1. Rewrite the equation $y'' + 2y' + 3y = 2t + 1$ as a system of first order equations.
2. Is the system of the previous problem autonomous? Is it linear? Explain.
3. Find the critical points of the system $\frac{dx}{dt} = x + y - 1$, $\frac{dy}{dt} = x - 2y$.
4. Find all constant solutions of $\frac{dx}{dt} = 2x + y - 1$, $\frac{dy}{dt} = x - 2y - 3$.
5. Discuss how you could tell if there are periodic solutions of $\frac{dx}{dt} = ax + by$, $\frac{dy}{dt} = cx + dy$.
6. Solve the system $\frac{dx}{dt} = 3x + 2y$, $\frac{dy}{dt} = -x + y$.
7. Is the system $\frac{dx}{dt} = 3x + 2y$, $\frac{dy}{dt} = -x + y$ stable or unstable? If stable, is it asymptotically stable? Explain.
8. Describe the geometric configuration of the solutions of $\frac{dx}{dt} = 3x + 2y$, $\frac{dy}{dt} = -x + y$ in the phase plane.
9. Solve the system $\frac{dx}{dt} = -3x + 2y$, $\frac{dy}{dt} = 2x - 3y$.
10. Is the system $\frac{dx}{dt} = -3x + 2y$, $\frac{dy}{dt} = 2x - 3y$ stable or unstable? If stable, is it asymptotically stable? Explain.
11. Describe the geometric configuration of the solutions of $\frac{dx}{dt} = -3x + 2y$, $\frac{dy}{dt} = 2x - 3y$ in the phase plane.
12. $x = \pi/4$ is a critical point of the differential equation $x' = \cos x - \sin x$. Is it stable or unstable?
13. $x = 5\pi/4$ is a critical point of the differential equation $x' = \cos x - \sin x$. Is it stable or unstable?
14. Find the real critical points of the system $x' = x^2 + y^2 - 2$, $y' = x^2 - y$.
15. Find the Jacobian matrix of the system $x' = x^2 + y^2 - 2$, $y' = x^2 - y$ at each critical point.
16. Are the critical points of the system $x' = x^2 + y^2 - 2$, $y' = x^2 - y$ stable or unstable? Explain.
17. Consider the system $x' = x^2 + y^2 - 2$, $y' = x^2 - y$. Show the linearized system at each critical point.
18. In the previous problem, classify the critical points as node, spiral point, center, or saddle point.
19. Write down the second order differential equation for a nonlinear pendulum. Then write it as a system of first order equations.
20. In the previous problem, find the critical points. Choose two of the critical points with different local behavior and discuss the geometric configuration of the solutions in the phase plane near the critical points, including stability.

ANSWER KEY

Zill Differential Equations 9e Chapter 10 Form A

1. $y' = u$, $u' = -2u - 3y + 2t + 1$
2. non-autonomous (because of the $2t + 1$ term), linear
3. $(2/3, 1/3)$
4. $x = 1$, $y = -1$
5. Find the eigenvalues (solutions of $\lambda^2 - (a + d)\lambda + ad - bc = 0$). If they are purely imaginary, then the solutions are periodic.
6. $x = 2c_1e^{2t} \cos t + 2c_2e^{2t} \sin t$, $y = -c_1e^{2t}(\cos t + \sin t) + c_2e^{2t}(\cos t - \sin t)$
7. unstable, the eigenvalues have positive real part
8. $(0, 0)$ is an unstable spiral point
9. $x = c_1e^{-t} + c_2e^{-5t}$, $y = c_1e^{-t} - c_2e^{-5t}$
10. asymptotically stable, both eigenvalues are negative
11. $(0, 0)$ is a stable node
12. stable
13. unstable
14. $(1, 1)$ and $(-1, 1)$
15. at $(1, 1)$, $J = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$; at $(-1, 1)$, $J = \begin{bmatrix} -2 & 2 \\ -2 & -1 \end{bmatrix}$
16. $(1, 1)$ is unstable (positive eigenvalue); $(-1, 1)$ is stable (both eigenvalues have negative real part)
17. at $(1, 1)$, $\frac{dx}{dt} = 2x + 2y$, $\frac{dy}{dt} = 2x - y$; at $(-1, 1)$, $\frac{dx}{dt} = -2x + 2y$, $\frac{dy}{dt} = -2x - y$
18. $(1, 1)$ is a saddle point; $(-1, 1)$ is a spiral point
19. $\frac{d^2\theta}{dt^2} + g \sin\theta/l = 0$ where θ is the angle from the vertical at time t, g is the gravitational acceleration, and l is the length. $\frac{d\theta}{dt} = u$, $\frac{du}{dt} = -g \sin\theta/l$
20. $(n\pi, 0)$ where n is an integer. $n = 0$ corresponds to a stable center point. $n = 1$ corresponds to an unstable saddle point.

1. Rewrite the equation $y'' + 5yy' + 4y = 0$ as a system of first order equations.
2. Is the system of the previous problem autonomous? Is it linear? Explain.
3. Find the critical points of the system $\frac{dx}{dt} = x^2 + y^2 - 1$, $\frac{dy}{dt} = x^2 - 2y^2$.
4. Find all constant solutions of $\frac{dx}{dt} = 3x - y + 12$, $\frac{dy}{dt} = 3x + 2y - 3$.
5. Find the periodic solutions of $\frac{dx}{dt} = 2x + 5y$, $\frac{dy}{dt} = -x - 2y$.
6. Solve the system $\frac{dx}{dt} = 5x + 4y$, $\frac{dy}{dt} = x + 2y$.
7. Is the system $\frac{dx}{dt} = 5x + 4y$, $\frac{dy}{dt} = x + 2y$ stable or unstable? If stable, is it asymptotically stable? Explain.
8. Describe the geometric configuration of the solutions of $\frac{dx}{dt} = 5x + 4y$, $\frac{dy}{dt} = x + 2y$ in the phase plane.
9. Solve the system $\frac{dx}{dt} = 3x + y$, $\frac{dy}{dt} = -5x + y$.
10. Is the system $\frac{dx}{dt} = 3x + y$, $\frac{dy}{dt} = -5x + y$ stable or unstable? If stable, is it asymptotically stable? Explain.
11. Describe the geometric configuration of the solutions of $\frac{dx}{dt} = 3x + y$, $\frac{dy}{dt} = -5x + y$ in the phase plane.
12. $x = 3\pi/4$ is a critical point of the differential equation $x' = \cos x + \sin x$. Is it stable or unstable?
13. $x = 7\pi/4$ is a critical point of the differential equation $x' = \cos x + \sin x$. Is it stable or unstable?
14. Find the critical points of the system $x' = x^2 + y^2 - 1$, $y' = 3y$.
15. Find the Jacobian matrix of the system $x' = x^2 + y^2 - 1$, $y' = 3y$ at each critical point.
16. Are the critical points of the system $x' = x^2 + y^2 - 1$, $y' = 3y$ stable or unstable? Explain.
17. Consider the system $x' = x^2 + y^2 - 1$, $y' = 3y$. Show the linearized system at each critical point.
18. In the previous problem, classify the critical points as node, spiral point, center, or saddle point.
19. Write down the Lotka–Volterra predator–prey differential equations. Make sure you define all variables.
20. In the previous problem, find the critical points. Discuss the geometric configuration of the solutions in the phase plane near the critical points, including stability.

ANSWER KEY

Zill Differential Equations 9e Chapter 10 Form B

1. $y' = u$, $u' = -5yu - 4y$
2. autonomous (no explicit t dependence), non-linear (because of the yy' term)
3. $(\sqrt{2/3}, \sqrt{1/3})$, $(\sqrt{2/3}, -\sqrt{1/3})$, $(-\sqrt{2/3}, \sqrt{1/3})$, $(-\sqrt{2/3}, -\sqrt{1/3})$
4. $x = -7/3$, $y = 5$
5. $x = 5c_1 \cos t + 5c_2 \sin t$, $y = c_1(-2\cos t - \sin t) + c_2(\cos t - 2\sin t)$
6. $x = c_1 e^t + 4c_2 e^{6t}$, $y = -c_1 e^t + c_2 e^{6t}$
7. unstable (positive eigenvalues)
8. $(0, 0)$ is a node
9. $x = c_1 e^{2t} \cos(2t) + c_2 e^{2t} \sin(2t)$, $y = c_1 e^{2t}(-\cos(2t) - 2\sin(2t)) + c_2 e^{2t}(2\cos(2t) - \sin(2t))$
10. unstable (positive real part eigenvalues)
11. spiral point
12. stable
13. unstable
14. $(1, 0)$ and $(-1, 0)$
15. at $(1, 0)$, $J = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$; at $(-1, 0)$, $J = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$
16. $(1, 0)$ is unstable (eigenvalues positive); $(-1, 0)$ is unstable (one positive eigenvalue)
17. at $(1, 0)$, $\frac{dx}{dt} = 2x$, $\frac{dy}{dt} = 3y$; at $(-1, 0)$, $\frac{dx}{dt} = -2x$, $\frac{dy}{dt} = 3y$
18. $(1, 0)$ is a node; $(-1, 0)$ is a saddle point
19. $x(t)$ is the size of the predator population at time t, $y(t)$ is the size of the prey population at time t, $\frac{dx}{dt} = -r_1 x + axy$, $\frac{dy}{dt} = r_2 y - bxy$, where r_1 is the predator death rate without prey, r_2 is the prey growth rate without predators, and a and b are interaction rates. All coefficients are positive.
20. $(0, 0)$ is an unstable saddle point; $(r_2/b, r_1/a)$ is a stable center point.

1. Which of the following systems are autonomous?

   Select all that apply.

   (a) $\frac{dx}{dt} = x + y^2$, $\frac{dy}{dt} = 2x - 3y$
   (b) $\frac{dx}{dt} = x + t$, $\frac{dy}{dt} = 2x - 3y$
   (c) $\frac{dx}{dt} = x + y$, $\frac{dy}{dt} = 2x - 3y$
   (d) $\frac{dx}{dt} = 0$, $\frac{dy}{dt} = 2\sin x - 3y$
   (e) $\frac{dx}{dt} = t + 1$, $\frac{dy}{dt} = 15x - 4y$

2. Which of the following systems are linear?

   Select all that apply.

   (a) $\frac{dx}{dt} = x + y^2$, $\frac{dy}{dt} = 2x - 3y$
   (b) $\frac{dx}{dt} = x + t$, $\frac{dy}{dt} = 2x - 3y$
   (c) $\frac{dx}{dt} = x + y$, $\frac{dy}{dt} = 2x - 3y$
   (d) $\frac{dx}{dt} = 0$, $\frac{dy}{dt} = 2\sin x - 3y$
   (e) $\frac{dx}{dt} = t + 1$, $\frac{dy}{dt} = 15x - 4y$

3. The differential equation $x'' - x'e^x = 0$ can be rewritten as the system

   Select the correct answer.

   (a) $x' = x$, $u' = xe^x$
   (b) $x' = x$, $u' = ue^x$
   (c) $x' = u$, $u' = xe^u$
   (d) $x' = u$, $u' = xe^x$
   (e) $x' = u$, $u' = ue^x$

4. The critical points of the system $\frac{dx}{dt} = 2x + y^2$, $\frac{dy}{dt} = 2x - 3y$ are

   Select the correct answer.

   (a) $y = 0$, $y = -3$
   (b) $x = 0$, $x = -9/2$
   (c) $(0, 0)$
   (d) $(0, 0)$, $(-9/2, -3)$
   (e) $(-9/2, -3)$

5. The only constant solution of $\frac{dx}{dt} = 5x - y - 4$, $\frac{dy}{dt} = x + 2y - 3$ is
   Select the correct answer.

   (a) $x = 1$, $y = 1$
   (b) $x = 1$
   (c) $x = -1$
   (d) $x = -1$, $y = -1$
   (e) $y = 1$

6. The values of c that make the system $\frac{dx}{dt} = 3x + 2y$, $\frac{dy}{dt} = -cx + y$ stable are
   Select the correct answer.

   (a) $c > -3/2$
   (b) $c < -3/2$
   (c) It is stable for all values of c.
   (d) It is unstable for all values of c.
   (e) $c > 0$

7. The solution of the system $\frac{dx}{dt} = 4x - y$, $\frac{dy}{dt} = x + 2y$ is
   Select the correct answer.

   (a) $x = c_1e^{-3t} + c_2(t+1)e^{-3t}$, $y = c_1e^{-3t} + c_2te^{-3t}$
   (b) $x = c_1e^{-3t} + c_2te^{-3t}$, $y = c_1e^{-3t} + c_2(t+1)e^{-3t}$
   (c) $x = c_1e^{3t} + c_2te^{3t}$, $y = c_1e^{3t} + c_2(t+1)e^{3t}$
   (d) $x = c_1e^{3t} + c_2(t+1)e^{3t}$, $y = c_1e^{3t} + c_2te^{3t}$
   (e) $x = c_1e^{3t} + c_2(t+1)e^{3t}$, $y = -c_1e^{3t} - c_2te^{3t}$

8. The critical point $(0, 0)$ of the system $\frac{dx}{dt} = 4x - y$, $\frac{dy}{dt} = x + 2y$ is
   Select all that apply.

   (a) asymptotically stable
   (b) stable but not asymptotically stable
   (c) unstable
   (d) an attractor
   (e) a repeller

9. The geometric configuration of the solutions of $\frac{dx}{dt} = 4x - y$, $\frac{dy}{dt} = x + 2y$ in the phase plane is

   Select the correct answer.

   (a) degenerate stable node
   (b) degenerate unstable node
   (c) stable spiral point
   (d) unstable spiral point
   (e) saddle point

10. The solution of the system $\frac{dx}{dt} = -6x - 5y$, $\frac{dy}{dt} = 3x + 2y$ is

    Select the correct answer.

    (a) $x = 3c_1e^{-3t} + c_2e^{-t}$, $y = -5c_1e^{-3t} - c_2e^{-t}$
    (b) $x = 3c_1e^{3t} - c_2e^{-t}$, $y = -5c_1e^{3t} + c_2e^{-t}$
    (c) $x = 3c_1e^{3t} + c_2e^{t}$, $y = -5c_1e^{3t} - c_2e^{t}$
    (d) $x = 5c_1e^{3t} + c_2e^{t}$, $y = -3c_1e^{3t} - c_2e^{t}$
    (e) $x = 5c_1e^{-3t} + c_2e^{-t}$, $y = -3c_1e^{-3t} - c_2e^{-t}$

11. The critical point $(0, 0)$ of the system $\frac{dx}{dt} = -6x - 5y$, $\frac{dy}{dt} = 3x + 2y$ is

    Select all that apply.

    (a) asymptotically stable
    (b) stable but not asymptotically stable
    (c) unstable
    (d) an attractor
    (e) a repeller

12. The geometric configuration of the solutions of $\frac{dx}{dt} = -6x - 5y$, $\frac{dy}{dt} = 3x + 2y$ in the phase plane is

    Select the correct answer.

    (a) stable node
    (b) unstable node
    (c) stable spiral point
    (d) unstable spiral point
    (e) saddle point

13. Consider the differential equation $x' = \tan x$. The point $x = 0$ is

    Select the correct answer.

    (a) a stable critical point
    (b) an unstable critical point
    (c) not a critical point

14. Consider the differential equation $x' = \cot x$. The point $x = \pi/2$ is

    Select the correct answer.

    (a) a stable critical point
    (b) an unstable critical point
    (c) not a critical point

15. The critical points of the system $x' = x^3 - y^3 - 8$, $y' = 5x$ are

    Select the correct answer.

    (a) $(2, 0)$, $(0, 2)$
    (b) $(-2, 0)$, $(0, -2)$
    (c) $(0, 2)$
    (d) $(0, -2)$
    (e) none of the above

16. The Jacobian matrix of the system $x' = x^3 - y^2 + 4$, $y' = 5x$ at the critical point $(0, 2)$ is

    Select the correct answer.

    (a) $A = \begin{bmatrix} 0 & -4 \\ 5 & 0 \end{bmatrix}$
    (b) $A = \begin{bmatrix} 0 & 4 \\ 5 & 0 \end{bmatrix}$
    (c) $A = \begin{bmatrix} 3x^2 & 2y \\ 5 & 0 \end{bmatrix}$
    (d) $A = \begin{bmatrix} 3x^2 & -2y \\ 5 & 0 \end{bmatrix}$
    (e) $A = \begin{bmatrix} 3x^2 & -2y \\ 0 & 5 \end{bmatrix}$

17. The critical point $(0, 2)$ of the system $x' = x^3 - y^2 + 4$, $y' = 5x$ is a

    Select the correct answer.

    (a) stable node
    (b) unstable node
    (c) stable spiral point
    (d) unstable spiral point
    (e) center point

18. The Jacobian matrix of the system $x' = x^3 - y^2 + 4$, $y' = 5x$ at the critical point $(0, -2)$ is

    Select the correct answer.

    (a) $A = \begin{bmatrix} 0 & -4 \\ 5 & 0 \end{bmatrix}$
    (b) $A = \begin{bmatrix} 0 & 4 \\ 5 & 0 \end{bmatrix}$
    (c) $A = \begin{bmatrix} 3x^2 & 2y \\ 5 & 0 \end{bmatrix}$
    (d) $A = \begin{bmatrix} 3x^2 & -2y \\ 5 & 0 \end{bmatrix}$
    (e) $A = \begin{bmatrix} 3x^2 & -2y \\ 0 & 5 \end{bmatrix}$

19. The critical point $(0, -2)$ of the system $x' = x^3 - y^2 + 4$, $y' = 5x$ is a

    Select the correct answer.

    (a) stable node
    (b) unstable node
    (c) stable spiral point
    (d) unstable spiral point
    (e) saddle point

20. Assume that $x(t)$ and $y(t)$ represent the populations of two competing species at time t. The Lotka–Volterra competition model is

    Select the correct answer.

    (a) $\frac{dx}{dt} = r_1x(K_1 + x - \alpha_{12}y)/K_1$, $\frac{dy}{dt} = r_2y(K_2 - y - \alpha_{21}y)/K_2$
    (b) $\frac{dx}{dt} = r_1x(K_1 - x + \alpha_{12}y)/K_1$, $\frac{dy}{dt} = r_2y(K_2 - y - \alpha_{21}y)/K_2$
    (c) $\frac{dx}{dt} = r_1x(K_1 - x - \alpha_{12}y)/K_1$, $\frac{dy}{dt} = r_2y(K_2 - y - \alpha_{21}y)/K_2$
    (d) $\frac{dx}{dt} = r_1x(K_1 - x - \alpha_{12}y)/K_1$, $\frac{dy}{dt} = r_2y(K_2 + y - \alpha_{21}y)/K_2$
    (e) $\frac{dx}{dt} = r_1x(K_1 - x - \alpha_{12}y)/K_1$, $\frac{dy}{dt} = r_2y(K_2 - y + \alpha_{21}y)/K_2$

ANSWER KEY
Zill Differential Equations 9e Chapter 10 Form C

1. a, c, d
2. b, c, e
3. e
4. d
5. a
6. d
7. d
8. c, e
9. b
10. e
11. a, d
12. a
13. b
14. a
15. d
16. a
17. e
18. b
19. e
20. c

1. Which of the following systems are autonomous?

   Select all that apply.

   (a) $\frac{dx}{dt} = x + y$, $\frac{dy}{dt} = 2t - 3\sin t$
   (b) $\frac{dx}{dt} = 3.5x + 2y$, $\frac{dy}{dt} = 2x - 5y$
   (c) $\frac{dx}{dt} = x + y$, $\frac{dy}{dt} = 2xe^t - 3y$
   (d) $\frac{dx}{dt} = 0$, $\frac{dy}{dt} = 2\cos x - 3y$
   (e) $\frac{dx}{dt} = t^2 + 1$, $\frac{dy}{dt} = 15x - 4y$

2. Which of the following systems are linear?

   Select all that apply.

   (a) $\frac{dx}{dt} = x + y$, $\frac{dy}{dt} = 2t - 3\sin t$
   (b) $\frac{dx}{dt} = 3.5x + 2y$, $\frac{dy}{dt} = 2x - 5y$
   (c) $\frac{dx}{dt} = x + 1/y$, $\frac{dy}{dt} = 2xe^t - 3y$
   (d) $\frac{dx}{dt} = 0$, $\frac{dy}{dt} = 2\cos x - 3y$
   (e) $\frac{dx}{dt} = t^2 + 1$, $\frac{dy}{dt} = 15x - 4y$

3. The initial value problem $x'' - x' + tx = 0$, $x(0) = 1$, $x'(0) = 2$ can be rewritten as the system

   Select the correct answer.

   (a) $x' = x$, $u' = u - tx$, $x(0) = 1$, $u(0) = 2$
   (b) $x' = x$, $u' = x - tu$, $x(0) = 1$, $u(0) = 2$
   (c) $x' = u$, $u' = u + tx$, $x(0) = 1$, $u(0) = 2$
   (d) $x' = u$, $u' = x + tu$, $x(0) = 1$, $u(0) = 2$
   (e) $x' = u$, $u' = u - tx$, $x(0) = 1$, $u(0) = 2$

4. The critical points of the system $\frac{dx}{dt} = 2x + y - 3$, $\frac{dy}{dt} = 2x - 3y - 7$ are

   Select the correct answer.

   (a) $y = -1$
   (b) $x = 2$
   (c) $(2, -1)$
   (d) $(-1, 2)$
   (e) $(-3, -7)$

5. The constant solutions of $\frac{dx}{dt} = 2x^2 + y^2 - 1$, $\frac{dy}{dt} = x - 2y$ are

   Select the correct answer.

   (a) $x = 2/3$, $y = 1/3$
   (b) $x = 2/3$, $y = 1/3$, and $x = -2/3$, $y = -1/3$
   (c) $x = -2/3$, $y = -1/3$
   (d) $x = 2/\sqrt{7}$, $y = 1/\sqrt{7}$
   (e) $x = 2/\sqrt{7}$, $y = 1/\sqrt{7}$ and $x = -2/\sqrt{7}$, $y = -1/\sqrt{7}$

6. The values of c that make the system $\frac{dx}{dt} = -3x + 2y$, $\frac{dy}{dt} = -cx - y$ stable are

   Select the correct answer.

   (a) $c > -3/2$
   (b) $c < -3/2$
   (c) It is locally stable for all values of c.
   (d) It is unstable for all values of c.
   (e) $c > 0$

7. The solution of the system $\frac{dx}{dt} = -3x + 2y$, $\frac{dy}{dt} = 4x - y$ is

   Select the correct answer.

   (a) $x = c_1e^t + c_2e^{-5t}$, $y = 2c_1e^t - c_2e^{-5t}$
   (b) $x = c_1e^t + c_2e^{5t}$, $y = 2c_1e^t - c_2e^{5t}$
   (c) $x = 2c_1e^t + c_2e^{-5t}$, $y = c_1e^t - c_2e^{-5t}$
   (d) $x = 2c_1e^t + c_2e^{5t}$, $y = c_1e^t - c_2e^{5t}$
   (e) $x = 2c_1e^t + c_2e^{-5t}$, $y = 2c_1e^t - c_2e^{-5t}$

8. The critical point $(0, 0)$ of the system $\frac{dx}{dt} = -3x + 2y$, $\frac{dy}{dt} = 4x - y$ is

   Select all that apply.

   (a) asymptotically stable
   (b) stable but not asymptotically stable
   (c) unstable
   (d) an attractor
   (e) a repeller

9. The geometric configuration of the solutions of $\frac{dx}{dt} = -3x + 2y$, $\frac{dy}{dt} = 4x - y$ in the phase plane is

   Select the correct answer.

   (a) stable node
   (b) unstable node
   (c) stable spiral point
   (d) unstable spiral point
   (e) saddle point

10. The solution of the system $\frac{dx}{dt} = 5x + y$, $\frac{dy}{dt} = 3x + 3y$ is

    Select the correct answer.

    (a) $x = c_1e^{-6t} + c_2e^{-2t}$, $y = c_1e^{-6t} - 3c_2e^{-2t}$
    (b) $x = c_1e^{6t} + c_2e^{2t}$, $y = c_1e^{6t} - 3c_2e^{2t}$
    (c) $x = c_1e^{-6t} - 3c_2e^{-2t}$, $y = c_1e^{-6t} + c_2e^{-2t}$
    (d) $x = c_1e^{6t} - 3c_2e^{-2t}$, $y = c_1e^{6t} + c_2e^{-2t}$
    (e) $x = c_1e^{6t} + c_2e^{-2t}$, $y = c_1e^{6t} - 3c_2e^{-2t}$

11. The critical point $(0, 0)$ of the system $\frac{dx}{dt} = 5x + y$, $\frac{dy}{dt} = 3x + 3y$ is

    Select all that apply.

    (a) asymptotically stable
    (b) stable but not asymptotically stable
    (c) unstable
    (d) an attractor
    (e) a repeller

12. The geometric configuration of the solutions of $\frac{dx}{dt} = 5x + y$, $\frac{dy}{dt} = 3x + 3y$ in the phase plane is

    Select the correct answer.

    (a) stable spiral point
    (b) unstable spiral point
    (c) stable node
    (d) unstable node
    (e) saddle point

13. The solution of the system $\frac{dx}{dt} = -6x - 5y$, $\frac{dy}{dt} = 4x + 2y$ is

    Select the correct answer.

    (a) $x = c_1e^{-2t}(2\cos(2t) + 4\sin(2t)) + c_2e^{-2t}(4\cos(2t) - 2\sin(2t))$,
    $y = 4c_1e^{-2t}\cos(2t) + 4c_2e^{-2t}\sin(2t)$

    (b) $x = c_1e^{2t}(2\cos(2t) + 4\sin(2t)) + c_2e^{2t}(4\cos(2t) - 2\sin(2t))$,
    $y = -4c_1e^{2t}\cos(2t) - 4c_2e^{2t}\sin(2t)$

    (c) $x = c_1e^{-2t}(4\cos(2t) + 2\sin(2t)) + c_2e^{-2t}(-2\cos(2t) + 4\sin(2t))$,
    $y = -4c_1e^{-2t}\cos(2t) - 4c_2e^{-2t}\sin(2t)$

    (d) $x = c_1e^{-2t}(2\cos(2t) + 4\sin(2t)) + c_2e^{-2t}(4\cos(2t) - 2\sin(2t))$,
    $y = -4c_1e^{-2t}\cos(2t) - 4c_2e^{-2t}\sin(2t)$

    (e) $x = c_1e^{2t}(4\cos(2t) + 2\sin(2t)) + c_2e^{2t}(-2\cos(2t) + 4\sin(2t))$,
    $y = -4c_1e^{2t}\cos(2t) - 4c_2e^{2t}\sin(2t)$

14. The critical point $(0, 0)$ of the system $\frac{dx}{dt} = -6x - 5y$, $\frac{dy}{dt} = 4x + 2y$ is

    Select all that apply.

    (a) asymptotically stable
    (b) stable but not asymptotically stable
    (c) unstable
    (d) an attractor
    (e) a repeller

15. The geometric configuration of the solutions of $\frac{dx}{dt} = -6x - 5y$, $\frac{dy}{dt} = 4x + 2y$ in the phase plane is

    Select the correct answer.

    (a) stable node
    (b) unstable node
    (c) stable spiral point
    (d) unstable spiral point
    (e) saddle point

16. Consider the differential equation $x' = \sec x$. The point $x = 0$ is

    Select the correct answer.

    (a) a stable critical point
    (b) an unstable critical point
    (c) not a critical point

17. Consider the differential equation $x' = \sin x + \cos x$. The point $x = 3\pi/4$ is

    Select the correct answer.

    (a) a stable critical point
    (b) an unstable critical point
    (c) not a critical point

18. The critical points of the system $x' = x^3 - y^2 - 8$, $y' = 8y$ are

    Select the correct answer.

    (a) $(2, 0)$, $(0, 2)$
    (b) $(-2, 0)$, $(0, -2)$
    (c) $(0, 2)$
    (d) $(0, -2)$
    (e) none of the above

19. The Jacobian matrix of the system $x' = x^3 - y^3 - 8$, $y' = 8y$ at the critical point $(2, 0)$ is

    Select the correct answer.

    (a) $A = \begin{bmatrix} 3x^2 & -3y^2 \\ 8 & 0 \end{bmatrix}$
    (b) $A = \begin{bmatrix} 3x^2 & -3y^2 \\ 0 & 8 \end{bmatrix}$
    (c) $A = \begin{bmatrix} 3x^2 & 3y^2 \\ 0 & 8 \end{bmatrix}$
    (d) $A = \begin{bmatrix} 12 & 0 \\ 0 & 8 \end{bmatrix}$
    (e) $A = \begin{bmatrix} 12 & 0 \\ 8 & 0 \end{bmatrix}$

20. Assume a bead of mass m slides along the curve $y = f(x)$. Also assume that there is a damping force acting in the direction opposite to the velocity and proportional to the velocity, with proportionality constant β. The differential equation that describes the horizontal position of the bead is

    Select the correct answer.

    (a) $m\frac{d^2x}{dt^2} = -mgf'(x)/(1 - [f'(x)]^2) - \beta\frac{dx}{dt}$
    (b) $m\frac{d^2x}{dt^2} = -mgf'(x)/(1 - [f'(x)]^2) + \beta\frac{dx}{dt}$
    (c) $m\frac{d^2x}{dt^2} = -mgf'(x)/(1 + [f'(x)]^2) - \beta\frac{dx}{dt}$
    (d) $m\frac{d^2x}{dt^2} = mgf'(x)/(1 + [f'(x)]^2) - \beta\frac{dx}{dt}$
    (e) $m\frac{d^2x}{dt^2} = mgf'(x)/(1 + [f'(x)]^2) + \beta\frac{dx}{dt}$

ANSWER KEY

Zill Differential Equations 9e Chapter 10 Form D

1. b, d
2. a, b, e
3. e
4. c
5. b
6. a
7. a
8. c
9. e
10. b
11. c, e
12. d
13. c
14. a, d
15. c
16. c
17. a
18. e
19. d
20. c

1. Rewrite the equation $y''' - 3yy'' + 3y' + y = 0$ as a system of first order equations.
2. Is the system of the previous problem autonomous? Is it linear? Explain.
3. The critical points of the system $\frac{dx}{dt} = 2x^2 + y - 20$, $\frac{dy}{dt} = 3x - 2y - 5$ are

   Select the correct answer.

   (a) $(2, 3)$, $(3, 2)$
   (b) $(-15/4, -65/8)$, $(3, 2)$
   (c) $(3, 2)$
   (d) $y = 2$
   (e) $x = 3$

4. The only constant solutions of the system $\frac{dx}{dt} = 2x + 6y - 20$, $\frac{dy}{dt} = 2x - 2y + 4$ are

   Select the correct answer.

   (a) $x = 1$
   (b) $y = 3$
   (c) $x = -1$
   (d) $x = 1$, $y = 3$
   (e) $x = -1$, $y = -3$

5. A system for a vector $\mathbf{X}(t)$ in Cartesian coordinates is changed to polar coordinates yielding $\frac{dr}{dt} = r - 2$, $\frac{d\theta}{dt} = 2$. Solve this system with the initial condition $\mathbf{X}(0) = (1, 0)$.
6. Solve the system $\frac{dx}{dt} = -3x + 2y$, $\frac{dy}{dt} = -2x + y$.
7. Is the system $\frac{dx}{dt} = -3x + 2y$, $\frac{dy}{dt} = -2x + y$ stable or unstable? If stable, is it asymptotically stable? Explain.
8. Describe the geometric configuration of the solutions of $\frac{dx}{dt} = -3x + 2y$, $\frac{dy}{dt} = -2x + y$ in the phase plane.
9. The solution of the system $\frac{dx}{dt} = 2x + 3y$, $\frac{dy}{dt} = x + 4y$ is

   Select the correct answer.

   (a) $x = 3c_1e^t - c_2e^{5t}$, $y = -c_1e^t + c_2e^{5t}$
   (b) $x = 3c_1e^t + c_2e^{5t}$, $y = -c_1e^t + c_2e^{5t}$
   (c) $x = 3c_1e^{-t} + c_2e^{-5t}$, $y = -c_1e^{-t} - c_2e^{-t}$
   (d) $x = c_1e^t + c_2e^{5t}$, $y = -3c_1e^t - c_2e^t$
   (e) $x = c_1e^{-t} - c_2e^{-5t}$, $y = -3c_1e^{-t} - c_2e^{-t}$

10. $x = 3\pi/4$ is a critical point of the differential equation $x' = \cos x + \sin x$. Is it stable or unstable?
11. $x = \pi/2$ is a critical point of the differential equation $x' = \cos x$. Is it stable or unstable?

12. $x = 3\pi/2$ is a critical point of the differential equation $x' = \cos x$. Is it stable or unstable?

13. The critical point $(0, 0)$ of the system $\frac{dx}{dt} = 2x + 3y$, $\frac{dy}{dt} = x + 4y$ is

    Select all that apply.

    (a) asymptotically stable
    (b) stable but not asymptotically stable
    (c) unstable
    (d) an attractor
    (e) a repeller

14. The geometric configuration of the solutions of $\frac{dx}{dt} = 2x+3y$, $\frac{dy}{dt} = x+4y$ in the phase plane is

    Select the correct answer.

    (a) stable node
    (b) unstable node
    (c) stable spiral point
    (d) unstable spiral point
    (e) saddle point

15. Write down the Lotka–Volterra competition model differential equations. Make sure you define all variables.

16. In the previous problem, find the critical points. Discuss the geometric configuration of the solutions in the phase plane near the critical points along the axes, including stability.

17. The Jacobian matrix of the system $x' = 5x^3 + y^3 + 4$, $y' = 3x - y$ at the critical point $(-1/2, -3/2)$ is

    Select the correct answer.

    (a) $J = \begin{bmatrix} 15/4 & 27/4 \\ 3 & -1 \end{bmatrix}$

    (b) $J = \begin{bmatrix} -15/4 & -27/4 \\ 3 & -1 \end{bmatrix}$

    (c) $J = \begin{bmatrix} 15x^2 & 3y^2 \\ 3 & -1 \end{bmatrix}$

    (d) $J = \begin{bmatrix} -15x^2 & -3y^2 \\ 3 & -1 \end{bmatrix}$

    (e) $J = \begin{bmatrix} -5/2 & -3/2 \\ 3 & -1 \end{bmatrix}$

18. The critical point $(-1/2, -3/2)$ of the system $x' = 5x^3 + y^3 + 4$, $y' = 3x - y$ is a

    Select the correct answer.

    (a) stable node
    (b) unstable node
    (c) stable spiral point
    (d) unstable spiral point
    (e) saddle point

19. The Jacobian matrix of the system $x' = x^2 + y^2 - 32, y' = x - y$ at the critical point $(4, 4)$ is

    Select the correct answer.

    (a) $J = \begin{bmatrix} 8 & 8 \\ -1 & 1 \end{bmatrix}$

    (b) $J = \begin{bmatrix} 48 & 8 \\ 1 & -1 \end{bmatrix}$

    (c) $J = \begin{bmatrix} 2x & 2y \\ -1 & 1 \end{bmatrix}$

    (d) $J = \begin{bmatrix} 2x & 2y \\ 1 & -1 \end{bmatrix}$

    (e) $J = \begin{bmatrix} 8 & 8 \\ 1 & -1 \end{bmatrix}$

20. The critical point $(4, 4)$ of the system $x' = x^2 + y^2 - 32$, $y' = x - y$ is a

    Select the correct answer.

    (a) stable node
    (b) unstable node
    (c) stable spiral point
    (d) unstable spiral point
    (e) saddle point

ANSWER KEY
Zill Differential Equations 9e Chapter 10 Form E

1. $y' = u$, $u' = v$, $v' = 3yv - 3u - y$
2. autonomous (no explicit t dependence), non-linear (because of the yy'' term)
3. b
4. d
5. $r = 2 - e^t$, $\theta = 2t$
6. $x = c_1e^{-t} + c_2te^{-t}$, $y = c_1e^{-t} + c_2(t + 1/2)e^{-t}$
7. stable and asymptotically stable
8. degenerate stable node
9. b
10. stable
11. stable
12. unstable
13. c, e
14. b
15. a
16. $\frac{dx}{dt} = r_1x(1 - x/K_1 - \alpha_{12}y)$, $\frac{dy}{dt} = r_2y(1 - y/K_2 - \alpha_{21}x)$, where $x(t)$ and $y(t)$ are the populations of the competing species at time t.
17. $(0, 0)$, $(0, K_2)$, $(K_1, 0)$, $((1/K_2-\alpha_{12})/(1/(K_1K_2)-\alpha_{12}\alpha_{21})$, $(1/K_1-\alpha_{21})/(1/(K_1K_2)-\alpha_{12}\alpha_{21}))$. The critical points $(0, K_2)$ and $(K_1, 0)$ are both stable nodes, assuming $\alpha_{12}K_2 > 1$, $\alpha_{21}K_1 > 1$.
18. e
19. e
20. e

1. Which of the following systems are autonomous?

   Select all that apply.

   (a) $\frac{dx}{dt} = x + y,\ \frac{dy}{dt} = 2z - 3\sin y,\ \frac{dz}{dt} = xyz$
   (b) $\frac{dx}{dt} = 3.5x + 2y,\ \frac{dy}{dt} = 2x - 5y,\ \frac{dz}{dt} = x + y + z$
   (c) $\frac{dx}{dt} = x + ze^t,\ \frac{dy}{dt} = 2xe^t - 3y,\ \frac{dz}{dt} = x - y + 2z$
   (d) $\frac{dx}{dt} = 0,\ \frac{dy}{dt} = 2\cos t - 3y,\ \frac{dz}{dt} = xyz$
   (e) $\frac{dx}{dt} = t^2 + 1,\ \frac{dy}{dt} = 15x - 4y,\ \frac{dz}{dt} = xyz$

2. Which of the following systems are linear?

   Select all that apply.

   (a) $\frac{dx}{dt} = x + y,\ \frac{dy}{dt} = 2z - 3\sin y,\ \frac{dz}{dt} = xyz$
   (b) $\frac{dx}{dt} = 3.5x + 2y,\ \frac{dy}{dt} = 2x - 5y,\ \frac{dz}{dt} = x + y + z$
   (c) $\frac{dx}{dt} = x + ze^t,\ \frac{dy}{dt} = 2xe^t - 3y,\ \frac{dz}{dt} = x - y + 2z$
   (d) $\frac{dx}{dt} = 0,\ \frac{dy}{dt} = 2\cos t - 3y,\ \frac{dz}{dt} = xyz$
   (e) $\frac{dx}{dt} = t^2 + 1,\ \frac{dy}{dt} = 15x - 4y,\ \frac{dz}{dt} = xyz$

3. Find the critical points of the system $\frac{dx}{dt} = x + y + 3,\ \frac{dy}{dt} = x - 2y - 1.$

4. Find all constant solutions of the system $\frac{dx}{dt} = -x + y - 3,\ \frac{dy}{dt} = 5x - 2y + 1.$

5. A periodic solution of an autonomous system

   Select all that apply.

   (a) is unstable
   (b) is locally stable
   (c) can occur in a linear system
   (d) can occur in a nonlinear system
   (e) none of the above

6. The values of c that make the system $\frac{dx}{dt} = 4x + y,\ \frac{dy}{dt} = -cx - 6y$ stable are

   Select the correct answer.

   (a) $c > 24$
   (b) $c < 24$
   (c) It is locally stable for all values of c.
   (d) It is unstable for all values of c.
   (e) $c > 0$

7. The solution of the system $\frac{dx}{dt} = 4x + y$, $\frac{dy}{dt} = -25x - 6y$ is

   Select the correct answer.

   (a) $x = c_1e^t + c_2te^t$, $y = -5c_1e^t + c_2(-5te^t + e^t)$
   (b) $x = c_1e^{-t} + c_2te^{-t}$, $y = -5c_1e^{-t} + c_2(-5te^{-t} + e^{-t})$
   (c) $x = -5c_1e^{-t} + c_2te^{-t}$, $y = c_1e^{-t} + c_2(-5te^{-t} + e^{-t})$
   (d) $x = -5c_1e^t + c_2te^t$, $y = c_1e^t + c_2(-5te^t + e^t)$
   (e) $x = c_1e^{-t} + c_2(te^{-t} + e^{-t})$, $y = -5c_1e^{-t} - 5c_2te^{-t}$

8. The critical point $(0, 0)$ of the system $\frac{dx}{dt} = 4x + y$, $\frac{dy}{dt} = -25x - 6y$ is

   Select all that apply.

   (a) asymptotically stable
   (b) stable but not asymptotically stable
   (c) unstable
   (d) an attractor
   (e) a repeller

9. The geometric configuration of the solutions of $\frac{dx}{dt} = 4x + y$, $\frac{dy}{dt} = -25x - 6y$ in the phase plane is

   Select the correct answer.

   (a) degenerate stable node
   (b) degenerate unstable node
   (c) stable spiral point
   (d) unstable spiral point
   (e) saddle point

10. Solve the system $\frac{dx}{dt} = -5x - 4y$, $\frac{dy}{dt} = -x - 2y$.

11. Is the system $\frac{dx}{dt} = -5x - 4y$, $\frac{dy}{dt} = -x - 2y$ stable or unstable? If stable, is it asymptotically stable? Explain.

12. Describe the geometric configuration of the solutions of $\frac{dx}{dt} = -5x - 4y$, $\frac{dy}{dt} = -x - 2y$ in the phase plane.

13. The solution of the system $\frac{dx}{dt} = 2x + 5y$, $\frac{dy}{dt} = -x - 2y$ is

    Select the correct answer.

    (a) $x = c_1(-2\cos t + \sin t)e^t + c_2(-\cos t - 2\sin t)e^t$, $y = c_1 \cos te^t + c_2 \sin te^t$
    (b) $x = c_1(-2\cos t + \sin t)e^{-t} + c_2(-\cos t - 2\sin t)e^{-t}$, $y = c_1 \cos te^{-t} + c_2 \sin te^{-t}$
    (c) $x = c_1(-2\cos t + \sin t) + c_2(-\cos t - 2\sin t)$, $y = c_1 \cos t + c_2 \sin t$
    (d) $x = c_1(\cos t + \sin t) + c_2(-\cos t + \sin t)$, $y = -2c_1 \cos t - 2c_2 \sin t$
    (e) $x = c_1(-2\cos t + \sin t) + c_2(-\cos t - 2\sin t)$, $y = -2c_1 \cos t - 2c_2 \sin t$

14. The critical point $(0, 0)$ of the system $\frac{dx}{dt} = 2x + 5y$, $\frac{dy}{dt} = -x - 2y$ is

    Select all that apply.

    (a) asymptotically stable
    (b) stable but not asymptotically stable
    (c) unstable
    (d) an attractor
    (e) a repeller

15. The geometric configuration of the solutions of $\frac{dx}{dt} = 2x + 5y$, $\frac{dy}{dt} = -x - 2y$ in the phase plane is

    Select the correct answer.

    (a) stable node
    (b) unstable node
    (c) stable spiral point
    (d) unstable spiral point
    (e) center point

16. Consider the differential equation $x' = \cos x - \sin x$. The point $x = \pi/4$ is

    Select the correct answer.

    (a) a stable critical point
    (b) an unstable critical point
    (c) not a critical point

17. Find the critical points of the system $x' = x^2 + y^2 - 8$, $y' = x - y$.

18. Find the Jacobian matrix of the system $x' = x^2 + y^2 - 8$, $y' = x - y$ at each critical point.

19. Are the critical points of the system $x' = x^2 + y^2 - 8$, $y' = x - y$ stable or unstable? Explain.

20. Classify the critical points of the system $x' = x^2 + y^2 - 8$, $y' = x - y$ as node, spiral point, center, or saddle point.

ANSWER KEY

Zill Differential Equations 9e Chapter 10 Form F

1. a, b

2. b, c

3. $(-5/3, -4/3)$

4. $x = 5/3, y = 14/3$

5. b, c, d

6. a

7. b

8. a, d

9. a

10. $x = c_1e^{-t} + 4c_2e^{-6t}$, $y = -c_1e^{-t} + c_2e^{-6t}$

11. stable and asymptotically stable

12. $(0, 0)$ is a stable node

13. c

14. b

15. e

16. a

17. $(2, 2)$, $(-2, -2)$

18. At $(2, 2)$, $J = \begin{bmatrix} 4 & 4 \\ 1 & -1 \end{bmatrix}$; at $(-2, -2)$, $J = \begin{bmatrix} -4 & -4 \\ 1 & -1 \end{bmatrix}$

19. $(2, 2)$ is unstable (one eigenvalue is positive); $(-2, -2)$ is stable (both eigenvalues have negative real part).

20. $(2, 2)$ is a saddle point; $(-2, -2)$ is a spiral point.

1. Rewrite the equation $y'''' - 3y''' + 3y'' + y' = e^t$ as a system of first order equations.

2. Is the system of the previous problem autonomous? Is it linear? Explain.

3. The critical points of the system $\frac{dx}{dt} = x^2 + y^2 - 25$, $\frac{dy}{dt} = x^2 - y^2 - 9$ are

   Select the correct answer.

   (a) $(\sqrt{17}, 2\sqrt{2})$
   (b) $(\sqrt{17}, 2\sqrt{2})$, $(-\sqrt{17}, -2\sqrt{2})$
   (c) $(\sqrt{17}, 2\sqrt{2})$, $(-\sqrt{17}, -2\sqrt{2})$, $(-\sqrt{17}, 2\sqrt{2})$, $(\sqrt{17}, -2\sqrt{2})$
   (d) $(-\sqrt{17}, 2\sqrt{2})$, $(\sqrt{17}, -2\sqrt{2})$
   (e) $(-\sqrt{17}, -2\sqrt{2})$

4. The constant solutions of $\frac{dx}{dt} = 2x^2 + y^2 - 1$, $\frac{dy}{dt} = x - y$ are

   Select the correct answer.

   (a) $x = 1/\sqrt{3}$, $y = 1/\sqrt{3}$
   (b) $x = -1/\sqrt{3}$, $y = -1/\sqrt{3}$
   (c) $x = 1/\sqrt{3}$, $y = 1/\sqrt{3}$ and $x = -1/\sqrt{3}$, $y = -1/\sqrt{3}$
   (d) $x = 1/\sqrt{3}$, $y = 1/\sqrt{3}$ and $x = -1/\sqrt{3}$, $y = -1/\sqrt{3}$ and $x = 1/\sqrt{3}$, $y = -1/\sqrt{3}$ and $x = -1/\sqrt{3}$, $y = 1/\sqrt{3}$
   (e) none of the above

5. Find the solution of $\frac{dx}{dt} = -y - x\sqrt{x^2 + y^2}$, $\frac{dy}{dt} = x - y\sqrt{x^2 + y^2}$, by changing to polar coordinates.

6. Solve the system $\frac{dx}{dt} = 4x + 2y$, $\frac{dy}{dt} = -x + y$.

7. Is the system $\frac{dx}{dt} = 4x + 2y$, $\frac{dy}{dt} = -x + y$ stable or unstable? If stable, is it asymptotically stable? Explain.

8. Describe the geometric configuration of the solutions of $\frac{dx}{dt} = 4x + 2y$, $\frac{dy}{dt} = -x + y$ in the phase plane.

9. The solution of the system $\frac{dx}{dt} = -3x + 2y$, $\frac{dy}{dt} = -5x + 3y$ is

   Select the correct answer.

   (a) $x = 3c_1 \cos t + 3c_2 \sin t$, $y = c_1(2\cos t - \sin t) + c_2(\cos t + 2\sin t)$
   (b) $x = 3c_1 \cos t - 3c_2 \sin t$, $y = c_1(2\cos t - \sin t) + c_2(\cos t + 2\sin t)$
   (c) $x = 2c_1 \cos t + c_2 \sin t$, $y = c_1(3\cos t - \sin t) + c_2(\cos t + 3\sin t)$
   (d) $x = c_1 \cos t + 2c_2 \sin t$, $y = c_1(3\cos t - \sin t) + c_2(\cos t + 3\sin t)$
   (e) $x = 2c_1 \cos t + 2c_2 \sin t$, $y = c_1(3\cos t - \sin t) + c_2(\cos t + 3\sin t)$

10. The critical point $(0, 0)$ of the system $\frac{dx}{dt} = -3x + 2y$, $\frac{dy}{dt} = -5x + 3y$ is

    Select all that apply.

    (a) asymptotically stable
    (b) stable but not asymptotically stable
    (c) unstable
    (d) an attractor
    (e) a repeller

11. The geometric configuration of the solutions of $\frac{dx}{dt} = -3x + 2y$, $\frac{dy}{dt} = -5x + 3y$ in the phase plane is

    Select the correct answer.

    (a) stable node
    (b) unstable node
    (c) stable spiral point
    (d) unstable spiral point
    (e) center

12. $x = 0$ is a critical point of the differential equation $x' = \sin x$. Is it stable or unstable?

13. $x = \pi$ is a critical point of the differential equation $x' = \sin x$. Is it stable or unstable?

14. The critical points of $\frac{dx}{dt} = 1 - 2xy$, $\frac{dy}{dt} = 2xy - y$ are

    Select the correct answer

    (a) $(1, 0)$
    (b) $(1, 0)$ and $(1/2, 1)$
    (c) $(1, 0)$ and $(-1/2, -1)$
    (d) $(1/2, 1)$
    (e) $(-1/2, -1)$

15. The Lotka–Volterra predator–prey system is $\frac{dx}{dt} = ax - bxy$, $\frac{dy}{dt} = -cy + dxy$, where $x(t)$ is the prey population at time t, $y(t)$ is the predator population at time t, and the constants a, b, c, and d are all positive. Classify the critical point at $(0, 0)$, including stability.

16. In the previous problem, identify the other critical point.

17. In the previous two problems, classify the other critical point, including stability.

18. The Jacobian of the system $\frac{dx}{dt} = x^2 + y^2 - 1$, $\frac{dy}{dt} = 3y$ at the critical point $(1, 0)$ is

    Select the correct answer

    (a) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

    (b) $\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$

    (c) $\begin{bmatrix} 2x & -2y \\ 0 & 3 \end{bmatrix}$

    (d) $\begin{bmatrix} 2x & 2y \\ 0 & 3 \end{bmatrix}$

    (e) $\begin{bmatrix} 2x & 2y \\ 0 & -3 \end{bmatrix}$

19. The critical point $(1, 0)$ of $\frac{dx}{dt} = x^2 + y^2 - 1$, $\frac{dy}{dt} = 3y$ is

    Select the correct answer

    (a) asymptotically stable
    (b) stable but not asymptotically stable
    (c) unstable
    (d) an attractor
    (e) a repeller

20. The critical point $(1, 0)$ of $\frac{dx}{dt} = x^2 + y^2 - 1$, $\frac{dy}{dt} = 3y$ is

    Select the correct answer

    (a) stable node
    (b) unstable node
    (c) stable spiral point
    (d) unstable spiral point
    (e) saddle point

ANSWER KEY
Zill Differential Equations 9e Chapter 10 Form G

1. $y' = u$, $u' = v$, $v' = w$, $w' = 3w - 3v - u + e^t$
2. non-autonomous (because of the e^t term), linear
3. c
4. c
5. $r = 1/(t + c_1)$, $\theta = t + c_2$
6. $x = c_1e^{2t} + 2c_2e^{3t}$, $y = -c_1e^{2t} - c_2e^{3t}$
7. unstable (positive eigenvalues)
8. $(0, 0)$ is a node
9. e
10. b
11. e
12. unstable
13. stable
14. d
15. $(0, 0)$ is an unstable saddle point
16. $(c/d, a/b)$
17. $(c/d, a/b)$ is a stable center point
18. a
19. c, e
20. b

1. Which of the following systems are autonomous?

   Select all that apply.

   (a) $\frac{dx}{dt} = x - ty,\ \frac{dy}{dt} = 2z^2 - 3\sin x,\ \frac{dz}{dt} = x + y - z$
   (b) $\frac{dx}{dt} = x + ze^t,\ \frac{dy}{dt} = 2xe^t - 3y,\ \frac{dz}{dt} = x - y + 2z$
   (c) $\frac{dx}{dt} = 3x + 2y,\ \frac{dy}{dt} = 2x - 5y,\ \frac{dz}{dt} = x + y + z$
   (d) $\frac{dx}{dt} = 0,\ \frac{dy}{dt} = 2\cos z - 3y,\ \frac{dz}{dt} = xyz$
   (e) $\frac{dx}{dt} = t^2 + 1,\ \frac{dy}{dt} = 15x - 4y,\ \frac{dz}{dt} = xyz$

2. Which of the following systems are linear?

   Select all that apply.

   (a) $\frac{dx}{dt} = x - ty,\ \frac{dy}{dt} = 2z^2 - 3\sin x,\ \frac{dz}{dt} = x + y - z$
   (b) $\frac{dx}{dt} = x + ze^t,\ \frac{dy}{dt} = 2xe^t - 3y,\ \frac{dz}{dt} = x - y + 2z$
   (c) $\frac{dx}{dt} = 3x + 2y,\ \frac{dy}{dt} = 2x - 5y,\ \frac{dz}{dt} = x + y + z$
   (d) $\frac{dx}{dt} = 0,\ \frac{dy}{dt} = 2\cos z - 3y,\ \frac{dz}{dt} = xyz$
   (e) $\frac{dx}{dt} = t^2 + 1,\ \frac{dy}{dt} = 15x - 4y,\ \frac{dz}{dt} = xyz$

3. Find the critical points of the system $\frac{dx}{dt} = x^2 + 4y + 1,\ \frac{dy}{dt} = x - 2y$.

4. Find all constant solutions of the system $\frac{dx}{dt} = x^2 + y^2 - 4,\ \frac{dy}{dt} = 5x^2 - 2y^2 + 1$.

5. The values of c that make the system $\frac{dx}{dt} = 2x + 2y,\ \frac{dy}{dt} = -cx - 4y$ stable are

   Select the correct answer.

   (a) $c > 4$
   (b) $c < 4$
   (c) It is locally stable for all values of c.
   (d) It is unstable for all values of c.
   (e) $c > 0$

6. The solution of the system $\frac{dx}{dt} = 2x + 2y,\ \frac{dy}{dt} = -5x - 4y$ is

   Select the correct answer.

   (a) $x = 2c_1e^t\cos t + 2c_2e^t\sin t,\ y = c_1e^t(-3\cos t - \sin t) + c_2e^t(\cos t - 3\sin t)$
   (b) $x = -3c_1e^t\cos t - 3c_2e^t\sin t,\ y = c_1e^t(2\cos t - \sin t) + c_2e^t(\cos t + 2\sin t)$
   (c) $x = -3c_1e^{-t}\cos t - 3c_2e^{-t}\sin t,\ y = c_1e^{-t}(2\cos t - \sin t) + c_2e^{-t}(\cos t + 2\sin t)$
   (d) $x = 2c_1e^{-t}\cos t + 2c_2e^{-t}\sin t,\ y = c_1e^{-t}(-3\cos t - \sin t) + c_2e^{-t}(\cos t - 3\sin t)$
   (e) $x = c_1e^{-t}\cos t + c_2e^{-t}\sin t,\ y = c_1e^{-t}(-3\cos t - \sin t) + c_2e^{-t}(\cos t - 3\sin t)$

7. The critical point $(0,0)$ of the system $\frac{dx}{dt} = 2x + 2y$, $\frac{dy}{dt} = -5x - 4y$ is

   Select all that apply.

   (a) asymptotically stable
   (b) stable but not asymptotically stable
   (c) unstable
   (d) an attractor
   (e) a repeller

8. The geometric configuration of the solutions of $\frac{dx}{dt} = 2x + 2y$, $\frac{dy}{dt} = -5x - 4y$ in the phase plane is

   Select the correct answer.

   (a) degenerate stable node
   (b) degenerate unstable node
   (c) stable spiral point
   (d) unstable spiral point
   (e) saddle point

9. Solve the system $\frac{dx}{dt} = -2x + y$, $\frac{dy}{dt} = -3y$.

10. Is the system $\frac{dx}{dt} = -2x+y$, $\frac{dy}{dt} = -3y$ stable or unstable? If stable, is it asymptotically stable? Explain.

11. Describe the geometric configuration of the solutions of $\frac{dx}{dt} = -2x + y$, $\frac{dy}{dt} = -3y$ in the phase plane.

12. The solution of the system $\frac{dx}{dt} = -6x - 4y$, $\frac{dy}{dt} = 4x + 2y$ is

    Select the correct answer.

    (a) $x = -c_1e^{2t} + c_2te^{2t}$, $y = c_1e^{2t} + c_2(-te^{2t} - e^{2t}/4)$
    (b) $x = c_1e^{2t} + c_2te^{2t}$, $y = -c_1e^{2t} + c_2(-te^{2t} - e^{2t}/4)$
    (c) $x = c_1e^{-2t} - c_2te^{-2t}$, $y = -c_1e^{-2t} + c_2(-te^{-2t} - e^{-2t}/4)$
    (d) $x = c_1e^{-2t} - c_2te^{-2t}$, $y = c_1e^{-2t} + c_2(te^{-2t} - e^{-2t}/4)$
    (e) $x = c_1e^{-2t} + c_2te^{-2t}$, $y = -c_1e^{-2t} + c_2(-te^{-2t} - e^{-2t}/4)$

13. The critical point $(0, 0)$ of the system $\frac{dx}{dt} = -6x - 4y$, $\frac{dy}{dt} = 4x + 2y$ is

    Select all that apply.

    (a) asymptotically stable
    (b) stable but not asymptotically stable
    (c) unstable
    (d) an attractor
    (e) a repeller

14. The geometric configuration of the solutions of $\frac{dx}{dt} = -6x - 4y$, $\frac{dy}{dt} = 4x + 2y$ in the phase plane is

    Select the correct answer.

    (a) degenerate stable node
    (b) degenerate unstable node
    (c) stable spiral point
    (d) unstable spiral point
    (e) saddle point

15. Consider the differential equation $x' = \sin x$. The point $x = \pi$ is

    Select the correct answer.

    (a) a stable critical point
    (b) an unstable critical point
    (c) not a critical point

16. Find the critical points of the system $x' = x^2 - y^2 - 48$, $y' = x - 2y$.

17. Find the Jacobian matrix of the system $x' = x^2 - y^2 - 48$, $y' = x - 2y$ at each critical point.

18. Are the critical points of the system $x' = x^2 - y^2 - 48$, $y' = x - 2y$ stable or unstable? Explain.

19. In the previous problem, classify all critical points as node, spiral point, center point, or saddle point.

20. Write down the differential equation for the angle of a nonlinear pendulum, and linearize it about one of its critical points.

ANSWER KEY
Zill Differential Equations 9e Chapter 10 Form H

1. c, d
2. b, c
3. $(-1, -1/2)$
4. $x = 1$, $y = \sqrt{3}$ and $x = 1$, $y = -\sqrt{3}$ and $x = -1$, $y = \sqrt{3}$ and $x = -1$, $y = -\sqrt{3}$
5. a
6. d
7. a, d
8. c
9. $x = c_1 e^{-2t} + c_2 e^{-3t}$, $y = -c_2 e^{-3t}$
10. stable and asymptotically stable (negative eigenvalues)
11. $(0, 0)$ is a stable node.
12. e
13. a, d
14. a
15. a
16. $(8, 4)$ and $(-8, -4)$
17. At $(8, 4)$, $J = \begin{bmatrix} 16 & -8 \\ 1 & -2 \end{bmatrix}$; at $(-8, -4)$, $J = \begin{bmatrix} -16 & 8 \\ 1 & -2 \end{bmatrix}$
18. $(8, 4)$ is unstable (one positive eigenvalue); $(-8, -4)$ is stable (negative eigenvalues)
19. $(8, 4)$ is a saddle point, $(-8, -4)$ is a node.
20. $\frac{d^2\theta}{dt^2} + g \sin\theta / l = 0$, $\frac{d^2\theta}{dt^2} + g\theta / l = 0$