Spring Semester 2007

MA2160 Final Exam - PART 2

Calculators are allowed on Part 2.

Name:(print) Solutions ID#: ____________

Score - Part 2: ___ / 50

Circle your section number:

Section	Time	Instructor	Section	Time	Instructor
01	08:05	S. Tao	07	11:05	A. Niu
02	12:05	H. Wang	08	12:05	S. Butler
03	09:05	R. Targove	09	14:05	K. Feigl
04	08:05	D. Yorgov	10	15:05	L. Erlebach
05	10:05	A. Roy	11	16:05	L. Erlebach
06	11:05	H. Wang	12	14:05	S. Butler

- Justify all answers and show all work! No work, no credit!
- Frame your answers.

9. (4 pts) Use all the data in the table below to estimate the value of $\int_1^7 f(x)\,dx$ using the Trapezoidal Rule.

x	1	3	5	7
$f(x)$	6	9	11	12

$\Delta x = 2, \quad n = 3$

$LEFT(3) = \Delta x[f(1) + f(3) + f(5)] = 2(6+9+11) = 52$

$RIGHT(3) = \Delta x[f(3) + f(5) + f(7)] = 2(9+11+12) = 64$

$$TRAP(3) = \frac{LEFT(3) + RIGHT(3)}{2} = \frac{52+64}{2} = \boxed{58}$$

10. (5 pts) Given the differential equation $\frac{dy}{dx} = 2x^2 + y$ with the initial condition $y(0) = 1$. Use Euler's method with a step size $\Delta x = 0.2$ to approximate the value of $y(0.4)$. Write your answer in decimal form, carrying at least 3 decimal places.

$$y_{k+1} = y_k + \Delta x\, f(x_k, y_k) = y_k + \Delta x\,(2x_k^2 + y_k)$$
$$= y_k + 0.2\,(2x_k^2 + y_k)$$

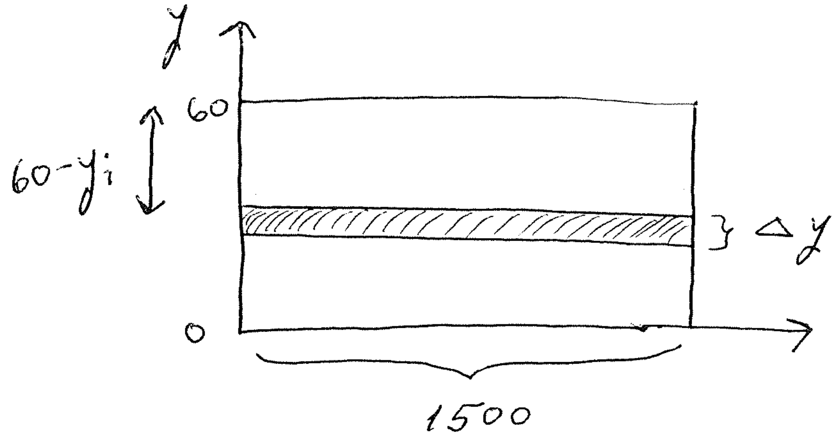
k	0	1	2
x_k	0	0.2	0.4
y_k	1	1.2	1.456
y'_k	1	1.28	

$y_1 = y_0 + 0.2\,(2x_0^2 + y_0)$
$= 1 + 0.2\,(0+1) = 1.2$

$y_2 = y_1 + 0.2\,(2x_1^2 + y_1)$
$= 1.2 + 0.2\,(2(0.2)^2 + 1.2)$
$= 1.2 + 0.2\,(1.28)$
$= 1.456$

$$\boxed{y(0.4) \approx 1.456}$$

11. (6 pts) A dam is a rectangular wall that is 1500 feet long and 60 feet high. Find a definite integral that represents the total force of the water on the dam. Assume water comes up to the top of the dam. Water weighs 62.4 lb/ft^3. (**Do not evaluate the integral.**)



<u>On slice i</u>: $F_i \approx P_i A_i$

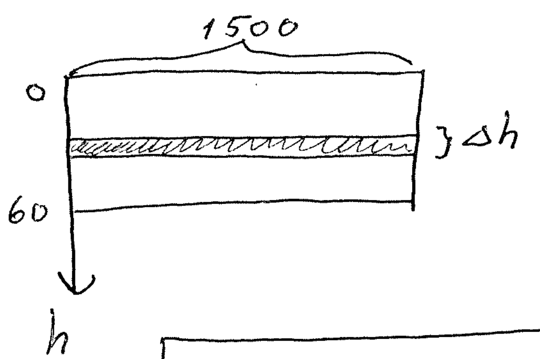
where $P_i \approx 62.4(60-y_i)$

$A_i = 1500\,\Delta y$

$\Rightarrow F_i \approx 62.4\,(1500)\,(60-y_i)\,\Delta y$

$$\Rightarrow \boxed{F = \int_0^{60} 62.4(1500)\,(60-y)\,dy = 93{,}600\int_0^{60}(60-y)\,dy}$$

OR

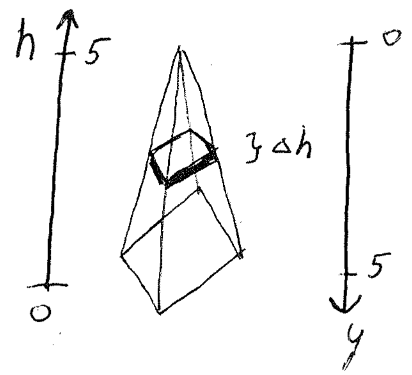


<u>On slice i</u>:

$F_i \approx P_i A_i \approx 62.4(15000)\,h_i\,\Delta h$

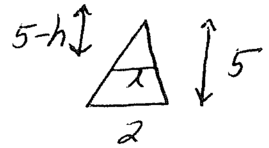
$$\Rightarrow \boxed{F = \int_0^{60} 62.4(1500)\,h\,dh = 93{,}600\int_0^{60} h\,dh}$$

12. (5 pts) Find a definite integral that represents the volume of a pyramid whose square base is 2 meters $\times$ 2 meters and whose height is 5 meters. (**Do not evaluate the integral.**)



Each slice perpendicular to h axis is a square of length $l(h)$

On slice i: $V_i \approx l_i^2 \Delta h$



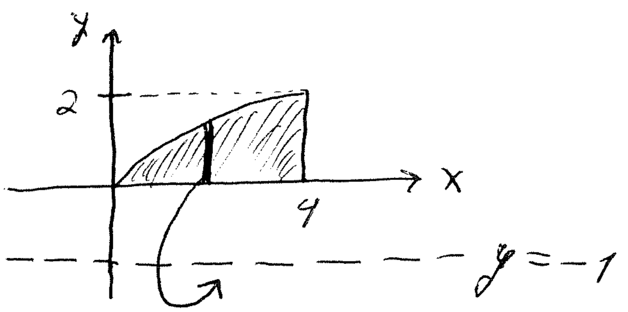
$\frac{2}{5} = \frac{l}{5-h} \Rightarrow l = \frac{2}{5}(5-h)$

$\Rightarrow V_i \approx \left[\frac{2}{5}(5-h_i)\right]^2 \Delta h$

$$\Rightarrow \boxed{V = \int_0^5 \frac{4}{25}(5-h)^2\,dh = \int_0^5 \frac{4}{5} y^2\,dy}$$

where $y = 5-h$

13. (5 pts) Find a definite integral that represents the volume of the solid obtained by rotating the region bounded by $y = \sqrt{x}$, the $x-$axis and the vertical line $x = 4$ about the line $y = -1$. (**Do not evaluate the integral.**)



$V = V_{out} - V_{in}$

$A_{out} = \pi r_{out}^2 = \pi(\sqrt{x}+1)^2$

$A_{in} = \pi r_{in}^2 = \pi(1)^2 = \pi$

$$\boxed{\begin{aligned} V &= \int_0^4 \pi(\sqrt{x}+1)^2\,dx - \int_0^4 \pi\,dx = \pi\int_0^4 \left[(\sqrt{x}+1)^2 - 1\right]dx \\ &= \pi \int_0^4 (x + 2\sqrt{x})\,dx \end{aligned}}$$

14. Let $f(x) = \cos x$.

(a) (4 pts) **Derive** the Taylor series of $f(x)$ near $x = \pi/2$. Write only the first four nonzero terms. Leave no uncomputed derivatives in your answer.

$f(x) = \cos x \qquad f(\pi/2) = 0$

$f'(x) = -\sin x \qquad f'(\pi/2) = -1$

$f''(x) = -\cos x \qquad f''(\pi/2) = 0$

$f'''(x) = \sin x \qquad f'''(\pi/2) = 1$

$$f(x) = f\left(\tfrac{\pi}{2}\right) + f'(\pi/2)(x-\pi/2) + \frac{f''(\pi/2)}{2!}(x-\pi/2)^2 + \cdots$$

$$\boxed{f(x) = -\left(x-\tfrac{\pi}{2}\right) + \tfrac{1}{3!}\left(x-\tfrac{\pi}{2}\right)^3 - \tfrac{1}{5!}\left(x-\tfrac{\pi}{2}\right)^5 + \tfrac{1}{7!}\left(x-\tfrac{\pi}{2}\right)^7 + \cdots}$$

(b) (2 pts) Write the general term of the above Taylor series and the starting value of the index.

$$\boxed{\frac{(-1)^k\left(x-\frac{\pi}{2}\right)^{2k-1}}{(2k-1)!}, \quad k \geq 1 \qquad \underline{\text{OK}} \qquad \frac{(-1)^{n+1}\left(x-\frac{\pi}{2}\right)^{2n+1}}{(2n+1)!}, \quad n \geq 0}$$

(c) (2 pts) Approximate $\cos(1.5)$ using the first degree Taylor polynomial of $f(x) = \cos x$ near $x = \pi/2$.

$$P_1(x) = -\left(x - \tfrac{\pi}{2}\right)$$

$$\cos(1.5) \approx P_1(1.5) = -\left(1.5 - \tfrac{\pi}{2}\right) \approx \boxed{0.070796}$$

(d) (2 pts) Use part (a) to find the Taylor series of $f(x) = 3 + x^2\cos(x)$ near $x = \pi/2$. Write only the first four nonzero terms.

$$f(x) = 3 + x^2\left[-\left(x-\tfrac{\pi}{2}\right) + \tfrac{1}{3!}\left(x-\tfrac{\pi}{2}\right)^3 - \tfrac{1}{5!}\left(x-\tfrac{\pi}{2}\right)^5 + \cdots\right]$$

$$\boxed{f(x) = 3 - x^2\left(x-\tfrac{\pi}{2}\right) + \frac{x^2}{3!}\left(x-\tfrac{\pi}{2}\right)^3 - \frac{x^2}{5!}\left(x-\tfrac{\pi}{2}\right)^5 + \cdots}$$

15. Hydrocodone bitartrate is used as a cough suppressant. After the drug is fully absorbed, the quantity of drug in the body decreases at a rate proportional to the amount left in the body. The half-life of hydrocodone bitartrate in the body is 3.8 hours, and the oral dose is 10 mg.

(a) (2 pts) Write a differential equation for the quantity, Q, of hydrocodone bitartrate in the body t hours after the drug was fully absorbed.

$$\boxed{\frac{dQ}{dt} = kQ, \quad k < 0}$$

or

$$\boxed{\frac{dQ}{dt} = -kQ, \quad k > 0}$$

(b) (3 pts) Solve the differential equation given in part (a), making use of the initial condition $Q(0) = 10$.

$$\frac{dQ}{Q} = -k\,dt, \quad Q \neq 0$$

$$\int \frac{dQ}{Q} = -\int k\,dt$$

$$\ln|Q| = -kt + C$$

$$|Q| = e^{C} e^{-kt}$$

$$|Q| = Be^{-kt}, \quad B > 0$$

$$Q = Ae^{-kt}, \quad A = \pm B \neq 0$$

Note: $Q(t) = 0$ is also a solution

IC: $10 = Q(0) = Ae^{0} = A$

$$\Rightarrow \boxed{Q(t) = 10e^{-kt}, \quad k > 0}$$

(c) (3 pts) Use the half-life to find the constant of proportionality in the differential equation.

$$\frac{1}{2}Q_0 = Q_0 e^{-k(3.8)}$$

$$\frac{1}{2} = e^{-3.8k}$$

$$\ln\left(\frac{1}{2}\right) = -3.8k$$

$$\boxed{k = -\frac{\ln(1/2)}{3.8} = \frac{\ln 2}{3.8} \approx 0.1824}$$

16. Consider the points $P = (1,4,1)$, $Q = (2,0,-1)$ and $R = (0,-1,3)$.

(a) (2 pts) Find the displacement vectors $\vec{PQ}$ and $\vec{PR}$.

$$\overrightarrow{PQ} = (2-1)\vec{i} + (0-4)\vec{j} + (-1-1)\vec{k}$$

$$\boxed{\overrightarrow{PQ} = \vec{i} - 4\vec{j} - 2\vec{k}}$$

$$\boxed{\overrightarrow{PR} = -\vec{i} - 5\vec{j} + 2\vec{k}}$$

(b) (5 pts) Find an equation for the plane that contains P, Q and R. Write the equation in the form $ax + by + cz = d$.

$$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -4 & -2 \\ -1 & -5 & 2 \end{vmatrix}$$

$$= (-8-10)\vec{i} - (2-2)\vec{j} + (-5-4)\vec{k}$$

$$\vec{n} = -18\vec{i} - 9\vec{k}$$

$$\overrightarrow{PP_0} = (x-1)\vec{i} + (y-4)\vec{j} + (z-1)\vec{k}$$

$$0 = \vec{n} \cdot \overrightarrow{PP_0} = -18(x-1) + 0(y-4) - 9(z-1)$$

$$= -18x + 18 - 9z + 9$$

$$= -18x - 9z + 27$$

$$\Rightarrow \boxed{18x + 9z = 27} \quad \underline{\underline{\text{or}}} \quad \boxed{2x + z = 3}$$