, in vector notation,

$$y(t) - \overline{\eta}(t) = \int_{t_n}^{t} K(t, x)(f(x, \overline{\eta}(x)) - \overline{\eta}'(x))dx,$$

where $K(t,x) = [K_{ij}(t,x)]_{i,j=1}^{m}$ is a variational matrix depending on the function $\overline{\eta}(t)$, such that $K(t,t) = I$ (the identity matrix) for all $t \geq t_n$. Hence,

$$
\begin{aligned}
y'(t) - \overline{\eta}'(t) = &\int_{t_n}^{t} \frac{\partial K(t,x)}{\partial t}(f(x,\overline{\eta}(x)) - \overline{\eta}'(x))dx \\
&+ f(t,\overline{\eta}(t)) - \overline{\eta}'(t),
\end{aligned} \tag{3.8}
$$

and for every matrix-valued function $G(x) = [G_{ij}(x)]_{i,j=1}^{m}$ we have

$$
\begin{aligned}
&\int_{t_n}^{t_{n+1}} G\,(x)(y'(x) - \overline{\eta}'(x))dx \\
&\quad = \int_{t_n}^{t_{n+1}} \int_{t_n}^{x} G(x)\frac{\partial K(x,\xi)}{\partial \xi}(f(\xi,\overline{\eta}(\xi)) - \overline{\eta}'(\xi))d\xi dx \\
&\quad\quad + \int_{t_n}^{t_{n+1}} G(x)(f(x,\overline{\eta}(x)) - \overline{\eta}'(x))dx.
\end{aligned}
$$

Changing the order of integration in the double integral on the right-hand side of the above equality yields the relation

$$
\begin{aligned}
&\int_{t_n}^{t_{n+1}} G\,(x)(y'(x) - \overline{\eta}'(x))dx \\
&\quad = \int_{t_n}^{t_{n+1}} \int_{x}^{t_{n+1}} G(\xi)\frac{\partial K(\xi,x)}{\partial \xi}(f(x,\overline{\eta}(x)) - \overline{\eta}'(x))d\xi dx \\
&\quad\quad + \int_{t_n}^{t_{n+1}} G(x)(f(x,\overline{\eta}(x)) - \overline{\eta}'(x))dx.
\end{aligned}
$$

This can be written in the form

$$
\int_{t_n}^{t_{n+1}} G(x)(y'(x) - \overline{\eta}'(x))dx = \int_{t_n}^{t_{n+1}} H(x)(f(x,\overline{\eta}(x)) - \overline{\eta}'(x))dx, \tag{3.9}
$$

where

$$
H(x) = G(x) + \int_{x}^{t_{n+1}} G(\xi)\frac{\partial K(\xi,x)}{\partial \xi}d\xi.
$$

The matrix-valued function $H(x) = [H_{ij}(x)]_{i,j=1}^{m}$ depends on $\overline{\eta}$ and is also uniformly bounded together with all its derivatives as $h \to 0$.

Writing (3.8) in componentwise form we obtain

$$
\begin{aligned}
y_i'(t) - \overline{\eta}_i'(t) &= \sum_{j=1}^{m} \int_{t_n}^{t} \frac{\partial K_{ij}(t,x)}{\partial t}(f_j(x,\overline{\eta}(x)) - \overline{\eta}_j'(x))dx + f_i(t,\overline{\eta}(t)) - \overline{\eta}_i'(t) \\
&= \sum_{j=1}^{m} \int_{t_n}^{t} \frac{\partial K_{ij}(t,x)}{\partial t}(f_j(x,\overline{\eta}(x)) \\
&\quad - f_j(x,\overline{\eta}_1(x),\ldots,\overline{\eta}_{j-1}(x),w_j(x),\overline{\eta}_{j+1}(x),\ldots,\overline{\eta}_m(x)) + w_j'(x) - \overline{\eta}_j'(x))dx \\
&\quad + f_i(t,\overline{\eta}(t)) - f_i(t,\overline{\eta}_1(t),\ldots,\overline{\eta}_{i-1}(t),w_i(t),\eta_{i+1}(t),\ldots,\eta_m(t)) \\
&\quad + w_i'(t) - \overline{\eta}_i'(t).
\end{aligned}
$$

Hence
(3.10)

$$y_i'(t) - \overline{\eta}_i'(t) = -\sum_{j=1}^{m} \int_{t_n}^{t} \frac{\partial K_{ij}(t,x)}{\partial t} \left(\frac{\partial f_j(x,\overline{\eta}(x))}{\partial y_j}(w_j(x) - \overline{\eta}_j(x)) - w_j'(x) + \eta_j'(x) \right) dx$$
$$- \frac{\partial f_i(t,\overline{\eta}(t))}{y_i}(w_i(t) - \overline{\eta}_i(t)) + w_i'(t) - \overline{\eta}'(t) + O(h^{2d+2}).$$

Similarly, writing (3.9) in componentwise form we obtain

$$\sum_{j=1}^{m} \int_{t_n}^{t_{n+1}} G_{ij}(x)(y_j'(x) - \overline{\eta}_j'(x))dx$$
$$= \sum_{j=1}^{m} \int_{t_n}^{t_{n+1}} H_{ij}(x)(f_j(x,\overline{\eta}(x)) - \overline{\eta}_j'(x))dx$$
$$= \sum_{j=1}^{m} \int_{t_n}^{t_{n+1}} H_{ij}(x)(f_j(x,\overline{\eta}(x)) - f_j(x,\overline{\eta}_1(x),\ldots,\overline{\eta}_{j-1}(x),w_j(x),\overline{\eta}_{j+1}(x),\ldots,\overline{\eta}_m(x))$$
$$+ w_j'(x) - \overline{\eta}_j'(x))dx.$$

Hence,

$$\sum_{j=1}^{m} \int_{t_n}^{t_{n+1}} G_{ij}(x)(y_j'(x) - \overline{\eta}_j'(x))dx$$
$$(3.11) \quad = -\sum_{j=1}^{m} \int_{t_n}^{t_{n+1}} H_{ij}(x) \left(\frac{\partial f_j(x,\overline{\eta}(x))}{\partial y_j}(w_j(x) - \overline{\eta}_j(x)) - w_j'(x) + \overline{\eta}_j'(x) \right) dx$$
$$+ O(h^{2d+3}).$$

Since all the functions appearing in (3.10) and (3.11) are uniformly bounded with all of their derivatives as $h \to 0$, (3.4), (3.6), and (3.10) yield (3.1). Moreover, since $2d \geq p$ (see [21]), (3.5), (3.7), and (3.11) yield (3.2). Finally, putting $G(x) = I$ in formula (3.2) it follows that the nodal order or DSRK method (2.8) is p. This completes the proof. □

4. Contractivity properties of WR iterations. In this section we investigate how to choose the CRK method (2.8) in such a way that the resulting WRGJRK, WRGSRK, and WRSORRK iterations have desirable contractivity properties with respect to the nonlinear systems (1.1), which are dissipative in the maximum norm. This means that if $y(t)$ and $z(t)$ are solutions to

$$(4.1) \quad \begin{cases} y'(t) = f(t,y(t)), & t \geq t_0, \\ y(t_0) = y_0, \end{cases}$$

and

$$(4.2) \quad \begin{cases} z'(t) = f(t,z(t)), & t \geq t_0, \\ z(t_0) = z_0, \end{cases}$$

then

$$\|y(t) - z(t)\|_\infty \le \|y_0 - z_0\|_\infty, \qquad t \ge t_0, \tag{4.3}$$

where for $y \in R^m$ the maximum norm is defined by $\|y\|_\infty = \max\{|y_i| : 1 \le i \le m\}$. We have the following result about the dissipativity of the system (1.1) in any norm $\|\cdot\|$ on R^m (in particular for the maximum norm $\|\cdot\|_\infty$).

THEOREM 4.1. *Assume that the function $f(t, y)$ in* (1.1) *is of class C^1 for $t \ge t_0$ and $y \in R^m$. Then the problem* (1.1) *is dissipative in a given norm $\|\cdot\|$ on R^m if and only if*

$$\mu(J(t, y)) \le 0,$$

where $J(t, y)$ is the Jacobian matrix

$$J(t, y) = \left[\frac{\partial f_i(t, y)}{\partial y_j}\right]_{i,j=1}^m$$

and where, for any matrix A, $\mu(A)$ is a logarithmic norm defined by

$$\mu(A) = \lim_{\epsilon \to 0+} \frac{\|I + \epsilon A\| - 1}{\epsilon}.$$

Proof. If $\mu(J(t, \mu)) \le 0$, then the system (1.1) is dissipative in a given norm $\|\cdot\|$ in view of Theorem 1.5.2 in Dekker and Verwer [8].

Assume next that (1.1) is dissipative in $\|\cdot\|$. Then it follows from Theorem 5.1 in Kraaijevanger [9] that

$$\|z - y - \epsilon(f(t, z) - f(t, y))\| \ge \|z - y\| \tag{4.4}$$

for all $\epsilon > 0, t \in R$, and all $y, z \in R^m$. For fixed y and $z = y + \delta u, \delta > 0$, we have

$$f_i(t, z) - f_i(t, y) = \sum_{j=1}^m \frac{\partial f_i(t, y + \delta\theta_i u)}{\partial y_j} \delta u_j,$$

$i = 1, 2, \ldots, m$. Hence,

$$f(t, z) - f(t, y) = \tilde{J}(t, y, \delta)\delta u, \tag{4.5}$$

where

$$\tilde{J}(t, y, \delta) := \left[\frac{\partial f_i(t, y + \delta\theta_i u)}{\partial y_j}\right]_{i,j=1}^m.$$

Substituting (4.5) into (4.4), dividing by δ and taking the limit as $\delta \to 0+$ we obtain

$$\|(I - \epsilon J(t, y))u\| \ge \|u\|,$$

$\epsilon > 0, y, u \in R^m$. This leads to

$$\|(I - \epsilon J(t, y))^{-1}\|^{-1} = \min_{\|u\|=1} \|(I - \epsilon J(t, y))u\| \ge 1$$

for sufficiently small $\epsilon > 0$, or

$$\|(I - \epsilon J(t,y))^{-1}\| \leq 1.$$

Expanding $(I - \epsilon J(t,y))^{-1}$ in powers of ϵ we obtain

$$\|I + \epsilon J(t,y)\| + O(\epsilon^2) \leq 1$$

for sufficiently small $\epsilon > 0$. Hence,

$$\frac{\|I + \epsilon J(t,y)\| - 1}{\epsilon} + O(\epsilon) \leq 0,$$

and taking the limit as $\epsilon \to 0+$ we obtain $\mu(J(t,y)) \leq 0$, which is our claim. □

Since the logarithmic norm $\mu_\infty(A)$ corresponding to $\|\cdot\|_\infty$ is

$$\mu_\infty(A) = \max_{1 \leq i \leq m} \left(a_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^{m} |a_{ij}| \right)$$

(compare [8]), it follows from Theorem 4.1 that the system (1.1) is dissipative in the maximum norm if and only if

$$\sum_{\substack{j=1 \\ j \neq i}}^{m} \left| \frac{\partial f_i(t,y)}{\partial y_j} \right| \leq -\frac{\partial f_i(t,y)}{\partial y_i}, \tag{4.6}$$

$i = 1, 2, \ldots, m$, for all $t \geq t_0$ and $y \in R^m$.

We denote the class of dissipative systems by $\mathcal{F}^*$. Important special subclasses of $\mathcal{F}^*$ are dissipative linear systems, both with constant and nonconstant coefficients (i.e., $f(t,y) = \Lambda y$ and $f(t,y) = \Lambda(t)y$, respectively). For these classes the matrices Λ and $\Lambda(t)$ are (weakly) diagonally dominant with negative diagonal elements. As already mentioned in the introduction such systems arise in applications, for example, from semidiscretization of parabolic partial differential equations (see [8]–[10] and [18]).

The numerical methods that inherit the property (4.3) are said to be contractive. To be more precise, we have the following definition.

DEFINITION 4.1. *A one-step method is said to be unconditionally contractive in the maximum norm if, whenever it is applied to the systems* (4.1) *and* (4.2) *with* $f \in \mathcal{F}^*$ *and with any stepsize* $h = h_1 > 0$, *it yields approximations* $\eta(t_0 + h)$ *and* $\xi(t_0 + h)$ *to* $y(t_0 + h)$ *and* $z(t_0 + h)$ *such that*

$$\|\eta(t_0 + h) - \xi(t_0 + h)\|_\infty \leq \|y_0 - z_0\|_\infty. \tag{4.7}$$

Unconditional contractivity is a very desirable but strong property of numerical methods for ODEs, and it should not be surprising that such methods are very difficult to construct. For example, it was demonstrated by Spijker [19] (see also Kraaijevanger [9]) that the requirement of unconditional contractivity (4.7) leads in case of RK methods to the very restrictive order barrier $p \leq 1$. It will be demonstrated that the new class of DSRK methods introduced in this paper break this order barrier (see §6 for examples of such methods of order 2 and 3).

To include in our discussion the explicit methods as well we also consider the set $\mathcal{F}^*(\rho) \subset \mathcal{F}^*$ of functions f which satisfy (4.6) and the additional condition

$$\frac{\partial f_i(t,y)}{\partial y_i} \geq -\rho,$$

$i = 1, 2, \ldots, m, t \geq t_0, y \in R^m$, for some $\rho > 0$. This class $\mathcal{F}^*(\rho)$ is included in the class $\mathcal{F}(\rho)$ defined by Kraaijevanger [9].

DEFINITION 4.2. *The region of (conditional) contractivity in the maximum norm of a one-step method for ODEs is the maximum interval* $[-R, 0]$ *such that whenever the method is applied to* (4.1) *and* (4.2) *with* $f \in \mathcal{F}^*(\rho), \rho > 0$, *and with any stepsize such that* $0 < \rho h < R$, *it yields approximations* $\eta(t_0 + h)$ *and* $\xi(t_0 + h)$ *to* $y(t_0 + h)$ *and* $z(t_0 + h)$ *for which* (4.7) *holds.*

It will be demonstrated that the WRGJRK, WRGSRK, and WRSORRK iterations are contractive if the underlying RK method (2.4a)–(2.4b) as well as the interpolant (2.4c) satisfy the so-called properties of $A_f(0)$-stability or $AN_f(0)$-stability. These properties are generalizations of A-stability and AN-stability and were recently introduced and investigated by Bellen and Zennaro [2] and Zennaro [23]. The relevant definitions are reproduced below.

DEFINITION 4.3. *The region of* $A_f(0)$*-stability of the RK method* (2.4a)–(2.4b) *is the maximal segment* $[-r, 0]$ *such that the matrix* $I - xA$ *is nonsingular and*

$$|1 + xb^T(I - xA)^{-1}e| + \|xb^T(I - xA)^{-1}\|_1 = 1$$

for $-r < x < 0$. *The method* (2.4a)–(2.4b) *is said to be* $A_f(0)$*-stable if* $r = +\infty$. *Here,* $A = [a_{ij}]_{i,j=1}^{\nu}$ *is the coefficient matrix of RK method* (2.4a)–(2.4b), $b = [b_1, b_2, \ldots, b_\nu]^T$ *is the vector of weights, and* $e := [1, 1, \ldots, 1]^T \in R^\nu$.

DEFINITION 4.4. *The region of* $AN_f(0)$*-stability of the RK method* (2.4a)–(2.4b) *is the maximum segment* $[-r, 0]$ *such that the matrix* $I - AX$ *is nonsingular and*

$$|1 + b^TX(I - AX)e| + \|b^TX(I - AX)^{-1}\|_1 = 1$$

for $-r < x_i < 0, i = 1, \ldots, \nu$, *where* $X = \operatorname{diag}(x_1, x_2, \ldots, x_\nu)$. *The method is said to be* $AN_f(0)$*-stable if* $r = +\infty$.

Remark. In [2] the above condition was formulated in the equivalent form

$$|1 + b^T(I - XA)^{-1}Xe| + \|b^T(I - XA)^{-1}X\|_1 = 1, \qquad -r < x_i < 0, i = 1, \ldots, \nu.$$

Similar definitions are given for the interpolants and for CRK methods.

DEFINITION 4.5. *The region of* $A_f(0)$*-stability of the CRK method* (2.4) *at the point* θ *is the maximum segment* $[-r(\theta), 0]$ *such that the matrix* $I - xA$ *is nonsingular and*

$$|1 + xb(\theta)^T(I - xA)^{-1}e| + \|xb(\theta)^T(I - xA)^{-1}\|_1 = 1,$$

for $-r(\theta) < x < 0$, *where* $b(\theta)^T = [b_1(\theta), b_2(\theta), \ldots, b_\nu(\theta)]^T$. *The CRK method is said to be* $A_f(0)$*-stable at the point* θ *if* $r(\theta) = +\infty$. *The segment* $[-R^*, 0]$, *where* $R^* = \min\{r(c_1), r(c_2), \ldots, r(c_\nu), r(1)\}$ *is said to be the region of semi* $A_f(0)$*-stability of the CRK method* (2.4). *The CRK method is said to be semi* $A_f(0)$*-stable if* $R^* = +\infty$. *Note that* $r(1) = r$ *of the Definition* 4.3.

DEFINITION 4.6. *The region of* $AN_f(0)$*-stability of the CRK method* (2.4) *at the point* θ *is the maximum segment* $[-r(\theta), 0]$ *such that the matrix* $I - AX$ *is nonsingular and*

$$|1 + b(\theta)^TX(I - AX)^{-1}e| + \|b(\theta)^TX(I - AX)^{-1}\|_1 = 1$$

for $-r(\theta) < x_i < 0, i = 1, \ldots, \nu$. *The CRK method is said to be* $AN_f(0)$*-stable at the point* θ *if* $r(\theta) = +\infty$. *The segment* $[-R^*, 0]$, *where* $R^* = \min\{r(c_1); r(c_2), \ldots, r(c_\nu),$

$r(1)\}$ *is said to be the region of semi* $AN_f(0)$*-stability of the CRK method* (2.4). *The CRK method is said to be semi* $AN_f(0)$*-stable if* $R^* = +\infty$. *Note that* $r(1) = r$ *of Definition* 4.4.

The above definitions are relevant to the scalar test equations with forcing terms

$$\text{(4.8)} \qquad \begin{cases} y'(t) = \lambda y(t) + g(t), & t \geq t_0, \\ y(t_0) = y_0, \end{cases}$$

and

$$\text{(4.9)} \qquad \begin{cases} y'(t) = \lambda(t) y(t) + g(t), & t \geq t_0, \\ y(t_0) = y_0, \end{cases}$$

respectively. It is known (compare [2]) that if $\lambda < 0$ and $\lambda(t) < 0$ their solutions satisfy the inequalities

$$\text{(4.10)} \qquad |y(t)| \leq \max\left\{|y_0|, \sup_{t_0 \leq x \leq t} \frac{|g(x)|}{-\lambda}\right\}, \qquad t \geq t_0,$$

and

$$\text{(4.11)} \qquad |y(t)| \leq \max\left\{|y_0|, \sup_{t_0 \leq x \leq t} \frac{|g(x)|}{-\lambda(x)}\right\}, \qquad t \geq t_0,$$

respectively. It was demonstrated by Bellen and Zennaro [2] that if $h\lambda \in (-r(\theta), 0)$, where $r(\theta)$ is as in Definition 4.5, then the numerical approximation $\eta(t_0 + \theta h)$ to the solution $y(t_0 + \theta h)$ of (4.8), furnished by the CRK method (2.4) satisfies

$$|\eta(t_0 + \theta h)| \leq \max\left\{|y_0|, \max_{1 \leq r \leq \nu} \frac{|g(t_0 + c_r h)|}{-\lambda}\right\},$$

which is a discrete analogue of (4.10). Similarly, if $h\lambda(t) \in (-r(\theta), 0)$, where $r(\theta)$ is as in Definition 4.6, then the numerical approximation $\eta(t_0 + \theta h)$ to the solution $y(t_0 + \theta h)$ of (4.9), furnished by CRK method (2.4), satisfies the inequality

$$|\eta(t_0 + \theta h)| \leq \max\left\{|y_0|, \max_{1 \leq r \leq \nu} \frac{|g(t_0 + c_r h)|}{-\lambda(t_0 + c_r h)}\right\},$$

which is the discrete analogue of (4.11).

To study contractivity properties of WR iterations, we will make the following assumptions.

(H_1) The function f in (4.1) is sufficiently smooth, $f \in \mathcal{F}^*(\rho)$ for some $\rho > 0$, and

$$\text{(4.12)} \qquad \frac{\partial f_i(t, y)}{\partial y_i} < 0, \qquad i = 1, 2, \ldots, m,$$

for all $t \geq t_0$ and $y \in R^m$.

(H_2) The mesh $\{t_0, t_1, \ldots, t_N\}$ is such that

$$\rho \max_{0 \leq n \leq N-1} h_{n+1} < R^*,$$

where $h_{n+1} = t_{n+1} - t_n$ and $[-R^*, 0]$ is the region of semi $AN_f(0)$-stability of the CRK method (2.4).

It follows from Theorem 2.1 that the WRGJRK, WRGSRK, and WRSORRK are well defined for sufficiently small stepsize h. Zennaro [23] proved a stronger result that these iterations are well defined if the function f satisfies (4.12) in H_1 and the mesh $\{t_0, t_1, \dots, t_N\}$ satisfies H_2. The next two theorems are concerned with contractivity properties of these WR iterations.

THEOREM 4.2. *Assume* $\mathrm{H}_1, \mathrm{H}_2$, *and that the initial approximations* η^0 *and* ξ^0 *to the solutions* y *and* z *of* (4.1) *and* (4.2), *respectively, are such that*

$$\|\eta^0(t) - \xi^0(t)\|_\infty \le \|y_0 - z_0\|_\infty$$

for $t \in [t_0, t_N]$. *Then the numerical approximations* η^k *and* ξ^k *generated by the* WRGJRK, WRGSRK, *and* WRSORRK, $0 < \omega \le 1$, *iterations satisfy the inequality*

$$\max_{\theta \in \{c_1, c_2, \dots, c_\nu, 1\}} \|\eta^k(t_n + \theta h) - \xi^k(t_n + \theta h)\|_\infty \le \|y_0 - z_0\|_\infty \tag{4.13}$$

for $n = 0, 1, \dots, N-1$. *In particular, if* $R^* = +\infty$ (*i.e., the underlying* CRK *method* (2.4) *is semi* $AN_f(0)$*-stable*) *then* (4.13) *holds for any problems* (4.1) *and* (4.2) *with* $f \in \mathcal{F}^*$ *satisfying* (4.12) *and for any mesh* $\{t_0, t_1, \dots, t_N\}$.

Proof. We will prove this theorem only for WRGJRK iteration by using the induction on the iteration index k and on the step index n. The other two cases can be treated in a similar way just by adding the induction on the component index i.

It is clear that (4.13) is satisfied for $k = 0$ and $0 \le n \le N-1$. Assume that (4.13) is true for fixed iteration index k and for all $0 \le n \le N-1$, and we will show that this is also true for the iteration index $k+1$ and $0 \le n \le N-1$. This will be accomplished if we are able to show that the inequality

$$\|\eta^{k+1}(t_n) - \xi^{k+1}(t_n)\|_\infty \le \|y_0 - z_0\|_\infty \tag{4.14}$$

implies (4.13), since (4.14) is clearly satisfied for $n = 0$.

Subtracting WRGJRK iterations applied to (4.1) and (4.2), respectively, we obtain

$$\begin{aligned} Y_{r,i}^{k+1} - Z_{r,i}^{k+1} &= \eta_i^{k+1}(t_n) - \xi_i^{k+1}(t_n) + h\sum_{s=1}^{\nu} a_{rs}\lambda_{iis}(Y_{s,i}^{k+1} - Z_{s,i}^{k+1}) \\ &\quad + h\sum_{s=1}^{\nu} a_{rs} \sum_{\substack{j=1 \\ j \ne i}}^{m} \lambda_{ijs}(\eta_j^k(t_n + c_s h) - \xi_j^k(t_n + c_s h)), \end{aligned}$$

$r = 1, 2, \dots, \nu, i = 1, 2, \dots, m$, and

$$\begin{aligned} &\eta_i^{k+1}(t_n + \theta h) - \xi_i^{k+1}(t_n + \theta h) \\ &\quad = \eta_i^{k+1}(t_n) - \xi_i^{k+1}(t_n) \\ &\quad\quad + h\sum_{s=1}^{\nu} b_s(\theta)\lambda_{iis}(Y_{s,i}^{k+1} - Z_{s,i}^{k+1}) \\ &\quad\quad + h\sum_{s=1}^{\nu} b_s(\theta) \sum_{\substack{j=1 \\ j \ne i}}^{m} \lambda_{ijs}(\eta_j^k(t_n + c_s h) - \xi_j^k(t_n + c_s h)), \end{aligned}$$

$i = 1, 2, \ldots, m, \theta \in [0, 1]$, where

$$\lambda_{ijs} = \frac{\partial f_i(t_n + c_s h, \gamma_{is})}{\partial y_j},$$

$i, j = 1, 2, \ldots, m, s = 1, 2, \ldots, \nu$, and the γ_{is}'s are suitable vectors that result from the application of the mean value theorem.

Put

$$\begin{aligned} X_{ij} &= h\text{diag}\,(\lambda_{ij1}, \lambda_{ij2}, \ldots, \lambda_{ij\nu}), \\ Y_i^{k+1} &= [Y_{1,i}^{k+1}, Y_{2,i}^{k+1}, \ldots, Y_{\nu,i}^{k+1}]^T, \\ Z_i^{k+1} &= [Z_{1,i}^{k+1}, Z_{2,i}^{k+1}, \ldots, Z_{\nu,i}^{k+1}]^T, \\ \eta_i^k &= [\eta_i^k(t_n + c_1 h), \eta_i^k(t_n + c_2 h), \ldots, \eta_i^k(t_n + c_\nu h)]^T, \\ \xi_i^k &= [\xi_i^k(t_n + c_1 h), \xi_i^k(t_n + c_2 h), \ldots, \xi_i^k(t_n + c_\nu h)]^T. \end{aligned}$$

Then

$$Y_i^{k+1} - Z_i^{k+1} = (\eta_i^{k+1}(t_n) - \xi_i^{k+1}(t_n))e$$
$$+AX_{ii}(Y_i^{k+1} - Z_i^{k+1}) + A\sum_{\substack{j=1 \\ j \neq i}}^{m} X_{ij}(\eta_j^k - \xi_j^k),$$

and

$$\eta_i^{k+1}(t_n + \theta h) - \xi_i^{k+1}(t_n + \theta h)$$
$$= \eta_i^{k+1}(t_n) - \xi_i^{k+1}(t_n)$$
$$+ b(\theta)^T X_{ii}(Y_i^{k+1} - Z_i^{k+1}) + b(\theta)^T \sum_{\substack{j=1 \\ j \neq i}}^{m} X_{ij}(\eta_j^k - \xi_j^k).$$

It follows from H_1 and H_2 that $h\lambda_{iir} \geq -h\rho > -R^*, r = 1, 2, \ldots, \nu$. This means that X_{ii} belongs to the region of semi $AN_f(0)$-stability of the underlying CRK method (2.4). In particular, the matrix $I - AX_{ii}$ is nonsingular and

$$Y_i^{k+1} - Z_i^{k+1} = (I - AX_{ii})^{-1} e(\eta_i^{k+1}(t_n) - \xi_i^{k+1}(t_n))$$
$$+(I - AX_{ii})^{-1} A \sum_{\substack{j=1 \\ j \neq i}}^{m} X_{ij}(\eta_j^k - \xi_j^k).$$

Substituting this relation into the expression for $\eta_i^{k+1}(t_n + \theta h) - \xi_i^{k+1}(t_n + \theta h)$, after some computations we get

$$\eta_i^{k+1}(t_n + \theta h) - \xi_i^{k+1}(t_n + \theta h) = (1 + b(\theta)^T X_{ii}(I - AX_{ii})^{-1} e)(\eta_i^{k+1}(t_n) - \xi_i^{k+1}(t_n))$$
$$+ b(\theta)^T X_{ii}(I - AX_{ii})^{-1} X_{ii}^{-1} \sum_{\substack{j=1 \\ j \neq i}}^{m} X_{ij}(\eta_j^k - \xi_j^k).$$

Using the Hölder inequality $|u^T v| \le \|u\|_1 \|v\|_\infty$ the above equation leads to the following estimate

$$\begin{aligned}
&|\eta_i^{k+1}(t_n+\theta h) - \xi_i^{k+1}(t_n+\theta h)| \\
&\quad \le |1 + b(\theta)^T X_{ii}(I - AX_{ii})^{-1} e| |\eta_i^{k+1}(t_n) - \xi_i^{k+1}(t_n)| \\
&\quad\quad + \|b(\theta)^T X_{ii}(I - AX_{ii})^{-1}\|_1 \|X_{ii}^{-1} \sum_{\substack{j=1\\ j\neq i}}^{m} X_{ij}(\eta_j^k - \xi_j^k)\|_\infty,
\end{aligned}$$

$i = 1, 2, \ldots, m, \theta \in [0, 1]$. Taking into account that

$$\|X_{ii}^{-1} X_{ij}\left(\eta_j^k - \xi_j^k\right)\|_\infty = \max_{1\le r\le \nu} \frac{\left|\sum_{j=1, j\neq i}^{m} \lambda_{ijr}(\eta_j^k(t_n + c_r h) - \xi_j^k(t_n + c_r h))\right|}{-\lambda_{iir}},$$

we get

$$\begin{aligned}
&|\eta_i^{k+1}(t_n+\theta h) - \xi_i^{k+1}(t_n+\theta h)| \\
&\le (|1 + b(\theta)^T X_{ii}(I - AX_{ii})^{-1} e| + \|b(\theta)^T X_{ii}(I - AX_{ii})^{-1}\|_1) \\
&\quad \times \max\left\{ |\eta_i^{k+1}(t_n) - \xi_i^{k+1}(t_n)|, \max_{1\le r\le\nu} \frac{\sum_{j=1, j\neq i}^{m} |\lambda_{ijr}| |\eta_j^k(t_n + c_r h) - \xi_j^k(t_n + c_r h)|}{-\lambda_{iir}} \right\},
\end{aligned}$$

$i = 1, 2, \ldots, m, \theta \in [0, 1]$. Since X_{ii} belongs to the region of semi $AN_f(\theta)$-stability of the method (2.4) we have

$$|1 + b(\theta)^T X_{ii}(I - AX_{ii})^{-1} e| + \|b(\theta)^T X_{ii}(I - AX_{ii})^{-1}\|_1 = 1,$$

$i = 1, 2, \ldots, m$, for $\theta \in \{c_1, c_2, \ldots, c_\nu, 1\}$. Hence,

$$\begin{aligned}
&|\eta_i^{k+1}(t_n+\theta h) - \xi_i^{k+1}(t_n+\theta h)| \\
&\quad \le \max\left\{ |\eta_i^{k+1}(t_n) - \xi_i^{k+1}(t_n)|, \max_{1\le r\le\nu} \frac{\sum_{j=1, j\neq i}^{m} |\lambda_{ijr}| |\eta_j^k(t_n + c_r h) - \xi_j^k(t_n + c_r h)|}{-\lambda_{iir}} \right\},
\end{aligned}$$

$i = 1, 2, \ldots, m, \theta \in \{c_1, c_2, \ldots, c_\nu, 1\}$. Taking into account that

$$\sum_{\substack{j=1\\ j\neq i}}^{m} |\lambda_{ijr}| \le -\lambda_{iir}, \qquad r = 1, 2, \ldots, \nu,$$

(recall that $f \in \mathcal{F}^*$), the induction hypothesis (4.13) and the condition (4.14) we obtain

$$|\eta_i^{k+1}(t_n+\theta h) - \xi_i^{k+1}(t_n+\theta h)| \le \|y_0 - z_0\|_\infty,$$

$i = 1, 2, \ldots, m, \theta \in \{c_1, c_2, \ldots, c_\nu, 1\}$. This proves (4.13) with k replaced by $k + 1$. It is also clear that if the CRK method is semi $AN_f(0)$-stable, then (4.13) is satisfied for any mesh $\{t_0, t_1, \ldots, t_N\}$. This completes the proof. □

If $\partial f_i/\partial y_i$ are negative and constant, we can obtain the desirable property of contractivity in the maximum norm requiring, instead of $AN_f(0)$-stability of CRK method, the weaker property of $A_f(0)$-stability. To be more precise, consider the following hypotheses.

(H_1^*) The function f in (4.1) is sufficiently smooth, $f \in \mathcal{F}^*(\rho)$ for some $\rho > 0$, and there exist negative constants λ_i such that

$$\frac{\partial f_i(t,y)}{\partial y_i} = \lambda_i, \qquad i = 1, 2, \ldots, m, \tag{4.15}$$

for all $t \geq t_0$ and $y \in R^m$.

(H_2^*) The mesh $\{t_0, t_1, \ldots, t_N\}$ is such that

$$\rho \max_{0 \leq n \leq N-1} h_{n+1} < R^*,$$

where $h_{n+1} = t_{n+1} - t_n$ and $[-R^*, 0]$ is the region of semi $A_f(0)$-stability of the CRK method (2.4).

We have the following analogue of Theorem 4.2.

THEOREM 4.3. *Assume* $\mathrm{H}_1^*, \mathrm{H}_2^*$, *and that*

$$\|\eta^0(t) - \xi^0(t)\|_\infty \leq \|y_0 - z_0\|_\infty$$

for $t \in [t_0, t_N]$. *Then the numerical approximations* η^k *and* ξ^k *generated by* WRGJRK, WRGSRK, *and* WRSORRK, $0 < \omega \leq 1$, *iterations satisfy* (4.13). *In particular, if* $R^* = +\infty$ (*i.e., the underlying* CRK *method* (2.4) *is semi* $A_f(0)$*-stable*), *then* (4.13) *holds for any problems* (4.1) *and* (4.2) *with* $f \in \mathcal{F}^*$ *satisfying* (4.15) *and for any mesh* $\{t_0, t_1, \ldots, t_N\}$.

Proof. Proceeding similarly as in the proof of Theorem 4.2, we obtain

$$\begin{aligned}
&|\eta_i^{k+1}(t_n + \theta h) - \xi_i^{k+1}(t_n + \theta h)| \\
&\leq (|1 + xb(\theta)^T(I - xA)^{-1}e| + \|xb(\theta)^T(I - xA)^{-1}\|_1) \\
&\times \max\left\{ |\eta_i^{k+1}(t_n) - \xi_i^{k+1}(t_n)|, \max_{1 \leq r \leq \nu} \frac{\sum_{j=1, j\neq i}^{m} |\lambda_{ijr}| |\eta_j^k(t_n + c_r h) - \xi_j^k(t_n + c_r h)|}{-\lambda_i} \right\},
\end{aligned}$$

$i = 1, 2, \ldots, m$, where $x = h\lambda_i$. Since $h\lambda_i \geq -\rho h > -R^*$ it follows that x belongs to the region of semi $A_f(0)$-stability of the method (2.4), and by the usual arguments the conclusion follows. □

Remark. The above results (Theorems 4.2 and 4.3) are independent of the length of the window $[t_0, t_N := T]$ and on the number of steps N. In fact, they are valid for sequences of mesh points $\{t_0, t_1, \ldots, t_n, \ldots\}$ on the unbounded window $[t_0, +\infty)$.

We conclude this section with the following result about the region of (conditional) contractivity of the DSRK method (2.8).

THEOREM 4.4. *Let the hypotheses of Theorem* 4.2 *be satisfied and let* $N = 1$. *Moreover, assume that the* DSRK *method is well defined and that the* WRGJRK, WRGSRK, *and* WRSORRK, $0 < \omega \leq 1$, *iterations applied to* (4.1) *and* (4.2) *converge to the numerical solution defined by* DSRK *method* (2.8). *Then the region of* (*conditional*) *contractivity* $[-R, 0]$ *of the* DSRK *method* (2.8) *includes the region* $[-R^*, 0]$ *of semi* $AN_f(0)$*-stability of the CRK method* (2.4). *In particular, if the* CRK *method is semi* $AN_f(0)$*-stable, then the* DSRK *method is unconditionally contractive.*

Proof. Computing the limit as $k \to \infty$ in (4.13) for $\theta = 1$ and $n = 0$ we obtain

$$\|\overline{\eta}(t_0 + h) - \overline{\xi}(t_0 + h)\|_\infty \leq \|y_0 - z_0\|_\infty,$$

which is the desired conclusion. □

Theorems 4.2–4.4 suggest ways to construct numerical schemes with good contractivity properties in the maximum norm. Indeed, at least two possibilities have been opened. One consists of implementing the numerical WR iterations along a certain window $[t_0, t_N]$, and the other is to advance in a step-by-step fashion with the new DSRK method. In the latter case, the resulting systems of nonlinear equations could be solved by some modification of the Newton method. Observe that the DSRK method is always implicit even if the underlying CRK method is explicit. Therefore, in this case the use of WR iterations seems to be more appropriate than the use of the corresponding DSRK method.

Time-point relaxation methods, in which a fixed number of iterations is performed, are discussed in [4] and [5].

5. Existence of solutions of the algebraic equations in DSRK schemes. It follows from Theorem 2.1 in §2 that the algebraic equations for the WRGJRK, WRGSRK, and WRSORRK iterations and for the limit DSRK method are well defined if the function f appearing in (1.1) is smooth enough and the stepsize h is sufficiently small. We have mentioned in §4 that in case of numerical WR iterations Zennaro [23] proved a more useful result that these equations have unique solutions if the assumptions of Theorem 4.2 are satisfied, i.e., if the function f satisfies H_1 and the grid $\{t_0, t_1, \ldots, t_N\}$ is such that H_2 holds. In this section we establish a similar result for DSRK method (2.8) imposing additional restrictions on the function f. We will also investigate the convergence of the numerical WR iterations to the numerical solution defined by DSRK methods.

Concerning the function f appearing in (4.1) and (4.2) assume that $f \in \mathcal{F}^*$ and that there exists a constant $q \in (0, 1)$ such that

$$\sum_{\substack{j=1 \\ j \neq i}}^{m} \left| \frac{\partial f_i(t, y)}{\partial y_j} \right| \leq -q \frac{\partial f_i(t, y)}{\partial y_i}, \tag{5.1}$$

$i = 1, 2, \ldots, m$, for $t \geq t_0$ and $y \in R^m$. Clearly, (5.1) is a stronger condition than (4.6). If $f(t, y) = \Lambda y$ or $f(t, y) = \Lambda(t) y$, the condition (5.1) means that the matrices Λ or $\Lambda(t)$ are strongly diagonally dominant.

We have the following theorem.

THEOREM 5.1. *Assume* $\mathrm{H}_1, \mathrm{H}_2$, *and that function f satisfies the condition* (5.1). *Then the limit* DSRK *method is well defined, i.e., the* $2m\nu \times 2m\nu$ *system* (2.8) *has a unique solution* $\{\overline{Y}_{r,i}\}_{r=1,}^{\nu}{}_{i=1}^{m}$ *and* $\{\overline{\eta}(t_n + c_r h)\}_{r=1,}^{\nu}{}_{i=1}^{m}$ *for all* $n = 0, 1, \ldots, N-1$. *Moreover, the* WRGJRK, WRGSRK, *and* WRSORRK, $0 < \omega \leq 1$, *iterations converge to the limit* DSRK *method at* $t_n + \theta h, \theta \in \{c_1, c_2, \ldots, c_\nu, 1\}$, *for* $n = 0, 1, \ldots, N-1$. *More precisely,*

$$\begin{aligned} \max_{0 \leq n \leq N-1} \max_{\theta \in \{c_1, c_2, \ldots, c_\nu, 1\}} & \|\eta^k(t_n + \theta h) - \overline{\eta}(t_n + \theta h)\|_\infty \\ & \leq \delta^k \max_{t_0 \leq t \leq t_N} \|\eta^0(t) - \overline{\eta}(t)\|_\infty, \end{aligned} \tag{5.2}$$

where $\delta = q$ *for* WRGJRK *and* WRGSRK *iterations and* $\delta = 1 - \omega(1 - q)$ *for* WRSORRK, $0 < \omega \leq 1$, *iterations.*

Proof. As in Theorem 4.2 we will prove the result only for WRGJRK iterations. The other two cases can be treated in a similar way by an additional induction step on the component index i.

Consider the mapping F, which assigns to the piecewise continuous function η^k of degree d (the degree of NCE in (2.4c)) the piecewise continuous function η^{k+1} of the same degree d, defined by WRGJRK iteration (2.5). The fact that this mapping is well defined was proved by Zennaro [23]. It is clear that we can view F as the mapping

$$F : R^{mN(\nu+1)} \to R^{mN(\nu+1)},$$

which assigns to the column vector

$$u^k = [(\eta^k(t_n + c_1 h), \dots, \eta^k(t_n + c_\nu h), \eta^k(t_{n+1}))_{n=0,1,\dots,N-1}]^T$$

the column vector

$$u^{k+1} = [(\eta^{k+1}(t_n + c_1 h), \dots, \eta^{k+1}(t_n + c_\nu h), \eta^{k+1}(t_{n+1}))_{n=0,1,\dots,N-1}]^T,$$

$\eta^k(t_0) = \eta^{k+1}(t_0) = y_0$. We will show that F is a contraction. Let $\xi^{k+1} = F(\xi^k)$, $\xi^k(t_0) = \xi^{k+1}(t_0) = y_0$, be given and denote by v^{k+1} and v^k the corresponding column vectors defined by the values of ξ^{k+1} and ξ^k at $t_n + \theta h, \theta \in \{c_1, c_2, \dots, c_\nu, 1\}, n = 0, 1, \dots, N-1$. We will use induction on the step index n and assume that

$$\begin{aligned} &\|\eta^{k+1}(t_n) - \xi^{k+1}(t_n)\| \\ &\qquad \le q \max_{0\le n\le N-1} \max_{\theta\in\{c_1,c_2,\dots,c_\nu,1\}} \|\eta^k(t_n+\theta h) - \xi^k(t_n+\theta h)\|_\infty. \end{aligned} \tag{5.3}$$

This assumption is clearly satisfied for $n = 0$. Proceeding as in the proof of Theorem 4.2, we obtain

$$\begin{aligned} &|\eta_i^{k+1}(t_n+\theta h) - \xi_i^{k+1}(t_n+\theta h)| \\ &\quad \le \max\left\{ |\eta_i^k(t_n) - \xi_i^k(t_n)|, \max_{1\le r\le\nu} \frac{\sum_{j=1, j\ne i}^{m} |\lambda_{ijr}||\eta_j^k(t_n+c_r h) - \xi_j^k(t_n+c_r h)|}{-\lambda_{iir}} \right\}, \end{aligned}$$

$i = 1, 2, \dots, m, \theta \in \{c_1, c_2, \dots, c_\nu, 1\}, n = 0, 1, \dots, N-1$, where λ_{ijr} are defined as before. Taking into account that

$$\sum_{\substack{j=1\\ j=1}}^{m} |\lambda_{ijr}| \le -q\lambda_{iir}$$

(recall that f satisfies (5.1)) and the induction hypothesis (5.3) it follows that

$$\begin{aligned} &\max_{0\le n\le N-1} \max_{\theta\in\{c_1,c_2,\dots,c_\nu,1\}} \|\eta^{k+1}(t_n+\theta h) - \xi^{k+1}(t_n+\theta h)\|_\infty \\ &\qquad \le q \max_{0\le n\le N-1} \max_{\theta\in\{c_1,c_2,\dots,c_\nu,1\}} \|\eta^k(t_n+\theta h) - \xi^k(t_n+\theta h)\|_\infty. \end{aligned} \tag{5.4}$$

This means that

$$\|F(u^k) - F(v^k)\|_\infty \le q\|u^k - v^k\|_\infty,$$

and in view of $q \in (0,1)$ the mapping F is a contraction. Hence, since $R^{mN(\nu+1)}$ is a complete metric space we can conclude that F has a unique fixed point u. This means that the DSRK method (2.8) is well defined. Moreover, it follows from (5.4) that the rate of the iterative process (2.5) is given by

$$\max_{0\le n\le N-1}\max_{\theta\in\{c_1,c_2,\ldots,c_\nu,1\}}\|\eta^{k+1}(t_n+\theta h)-\overline{\eta}(t_n+\theta h)\|_\infty$$
$$\le q\max_{0\le n\le N-1}\max_{\theta\in\{c_1,c_2,\ldots,c_\nu,1\}}\|\eta^{k}(t_n+\theta h)-\overline{\eta}(t_n+\theta h)\|_\infty,$$

which implies (5.2). This completes the proof. □

Remark. It is clear that the conclusion of Theorem 5.1 holds if we assume H_1^* and H_2^* instead of H_1 and H_2; compare with the proof of Theorem 4.3. Moreover, we would like to stress again that the above result is independent of the length of the windows $[t_0, t_N]$ and on the number of steps N.

It follows from the results of [21] that the degree d of the NCE (2.4c) satisfies the inequality $d \le \nu^*$, where ν^* is the cardinality of the set $\{c_1, c_2, \ldots, c_\nu\}$. Therefore, unless $d = \nu^*$ and $0, 1 \in \{c_1, c_2, \ldots, c_\nu\}$, the polynomial $\overline{\eta}(t_n + \theta h)$ is determined by interpolation at the points $t_n + \theta h$, $\theta \in \{0, c_1, \ldots, c_\nu, 1\}$. In this case we have

$$\max_{t_n\le t\le t_{n+1}}\|\overline{\eta}(t)\| \le C\max_{\theta\in\{0,c_1,\ldots,c_\nu,1\}}\|\overline{\eta}(t_n+\theta h)\|_\infty,$$

where C is the norm of the relevant interpolation projector. As a consequence, by the linearity of the projector, the condition (5.2) implies

$$\max_{t_0\le t\le t_N}\|\eta^k(t)-\overline{\eta}(t)\|_\infty \le C\delta^k\max_{t_0\le t\le t_N}\|\eta^0(t)-\overline{\eta}(t)\|_\infty. \tag{5.5}$$

Moreover, even if $d = \nu^*$ and $0, 1 \in \{c_1, c_2, \ldots, c_\nu\}$, the estimate (5.5) may still hold. To this purpose, it is sufficient to assume, for example, that the interpolant is $AN_f(0)$- or $A_f(0)$-stable for any $\theta \in [0, 1]$ and not only for $\theta \in \{c_1, c_2, \ldots, c_\nu, 1\}$. In this case the inequality (5.5) is satisfied with $C = 1$.

As for iterative solutions of linear systems, it is possible to get results about the unique solvability of (2.8) and the convergence of numerical WR iterations under weaker conditions than those given in Theorem 5.1. Furthermore, it is possible to obtain the results about the different rates of convergence for the three types of WR iterations. In this paper we confine ourselves to the simple result given in Theorem 5.1 and the generalizations mentioned above will be reported elsewhere.

6. Examples of contractive methods. In this section we give some examples of unconditionally contractive numerical WR iterations and DSRK methods of order 1, 2, and 3 generated by CRK methods (2.4) of the same order. Moreover, we also give some results about explicit methods up to order 4. All the examples listed here are taken from [2] and [23].

We begin with unconditionally contractive methods and recall that we have assumed $c_r \in [0, 1], r = 1, \ldots, \nu$ for the CRK method (2.4).

6.1. One-stage methods of order 1. The only $A_f(0)$-stable and $AN_f(0)$-stable RK method in this class is backward Euler. Since $c_1 = 1$, no interpolation is necessary for it. However, this method is a collocation method and, therefore, as we already observed in §2, the corresponding DSRK method (2.8) coincides with the backward Euler method itself. Thus, we have proved again the unconditional contractivity of the backward Euler method (see [9]). Furthermore, the corresponding numerical WR iterations are unconditionally contractive.

6.2. Two-stage methods of order 2. The $A_f(0)$-stable RK methods in this class are also $AN_f(0)$-stable and are characterized by the following Butcher tableau:

$$\begin{array}{c|cc} c_1 & a_{11} & c_1 - a_{11} \\ 1 & b_1 & b_2 \\ \hline & b_1 & b_2 \end{array},$$

where $0 \leq c_1 \leq \frac{1}{2}, b_1 = \frac{1}{2(1-c_1)}, b_2 = 1 - b_1$, and $a_{11} \geq b_1$ (see [2]). The linear interpolant, given by

$$b_1(\theta) = b_1\theta, \quad b_2(\theta) = b_2\theta, \tag{6.1}$$

is an NCE of these methods and is also semi $AN_f(0)$-stable (more precisely, it is even $AN_f(0)$-stable). Therefore, in view of Theorems 4.2 and 4.4, by choosing the same parameters in (2.8), we have a class of unconditionally contractive 2-stage DSRK methods of order 2. Moreover, the corresponding numerical WR iterations are unconditionally contractive.

6.3. Three-stage methods of order 3. Within this class, we give the following example of the $AN_f(0)$-stable RK method:

$$\begin{array}{c|ccc} 0 & \frac{5}{2} & -2 & -\frac{1}{2} \\ \frac{1}{2} & -1 & 2 & -\frac{1}{2} \\ 1 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ \hline & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array};$$

see [23]. An $AN_f(0)$-stable NCE of degree 2 of this method is given by

$$b_1(\theta) = -\frac{5}{6}\theta^2 + \theta, \quad b_2(\theta) = \frac{2}{3}\theta^2, \quad b_3(\theta) = \frac{1}{6}\theta^2. \tag{6.2}$$

Thus, the corresponding DSRK method (2.8) as well as the numerical WR iterations (2.5), (2.6), and (2.7) are unconditionally contractive.

Next we consider the explicit RK methods and their regions of (conditional) contractivity.

6.4. One-stage methods of order 1. The only explicit RK method in this class is the forward Euler method. As for the implicit case, since $c_1 = 0$, no interpolation is necessary for it. Moreover, since it is again a collocation method, the corresponding DSRK method (2.8) coincides with the forward Euler method itself. Now, from [9] we know that its region of contractivity is $[-1, 0]$ and we can observe that, if we apply Theorem 4.4, we get the same result. In fact, its region of $AN_f(0)$-stability is exactly $[-1, 0]$.

6.5. Two-stage methods of order 2. The RK methods in this class having nonempty region of $A_f(0)$-stability are characterized by the following Butcher tableau:

$$\begin{array}{c|cc} 0 & 0 & 0 \\ c_2 & c_2 & 0 \\ \hline & b_1 & b_2 \end{array}$$

(see [23]), where $\frac{1}{2} < c_2 \leq 1, b_2 = 1/2c_2$ and $b_1 = 1 - b_2$. For such methods, with the linear interpolant (6.1), the region of semi $A_f(0)$-stability coincides with the region of semi $AN_f(0)$-stability and is the segment $[-2 + \frac{1}{c_2}, 0]$. Hence, by Theorems 4.3 and 4.4, the region of contractivity of the corresponding DSRK method (2.8) and of the numerical WR iterations contains the segment $[-2 + \frac{1}{c_2}, 0]$.

Observe that the best result is obtained for $c_2 = 1$, that is, for the Heun method. In this case, the region of contractivity contains the segment $[-1, 0]$, which is just the best we can get with classical two-stage RK methods of order 2 (see [9]).

6.6. Three-stage methods of order 3. Within this class, we consider the following RK method:

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \\ \hline & \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \end{array},$$

which, as has been proved in [9], has the maximum region of contractivity equal to $[-1, 0]$. This method with the NCE given by (6.2) is a CRK method whose region of semi $A_f(0)$-stability is $[-1, 0]$ and coincides with the region of semi $AN_f(0)$-stability. Therefore, by Theorems 4.2 or 4.3 and Theorem 4.4 we find again that the region of contractivity of the corresponding DSRK method (2.8) and of the numerical WR iterations contains the segment $[-1, 0]$.

We also consider the other well-known method:

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & -1 & 2 & 0 \\ \hline & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array},$$

which, since $a_{31} < 0$, in view of the results in [9], has an empty region of contractivity. However, this method, together with the NCE of degree 2,

$$\begin{aligned} b_1(\theta) &= -\frac{3}{4}\theta^2 + \frac{11}{12}\theta, \\ b_2(\theta) &= \frac{1}{2}\theta^2 + \frac{1}{6}\theta, \\ b_3(\theta) &= \frac{1}{4}\theta^2 - \frac{1}{12}\theta, \end{aligned} \tag{6.3}$$

is a CRK method whose region of semi $A_f(0)$-stability is the segment $[-1.59607\ldots, 0]$ and whose region of semi $AN_f(0)$-stability is the segment $[-0.5, 0]$. Therefore, in this case, the region of contractivity of the corresponding DSRK method (2.8), and of the numerical WR iterations, contains the segment $[-0.5, 0]$. Moreover, for problems satisfying the hypothesis H_1^* this region is enlarged to the segment $[-1.59607\ldots, 0]$.

6.7. Four-stage methods of order 4. In [9] it is proved that, within this class, all classical RK methods have an empty region of contractivity. Unfortunately, we are not able to remove this negative result with our WR techniques. For example, whereas the well-known method

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}$$

has the region of $A_f(0)$-stability equal to the segment $[-1.29559\ldots, 0]$ and the region of $AN_f(0)$-stability is equal to the segment $[-1, 0]$, all its NCEs have an empty region of $A_f(0)$-stability at the point $\theta = \frac{1}{2}$, and, hence, we cannot get any positive result.

This situation is common to all explicit four-stage RK methods of order 4. Although there is clearly a potential for the discrete methods, the NCEs prevent us from getting DSRK methods with nonempty regions of contractivity.

7. Concluding remarks. We have demonstrated that by using the waveform relaxation techniques it is possible to improve the contractivity properties of the Runge–Kutta methods. As indicated by the examples given in §6, this is true for both implicit and explicit methods, although this is more evident in the former case. The examples of explicit methods also seem to indicate that the limits of the proposed technique come out from the interpolation (2.4c) rather than from the discrete method (2.4a)–(2.4b) itself. This view is further supported by the result proved in [23] by using the results in [9]: the upper bound on the order of DSRK method (2.8) (where (2.8c) is an NCE) with nonempty region of contractivity is 4. Once again, the bound is caused by the use of the NCEs.

In view of the above remarks it seems to be of interest to extend the techniques of this paper to DSRK methods with other types of interpolation. Turning the problem around, we could also try to look for new interpolants that would lead to DSRK methods with optimal contractivity properties.

Apart from stability and contractivity considerations other types of interpolation are of interest in many situations. This is the case if, for example, various components of the system (1.1) change at different rates and should be integrated by using different stepsizes for each component to improve the efficiency of the algorithm. In such cases NCEs are not the best choice since, in general, the uniform order of NCE is lower than the nodal order of the underlying discrete Runge–Kutta method.

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