Spring 2002 **MA2160 Final Exam** **Part One**

Name: Answer Key **ID #:** ____________
(please print)

<u>Please circle your section number.</u>

Section	Time	Instructor	Page	Score
01	8:05	K. Bonn	2	____ /6
03	9:05	C. Sarami	3	____ /8
04	9:05	P. Balachowski	4	____ /9
05	10:05	A. Baartmans	5	____ /10
06	11:05	L. Erlebach		
07	11:05	V. Puente Vera		
08	12:05	O. Paez Osuna		
09	2:05	M. Gopal	Part One Total	____ /33
10	8:05	A. Andreev		
11	3:05	B. Francis	Part Two Total	____ /67
12	3:05	S. Sy		
13	4:05	S. Sy	**TOTAL**	____ /100

Instructions Regarding Calculator Use:
- Part One - Calculators are <u>NOT</u> allowed.
- Part Two - You may work on Part Two at any time, **but you must hand in Part One before taking out your calculator**. In other words, please keep your calculator off your desk (e.g., under your chair) until after you've handed in Part One.

Note: **Once you've handed in Part One, you may not return to it later on during the exam period.**

Part One — Calculators are NOT allowed

Justify all answers and show all work. **Circle your final answer.** No work, no credit!

1. Find a unit vector in the same direction as $\bar{v} = 2\bar{i} - \bar{j} + 3\bar{k}$.

$\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|}$; $\|\vec{v}\| = \sqrt{4+1+9} = \sqrt{14}$

$\vec{u} = \dfrac{1}{\sqrt{14}}(2\vec{i} - \vec{j} + 3\vec{k})$

____/2 pts.

2. Find a non-zero vector perpendicular to both $\bar{u} = \bar{i} - 3\bar{k}$ and $\bar{v} = 2\bar{i} - \bar{j} - \bar{k}$.

one choice: $\bar{u} \times \bar{v} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 0 & -3 \\ 2 & -1 & -1 \end{vmatrix} = \bar{i}\begin{vmatrix} 0 & -3 \\ -1 & -1 \end{vmatrix} - \bar{j}\begin{vmatrix} 1 & -3 \\ 2 & -1 \end{vmatrix} + \bar{k}\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix}$

$= -3\bar{i} - 5\bar{j} - \vec{k}$

or

another choice: $\bar{v} \times \bar{u} = 3\bar{i} + 5\bar{j} + \bar{k}$

____/4 pts.

3. Evaluate $\int (3x^3+1)^5 x^2\,dx$

Let $u = 3x^3 + 1$.

Then $du = 9x^2\,dx$ so $x^2\,dx = \frac{1}{9}\,du$.

$$I = \frac{1}{9}\int u^5\,du$$

$$= \frac{u^6}{54}$$

$$= \frac{(3x^3+1)^6}{54} + C.$$

____ /4 pts.

4. Evaluate $\int_0^{\frac{\pi}{2}} (\sin x)(\cos^2 x)\,dx$

Let $\boxed{u = \cos x}$. Then $du = -\sin x\,dx$.

$$I = -\int u^2\,du$$

$$= -\frac{u^3}{3} = -\frac{1}{3}\cdot(\cos^3 x)\Big|_{x=0}^{\pi/2}$$

$$= -\frac{1}{3}\left[0 - 1\right]$$

$$= 1/3$$

____ /4 pts.

5. Evaluate $\int_0^{\infty} e^{-x}\,dx$. Improper integral.

$$= \lim_{b\to\infty}\left(\int_0^b e^{-x}\,dx\right)$$

$$\int_0^b e^{-x}\,dx = -e^{-x}\Big|_0^b = 1-e^{-b}$$

$$= \lim_{b\to\infty}\left(1-e^{-b}\right)$$

$$= 1 \quad : \quad \text{(Convergent)}$$

____/4 pts.

6. Evaluate $\int x^2 e^x\,dx$ Integration by Parts

$$= \int u\,dv \quad \text{where} \quad u = x^2,\ dv = e^x dx,\ v = e^x$$

$$= uv - \int v\,du$$

$$= x^2e^x - \int 2xe^x\,dx. \quad \text{———— (1)}$$

Repeating the process again:

$$\int xe^x\,dx = \int u_1 dv_1, \quad u_1 = x, \quad dv_1 = e^x dx, \text{ so } v_1 = e^x$$

$$= u_1v_1 - \int v_1\,du_1$$

$$= xe^x - \int e^x dx$$

$$= xe^x - e^x \quad \text{———— (2)}$$

____/5 pts.

Substitute back in (1) to get

$$I = x^2e^x - 2(xe^x - e^x) + C$$

$$= x^2e^x - 2xe^x + 2e^x + C.$$

7. Find the Taylor polynomial of degree 3, approximating the function

$f(x) = \frac{1}{x}$ for x near 2.

$$P_3(x) = f(2) + \frac{f'(2)}{1!}(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3.$$

$$f(x) = 1/x \longrightarrow f(2) = 1/2$$
$$f'(x) = -1/x^2 \longrightarrow f'(2) = -1/4$$
$$f''(x) = 2/x^3 \longrightarrow f''(2) = 1/4$$
$$f'''(x) = -6/x^4 \longrightarrow f'''(2) = -6/16 = -3/8.$$

$$P_3(x) = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1/4}{2}(x-2)^2 - \frac{3/8}{6}(x-2)^3.$$
$$= \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^3 - \frac{1}{16}(x-2)^3.$$

____ /6 pts.

8. Find the general solution (in terms of real functions only) for the differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 13y = 0$$

In operator notation: $(D^2 + 4D + 13)y = 0$, $D = \frac{d}{dt}$.

Characteristic eqn: $r^2 + 4r + 13 = 0.$

Zeros: $r = -2 \pm 3i$

General solution: $y = c_1 e^{-2t}\cos 3t + c_2 e^{-2t}\sin 3t$.

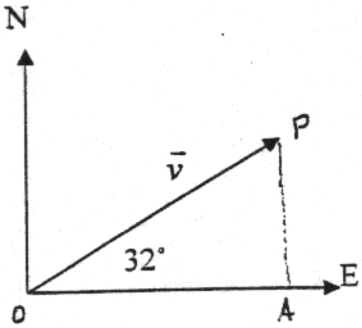
____ /4 pts.

Part Two — Calculators Allowed

Justify all answers and show all work. **Circle your final answer.** No work, no credit! (If you obtain an answer or part of an answer with your calculator, *please indicate what you punched into your calculator and what the output was*.)

You may work on this section at any time, **but you must hand in Part I before taking out your calculator**.

9. The velocity vector $\vec{v}$ of an airplane is shown below. Write $\vec{v}$ in component form if the speed of the airplane is 300 mph. Answer to 2 decimal places.



Given: $\|\vec{v}\| = 300.$

$\vec{v} = \vec{OP} = \vec{OA} + \vec{AP}$

$= 300 \cos 32^\circ \vec{i} + 300 \sin 32^\circ \vec{j}$

$\approx 254.41 \vec{i} + 158.98 \vec{j}$

____ /4 pts.

10. Find the equation of a plane through the point $(1, -1, 4)$ **parallel** to the plane that has equation $3x + 2y - z = 1$.

Eqn of any parallel plane : $3x + 2y - z = k$ (k: const).

$(1, -1, 4)$ point on this plane $\Rightarrow 3(1) + 2(-1) - 4 = k$

ie $k = -3$

ans: $3x + 2y - z = -3.$

____ /4 pts.

11. For each of the following **three** parts, use the information given about vectors $\bar{a}$, $\bar{b}$, $\bar{c}$, and $\bar{d}$; circle one of the given choices, and then give a supporting statement for your choice.

$$\bar{a} = \bar{i} - \bar{j} + 2\bar{k},\ \bar{b} = -3\bar{i} + 3\bar{j} + 3\bar{k},\ \bar{c} = -2\bar{i} + 2\bar{j} - 4\bar{k},\ \text{and } \bar{d} = \bar{i} + \bar{j} - \bar{k}$$

I. $\bar{a}$ and $\bar{b}$ are perpendicular, parallel, neither (circle one)

Because: $\bar{a} \cdot \bar{b} = -3 - 3 + 6 = 0.$

____ /2 pts.

II. $\bar{a}$ and $\bar{c}$ are perpendicular, parallel, neither (circle one)

Because: $\vec{c} = -2\vec{a}.$ (scalar multiple of $\bar{a}$).

____ /2 pts.

III. The angle formed by $\bar{a}$ and $\bar{d}$ is $< 90°$, $> 90°$, $= 90°$ (circle one)

Because: $\vec{a} \cdot \vec{d} = -2 < 0.$

____ /2 pts.

12. The table below gives values for a function f.

x	0	2	4	6	8	10
$f(x)$	2	3	7	12	25	39

a. Use the **Trapezoid Rule** with $n = 5$ to estimate $\int_0^{10} f(x)dx$

$\Delta x = 2.$

$$\text{Trap}(5) = \frac{1}{2}\left[\text{Left}(5) + \text{Right}(5)\right].$$

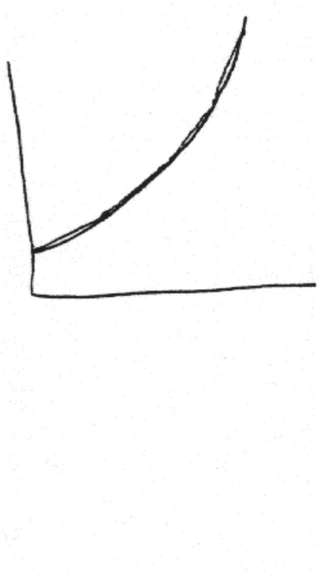
$$= \cancel{\tfrac{1}{2}}\left[(2+3+7+12+25)\cdot\cancel{2} + (3+7+12+25+39)\cdot\cancel{2}\right]$$

$$= 49 + 86$$

$$= 135$$

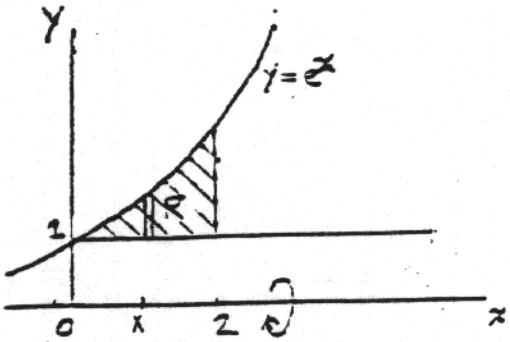
____ /4 pts.

b. If you plot a graph for $y = f(x)$, it is concave up. Is the estimate from part (a) an underestimate or an overestimate? **Circle your answer.**



____ /2 pts.

13. Set up (but DO NOT evaluate) the integral for the volume of the solid of revolution generated by rotating the plane region **R** bounded by $y = e^x$, $y = 1$ and $x = 2$ about the x-axis.



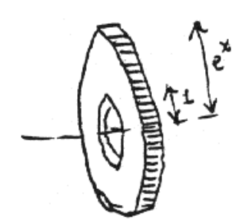
$$V = \lim \sum \Delta V$$

where ΔV = elem. volume

$= \begin{cases} \text{vol. of a 'washer', thickness } \Delta x, \\ \text{outer radius : } r_o = e^x \\ \text{inner radius } r_i = 1 \end{cases}$

$= \pi (r_o^2 - r_i^2)\, \Delta x$

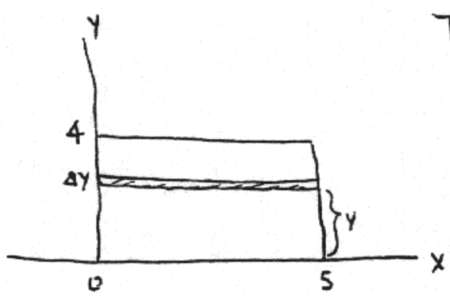
$= \pi (e^{2x} - 1)\, \Delta x$



$$\therefore \quad V = \int_{x=0}^{1} \pi (e^{2x} - 1)\, dx.$$

____ /6 pts.

14. A thin rectangular plate of dimensions 5 cm × 4 cm is bounded by the lines $x = 0$, $x = 5$, $y = 0$, and $y = 4$. The mass density at the point $P(x,y)$ is $\delta(y) = \dfrac{3y}{1+y^3}$ gm/cm^2.
Set up (but DO NOT evaluate) the integral for the total mass of this plate. **Include a sketch.**



Total mass $m = \lim \sum_i \Delta m$

where Δm = elem. mass

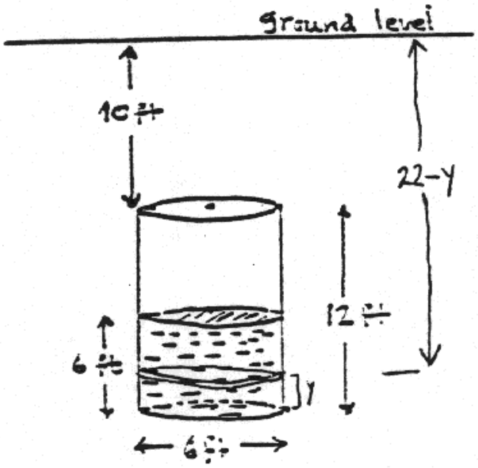
$= (\text{density})(\text{elem. area } \Delta A)$

$= \dfrac{3y}{1+y^3} \cdot 5\, \Delta y$

Thus: $m = 15 \displaystyle\int_0^4 \frac{y\, dy}{1+y^3}.$

____ /6 pts.

15. A fuel oil tank is an upright cylinder and is buried so that its circular top is 10 ft. below ground level. The tank has a radius of 3 ft., is 12 ft. high, and is half full of oil. Set up (but DO NOT evaluate) the integral for the work done in pumping all the oil to the surface. Oil weighs 50 lbs/ft³.



Horizontal slice at height y from the bottom.

Total work $W = \lim \sum \Delta W$

where $\Delta W =$ elementary work

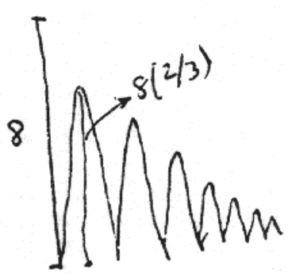
$= (\text{elem. weight})(\text{distance moved})$

$= (\pi \cdot 3^2 \Delta y \cdot 50)(22-y)$

$\therefore \; W = 450\pi \int_{y=0}^{6} (22-y)\, dy$ ft-lbs.

____/6 pts.

16. A ball is dropped from a height of 8 ft. and starts bouncing. Each bounce is 2/3 of the height of the bounce before. **Use a geometric series** to find the total vertical distance traveled by the ball.



Total vertical distance traveled

$= 8 + 2\left[8\left(\frac{2}{3}\right) + 8\left(\frac{2}{3}\right)^2 + \cdots\right]$

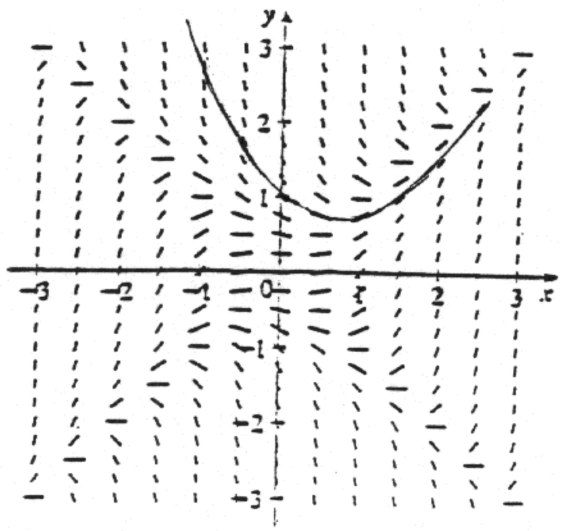
$= 8 + 2 \cdot \dfrac{16/3}{1 - 2/3}$

$= 8 + \dfrac{32/3}{1/3}$

$= 40$ ft

____/6 pts.

17a. A slope field for the differential equation $y' = x^2 - y^2$ is shown. Sketch the solution of the initial-value problem $y' = x^2 - y^2$, $y(0) = 1$. **Use your graph** to estimate the value of $y(1)$.



$y(1) \approx 0.7$ (eyeballing)

$y(1)$ is approximately 0.7

____ /4 pts.

17b. Referring to the initial value problem in part (a) above, **use Euler's method** with step size $\Delta x = 0.5$ to estimate $y(1)$, starting at the point (0, 1).

P	x	y	$m = y'$ ($x^2 - y^2$)	$\Delta y = m\,\Delta x$, $\Delta x = 0.5$
P_0	0	1	−1	$-0.5 = (-1)(0.5)$
P_1	0.5	0.5	0	$0 = (0)(0.5)$
P_2	1	0.5		

$y(1) \approx 0.5$ (Euler)

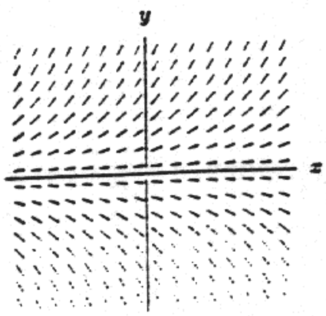
$y(1)$ is approximately 0.5

____ /5 pts.

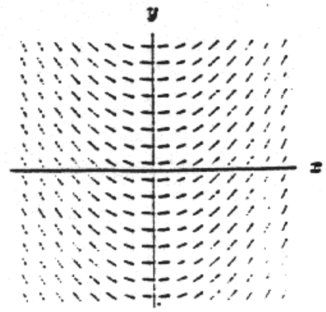
18. Select the slope field (I, II, or III) that matches the differential equation $\frac{dy}{dx} = \frac{1}{y}$

Explain why.

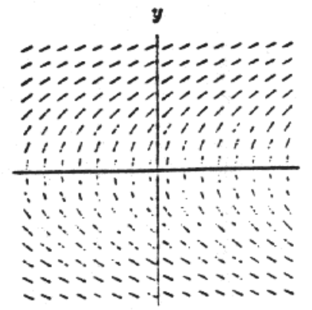
I.



II.



III.

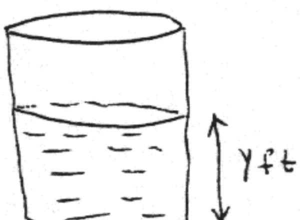


Slope Field: III Explanation: ① $y\,dy = dx \Rightarrow x = y^2 + C$ ∴ Family of parabolas, opening to the right. OR ② note that $y' = 1$ when $y = 1$, undefined at $y = 0$.

____ /4 pts.

19. Set up (but DO NOT evaluate) the differential equation for the following problem. Define all variables.

Water leaks out of an upright cylindrical tank at a rate proportional to the square root of the depth of the water at that time.



Let depth be y ft (at any time t). (t in minutes, for example)

Then $\frac{dy}{dt} < 0$ since y decr. as t incr.

Thus: $\frac{dy}{dt} = -k\sqrt{y}, \quad k > 0.$

differential equation: $\frac{dy}{dt} = -k\sqrt{y}, \quad k > 0.$

define all variables: y ft = depth at time t.

____ /4 pts.

20. A yam is put in a 200°C oven at 2:00 p.m. and heats up according to the differential equation

$$\frac{dH}{dt} = -0.0288(H-200) \qquad [t \text{ in minutes}]$$

If the temperature of the yam was 10°C when it was put in the oven, find the temperature of the yam at 2:45 p.m.

Separating variables :

$$\frac{dH}{H-200} = -0.0288\, dt.$$

Integrating: $\displaystyle\int \frac{dH}{H-200} = -0.0288t + C$

ie $\ln|H(t) - 200| = -0.0288t + C.$

ie $\underline{H(t) - 200 = Ae^{-.0288t}}$

$H(0) = 10°C \Rightarrow 10 - 200 = A.$ ie $\underline{A = -190}$

Thus: $\underline{H(t) = 200 - 190e^{-.0288t}}$

At 2:45 PM: $t = 45$:

$H(45) = 200 - 190e^{-.0288.(45)}$

$\approx 148.01°C.$

____/6 pts.