1. Calculate the inner product of $f_1(x) = 1$ and $f_2(x) = x$ on the interval $[0, 3]$. Are the functions orthogonal on the interval?

2. Calculate the inner product of $f_1(x) = \cos x$ and $f_2(x) = \sin x$ on the interval $[0, \pi]$. Are the functions orthogonal on the interval?

3. Show that the set of functions $\{\sin x, \sin(3x), \sin(5x), \ldots\}$ is orthogonal on the interval $[0, \pi/2]$.

4. Is the set of functions $\{\sin(nx)\}$, $n = 1, 2, 3, \ldots$, orthogonal on the interval $[-\pi, \pi]$? Explain.

5. Is the set of functions $\{\sin(nx)\}$, $n = 1, 2, 3, \ldots$, complete on the interval $[-\pi, \pi]$? Explain.

6. What conditions on a function $f$ on an interval ensure that the Fourier Series of $f$ converges to $f$ on the interval?

7. Find the Fourier coefficients of $f(x) = 1 - x$ on $[-1, 1]$.

8. Find the Fourier series of $f(x) = 1 - x$ on $[-1, 1]$.

9. Find the Fourier series of $f(x) = \cos(5x)$ on the interval $[-\pi, \pi]$.

10. Let $f(x) = \left\{ \begin{array}{ccc} 0 & \text{if} & -\pi \le x < 0 \\ \pi - x & \text{if} & 0 \le x \le \pi \end{array} \right\}$. Find the Fourier coefficients of $f$.

11. In the previous problem, what is the Fourier series of $f$?

12. In the previous two problems, to what value does the Fourier series converge at $x = 0$?

13. In the previous three problems, to what value does the Fourier series converge at $x = 1$?

14. In the previous four problems, to what value does the Fourier series converge at $x = \pi$?

15. Let $f(x) = \left\{ \begin{array}{ccc} 0 & \text{if} & -\pi \le x < 0 \\ x^2 & \text{if} & 0 \le x \le \pi \end{array} \right\}$. Find the Fourier coefficients of $f$.

16. In the previous problem, what is the Fourier series of $f$?

17. Use the previous two problems to find an infinite series that converges to $\pi^2/6$.

18. Use the previous three problems to find an infinite series that converges to $\pi^2/12$.

19. Use the previous two problems to find an infinite series that converges to $\pi^2/8$.

20. Let $f(x) = \sin(4x)$ if $0 \le x \le \pi$. Find the Fourier cosine series of $f$.

**ANSWER KEY**

***Zill Differential Equations 9e Chapter 11 Form A***

1. 9/2, no

2. 0, yes

3. $\int_0^{\pi/2} \sin(mx)\sin(nx)dx = (\sin((m-n)x)/(m-n) - \sin((m+n)x)/(m+n))/2 \,|_0^{\pi/2} = 0$, since $m-n$ and $m+n$ are even when $m$ and $n$ are odd.

4. $\int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx = (\sin((m-n)x)/(m-n) - \sin((m+n)x)/(m+n))/2 \,|_{-\pi}^{\pi} = 0$ for all $m \neq n$.

5. No. For example the function $f(x) = 1$ is orthogonal to each element of the set.

6. $f$ is continuous and piecewise smooth on the interval, and its values at the endpoints match.

7. $a_0 = 2$, $a_n = 0$, $b_n = 2(-1)^n/(n\pi)$, $n = 1, 2, 3, \ldots$

8. $f(x) = 1 - x \sim 1 + \sum_{n=1}^{\infty} 2(-1)^n \sin(n\pi x)/(n\pi)$

9. $f(x) = \cos(5x) \sim \cos(5x)$

10. $a_0 = \pi/2$, $a_n = (1 - (-1)^n)/(n^2\pi)$, $b_n = 1/n$

11. $f(x) \sim \pi/4 + \sum_{n=1}^{\infty}[(1 - (-1)^n)\cos(nx)/(n^2\pi) + \sin(nx)/n]$

12. $\pi/2$

13. $\pi - 1$

14. 0

15. $a_0 = \pi^2/3$, $a_n = 2(-1)^n/n^2$, $b_n = ((-1)^n(2 - n^2\pi^2) - 2)/(n^3\pi)$

16. $f(x) \sim \pi^2/6 + \sum_{n=1}^{\infty}[2(-1)^n \cos(nx)/n^2 + ((-1)^n(2 - n^2\pi^2) - 2)\sin(nx)/(n^3\pi)]$

17. Evaluate Fourier series at $x = \pi$, $\pi^2/6 = \sum_{n=1}^{\infty} 1/n^2$

18. Evaluate Fourier series at $x = 0$, $\pi^2/12 = \sum_{n=1}^{\infty}(-1)^{n+1}/n^2$

19. Add the two previous series, $\pi^2/8 = \sum_{n=1}^{\infty} 1/(2n-1)^2$

20. $\sum_{n=1}^{\infty} 16\cos((2n-1)x)/(\pi(16 - (2n-1)^2))$

1. Calculate the inner product of $f_1(x) = x$ and $f_2(x) = x - x^2$ on the interval $[0, 1]$. Are the functions orthogonal on the interval?

2. The functions $f_1(x) = \cos x$ and $f_2(x) = \sin x$ are orthogonal on the interval $[0, \pi]$. Form an orthonormal set from them.

3. The set of functions $\{\sin x, \sin(3x), \sin(5x), \ldots\}$ is orthogonal on the interval $[0, \pi/2]$. Find a corresponding orthonormal set.

4. Is the set of functions $\{\cos(nx)\}$, $n = 1, 2, 3, \ldots$, orthogonal on the interval $[-\pi, \pi]$? Explain.

5. Is the set of functions $\{\cos(nx)\}$, $n = 1, 2, 3, \ldots$, complete on the interval $[-\pi, \pi]$? Explain.

6. What conditions on a function $f$ ensure that the Fourier Series of $f$ converges to $(f(x+) + f(x-))/2$ on an interval?

7. Find the Fourier series of $f(x) = \sin(2x)$ on the interval $[-\pi, \pi]$.

8. Find the Fourier coefficients of $f(x) = x^2$ on $[-1, 1]$.

9. Find the Fourier series of $f(x) = x^2$ on $[-1, 1]$.

10. Let $f(x) = \left\{ \begin{array}{ccc} x & \text{if} & -1 \leq x < 0 \\ 1 - x & \text{if} & 0 \leq x \leq 1 \end{array} \right\}$. Find the Fourier coefficients of $f$.

11. In the previous problem, what is the Fourier series of $f$?

12. In the previous two problems, to what value does the Fourier series converge at $x = 0$?

13. In the previous three problems, to what value does the Fourier series converge at $x = 1/2$?

14. In the previous four problems, to what value does the Fourier series converge at $x = 1$?

15. Find the Fourier coefficients of $f(x) = \pi + x$ on $[-\pi, \pi]$.

16. Find the Fourier series of $f(x) = \pi + x$ on $[-\pi, \pi]$.

17. Use the previous problem to find an infinite series that converges to $\pi/4$

18. Write down the general form of a regular Sturm-Liouville problem. State all conditions that are necessary for the problem to be regular.

19. State four properties of a regular Sturm-Liouville problem.

20. Define the term "separated boundary condition". Give an example.

**ANSWER KEY**
***Zill Differential Equations 9e Chapter 11 Form B***

1. 1/12, no
2. $\{\sqrt{2/\pi}\cos x,\ \sqrt{2/\pi}\sin x\}$
3. $\{2\sin x/\sqrt{\pi},\ 2\sin(3x)/\sqrt{\pi},\ 2\sin(5x)/\sqrt{\pi},\ \ldots\}$
4. Yes, $\int_{-\pi}^{\pi}\cos(nx)\cos(mx)dx = 0$ for positive integers $n \neq m$
5. No, for example, $f(x) = 1$ is orthogonal to each function in the set.
6. $f$ and $f'$ are piecewise continuous on the interval
7. $f(x) \sim \sin(2x)$
8. $a_0 = 2/3$, $a_n = 4(-1)^n/(n\pi)^2$, $b_n = 0$
9. $f(x) \sim 1/3 + \sum_{n=1}^{\infty} 4(-1)^n \cos(n\pi x)/(n\pi)^2$
10. $a_0 = 0$, $a_n = 2(1-(-1)^n)/(n\pi)^2$, $b_n = (1-(-1)^n)/(n\pi)$
11. $f(x) \sim \sum_{n=1}^{\infty}[2(1-(-1)^n)\cos(n\pi x)/(n\pi)^2 + (1-(-1)^n)\sin(n\pi x)/(n\pi)]$
12. 1/2
13. 1/2
14. $-1/2$
15. $a_0 = 2\pi$, $a_n = 0$, $b_n = 2(-1)^{n+1}/n$
16. $f(x) \sim \pi + \sum_{n=1}^{\infty} 2(-1)^{n+1}\sin(nx)/n$
17. Evaluate the previous series at $x = \pi/2$, $\pi/4 = \sum_{n=1}^{\infty}(-1)^{n+1}/(2n-1)$
18. $\frac{d}{dx}[r(x)\frac{dy}{dx}] + (q(x) + \lambda p(x))y = 0$, $A_1 y(a) + B_1 y'(a) = 0$, $A_2 y(b) + B_2 y'(b) = 0$; $p$, $q$, $r$, $r'$ are continuous on $[a, b]$; $r > 0$, $p > 0$ on $[a, b]$; $A_1^2 + B_1^2 \neq 0$, $A_2^2 + B_2^2 \neq 0$
19. (1) There is an infinite number of real eigenvalues which can be ordered, $\lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_n < \ldots$, such that $\lambda_n \to \infty$ as $n \to \infty$. (2) For each eigenvalue, there is only one (up to constant multiples) eigenfunction. (3) Eigenfunctions corresponding to distinct eigenvalues are linearly independent. (4) The set of eigenfunctions corresponding to the set of eigenvalues is orthogonal with respect to the weight function $p$ on $[a, b]$.
20. Separated boundary conditions are boundary conditions that apply at only one point. For example, $y(0) = 0$, $y'(1) = 0$, but not $y(0) + y'(1) = 0$.

1. The square norm of the function $f(x) = \sin x$ on the interval $[0, \pi]$ is

   Select the correct answer.

   (a) 1
   (b) $\pi$
   (c) $\pi/2$
   (d) $\pi/4$
   (e) 0

2. The square norm of the function $f(x) = 1 - x$ on the interval $[0, 2]$ is

   Select the correct answer.

   (a) 2/3
   (b) $\sqrt{2/3}$
   (c) 1/3
   (d) $\sqrt{1/3}$
   (e) 0

3. The Fourier Series of a function $f$ defined on $[-p, p]$ is

   $f(x) = a_0/2 + \sum_{n=1}^{\infty}(a_n \cos(n\pi x/p) + b_n \sin(n\pi x/p))$ where

   Select all that apply.

   (a) $a_0 = \int_{-p}^{p} f(x)dx/p$
   (b) $a_n = \int_{-p}^{p} f(x)\cos(n\pi x/p)dx/p$
   (c) $a_n = \int_{-p}^{p} f(x)\sin(n\pi x/p)dx/p$
   (d) $b_n = \int_{-p}^{p} f(x)\cos(n\pi x/p)dx/p$
   (e) $b_n = \int_{-p}^{p} f(x)\sin(n\pi x/p)dx/p$

4. In order to be assured by a theorem that the Fourier Series of $f$ on $[a, b]$ converges to $f$, which of the following conditions need to be satisfied?

   Select all that apply.

   (a) $f$ is continuous on $[a, b]$
   (b) $f'$ is continuous on $[a, b]$
   (c) $f$ is piecewise continuous on $[a, b]$
   (d) $f'$ is piecewise continuous on $[a, b]$
   (e) $f$ is integrable on $[a, b]$

5. The function $f(x) = \left\{ \begin{array}{lll} 0 & \text{if} & x < 0 \\ 1 & \text{if} & x > 0 \end{array} \right\}$ has a Fourier series on $[-2, 2]$ that converges at $x = 0$ to

   Select the correct answer.

   (a) 0
   (b) 1
   (c) 1/2
   (d) −1
   (e) unknown

6. The function $f(x) = \left\{ \begin{array}{l} 0 \text{ if } x < 0 \\ 1 \text{ if } x > 0 \end{array} \right\}$ has a Fourier series on $[-2, 2]$ that converges at $x = 1$ to

   Select the correct answer.

   (a) 0
   (b) 1
   (c) 1/2
   (d) −1
   (e) unknown

7. The Fourier series of an odd function might

   Select all that apply.

   (a) contain sine terms
   (b) contain cosine terms
   (c) contain a constant term
   (d) contain sine and cosine terms
   (e) contain sine, cosine, and constant terms

8. The function $f(x) = |x|$ is

   Select all that apply.

   (a) odd
   (b) even
   (c) neither even nor odd
   (d) continuous on $[-\pi, \pi]$
   (e) discontinuous on $[-\pi, \pi]$

9. The Fourier series of the function $f(x) = |x|$ on $[-2, 2]$

   Select all that apply.

   (a) contains cosine terms
   (b) contains sine terms
   (c) contains sine and cosine terms
   (d) contains a constant term
   (e) contains sine, cosine, and constant terms

10. The Fourier coefficients of the function $f(x) = x$ on $[-1, 1]$ are

    Select the correct answer.

    (a) $a_0 = -2/\pi$, $b_n = 0$, $a_n = 0$
    (b) $a_0 = 0$, $b_n = (-1)^n\, 2/(n\pi)$, $a_n = 0$
    (c) $a_0 = 0$, $b_n = (-1)^{n+1} 2/(n\pi)$, $a_n = 0$
    (d) $a_0 = 0$, $b_n = 0$, $a_n = (-1)^n\, 2/(n\pi)$
    (e) $a_0 = 0$, $b_n = 0$, $a_n = (-1)^{n+1}\, 2/(n\pi)$

11. The Fourier series of the function $f(x) = x$ on $[-1, 1]$ is

    Select the correct answer.

    (a) $\sum_{n=0}^{\infty}(-1)^n\, 2\cos(n\pi x)/(n\pi)$
    (b) $\sum_{n=0}^{\infty}(-1)^{n+1}\, 2\cos(n\pi x)/(n\pi)$
    (c) $\sum_{n=1}^{\infty}(-1)^n\, 2\sin(n\pi x)/(n\pi)$
    (d) $\sum_{n=1}^{\infty}(-1)^{n+1}\, 2\sin(n\pi x)/(n\pi)$
    (e) $\sum_{n=1}^{\infty}(-1)^{n+1}\, \sin(n\pi x)/(n\pi)$

12. Consider the differential equation $y'' + \lambda y = 0$. Examples of boundary conditions for this equation that make a regular Sturm-Liouville problem are

    Select all that apply.

    (a) $y(0) = 0$, $y(1) = 0$
    (b) $y(0) = 0$, $y'(1) = 0$
    (c) $y(0) = 0$, $y'(0) = 0$
    (d) $y'(0) = 0$, $y(1) + y'(1) = 0$
    (e) $y(0) + y(1) = 0$, $y'(1) = 0$

13. The problem $\frac{d}{dx}[r(x)y']+(q(x)+\lambda p(x))y=0$, $A_1y(a)+B_1y'(a)=0$, $A_2y(b)+B_2y'(b)=0$ is a regular Sturm-Liouville problem under certain conditions, including

    Select all that apply.

    (a) $p$, $q$, $r$ are continuous on $[a,b]$
    (b) $r(x)>0$ and $p(x)<0$ on $[a,b]$
    (c) $r(x)<0$ and $p(x)>0$ on $[a,b]$
    (d) $A_1^2+B_1^2\neq 0$
    (e) $A_1A_2\neq 0$

14. The problem $\frac{d}{dx}[r(x)y']+(q(x)+\lambda p(x))y=0$, $A_1y(a)+B_1y'(a)=0$, $A_2y(b)+B_2y'(b)=0$ is not a regular Sturm-Liouville problem under which of the following conditions.

    Select all that apply.

    (a) $r=1/(x-a)$ on $[a,b]$
    (b) $p(x)=x-a$ on $[a,b]$
    (c) $q(x)=0$ on $[a,b]$
    (d) $A_1A_2=0$
    (e) $A_1^2+B_1^2=0$

15. Which of the following differential equations are in self-adjoint form?

    Select all that apply.

    (a) $r'(x)y''+r(x)y'+\lambda y=0$
    (b) $y''+y'+\lambda y=0$
    (c) $(1-x^2)y''-2xy'+\lambda y=0$
    (d) $x^2y''+xy'+(x^2-n^2)y=0$
    (e) $y''+\lambda y=0$

16. The solution of the eigenvalue problem $y''+\lambda y=0$, $y(0)=0$, $y(1)=0$ is

    Select the correct answer.

    (a) $\lambda=n\pi$, $y=\cos(n\pi x)$, $n=0,1,2,\ldots$
    (b) $\lambda=n\pi$, $y=\sin(n\pi x)$, $n=1,2,3,\ldots$
    (c) $\lambda=n^2\pi^2$, $y=\cos(n\pi x)$, $n=0,1,2,\ldots$
    (d) $\lambda=n^2\pi^2$, $y=\sin(n\pi x)$, $n=1,2,3,\ldots$
    (e) $\lambda=n\pi$, $y=\cos(n\pi x)+\sin(n\pi x)$, $n=1,2,3,\ldots$

17. The solution of the eigenvalue problem $y'' + \lambda y = 0$, $y'(0) = 0$, $y'(\pi) = 0$ is

    Select the correct answer.

    (a) $\lambda = n$, $y = \cos(nx)$, $n = 0, 1, 2, \ldots$
    (b) $\lambda = n$, $y = \sin(nx)$, $n = 1, 2, 3, \ldots$
    (c) $\lambda = n^2$, $y = \cos(nx)$, $n = 0, 1, 2, \ldots$
    (d) $\lambda = n^2$, $y = \sin(nx)$, $n = 1, 2, 3, \ldots$
    (e) $\lambda = n$, $y = \cos(nx) + \sin(nx)$, $n = 1, 2, 3, \ldots$

18. The solution of the eigenvalue problem $rR'' + R' + r\lambda R = 0$, $R(0)$ is bounded , $R'(3) = 0$ is $(J_0'(z_n) = 0)$

    Select the correct answer.

    (a) $\lambda = z_n^2$, $R = J_0(z_n r)$, $n = 1, 2, 3, \ldots$
    (b) $\lambda = z_n^2/9$, $R = J_0(z_n r/3)$, $n = 1, 2, 3, \ldots$
    (c) $\lambda = z_n$, $R = J_0(z_n r)$, $n = 1, 2, 3, \ldots$
    (d) $\lambda = z_n/3$, $R = J_0(z_n r/3)$, $n = 1, 2, 3, \ldots$
    (e) $\lambda = z_n$, $R = J_0(r)$, $n = 1, 2, 3, \ldots$

19. Consider the parameterized Bessel's differential equation $x^2y'' + xy' + (\alpha^2 x^2 - n^2)y = 0$ along with the conditions $y(0)$ is bounded, $y(2) = 0$. The solution of this eigenvalue problem is $(J_n(z_n) = 0)$

    Select the correct answer.

    (a) $\alpha = z_n/2$, $y = J_n(z_n x/2)$, $n = 1, 2, 3, \ldots$
    (b) $\alpha = z_n^2/4$, $y = J_n(z_n x/2)$, $n = 1, 2, 3, \ldots$
    (c) $\alpha = z_n$, $y = J_n(\sqrt{z_n/2}x)$, $n = 1, 2, 3, \ldots$
    (d) $\alpha = z_n/2$, $y = J_n(\sqrt{z_n/2}x)$, $n = 1, 2, 3, \ldots$
    (e) $\alpha = z_n^2/4$, $y = J_n(\sqrt{z_n/2}x)$, $n = 1, 2, 3, \ldots$

20. Using the eigenfunctions of the previous problem, written as $g_n(x)$, the Fourier-Bessel series for the function $f(x)$ is $\sum_{n=1}^{\infty} c_n g_n(x)$, where

    Select the correct answer.

    (a) $c_n = \int_0^2 f(x)g_n(x)dx / \int_0^2 g_n^2(x)dx$
    (b) $c_n = \int_0^2 xf(x)g_n(x)dx / \int_0^2 g_n^2(x)dx$
    (c) $c_n = \int_0^2 f(x)g_n(x)dx / \int_0^2 xg_n^2(x)dx$
    (d) $c_n = \int_0^2 xf(x)g_n(x)dx / \int_0^2 xg_n^2(x)dx$
    (e) $c_n = \int_0^2 x^2f(x)g_n(x)dx / \int_0^2 x^2g_n^2(x)dx$

**ANSWER KEY**
***Zill Differential Equations 9e Chapter 11 Form C***

1. c
2. a
3. a, b, e
4. a, c, d, e
5. c
6. b
7. a
8. b, d
9. a, d
10. c
11. d
12. a, b, d
13. a, d
14. a, b, e
15. c, e
16. d
17. c
18. b
19. a
20. d

1. The square norm of the function $f(x) = \cos(3x)$ on the interval $[0, \pi/2]$ is

   Select the correct answer.

   (a) 1
   (b) $\pi$
   (c) $\pi/2$
   (d) $\pi/4$
   (e) 0

2. The square norm of the function $f(x) = x^2$ on the interval $[0, 1]$ is

   Select the correct answer.

   (a) 1/2
   (b) 1/3
   (c) 1/5
   (d) 1
   (e) 0

3. The Fourier Series of a $f(x) = x$ defined on $[-1, 1]$ is
   $f(x) = a_0/2 + \sum_{n=1}^{\infty}(a_n \cos(n\pi x) + b_n \sin(n\pi x))$ where

   Select all that apply.

   (a) $a_0 = 1$
   (b) $a_n = 0$
   (c) $a_n = \int_{-1}^{1} x \sin(n\pi x)dx$
   (d) $b_n = 0$
   (e) $b_n = \int_{-1}^{1} x \sin(n\pi x)dx$

4. In order to be assured by a theorem that the Fourier Series of $f$ on $[a, b]$ converges at $x$ to $(f(x+) + f(x-))/2$, which of the following conditions need to be satisfied?

   Select all that apply.

   (a) $f$ is continuous on $[a, b]$
   (b) $f'$ is continuous on $[a, b]$
   (c) $f$ is piecewise continuous on $[a, b]$
   (d) $f'$ is piecewise continuous on $[a, b]$
   (e) $f$ is integrable on $[a, b]$

5. The function $f(x) = \left\{ \begin{array}{ccc} x & \text{if} & x < 0 \\ 2+5x & \text{if} & x > 0 \end{array} \right\}$ has a Fourier series on $[-2, 2]$ that converges at $x = 0$ to

   Select the correct answer.

   (a) 0
   (b) 1
   (c) 1/2
   (d) 2
   (e) unknown

6. The function $f(x) = \left\{ \begin{array}{ccc} x & \text{if} & x < 0 \\ 2+5x & \text{if} & x > 0 \end{array} \right\}$ has a Fourier series on $[-2, 2]$ that converges at $x = 1$ to

   Select the correct answer.

   (a) 7
   (b) 1
   (c) 1/2
   (d) $-3$
   (e) unknown

7. The Fourier series of an even function might

   Select all that apply.

   (a) contain sine terms
   (b) contain cosine terms
   (c) contain a constant term
   (d) contain sine and cosine terms
   (e) contain sine, cosine, and constant terms

8. The function $f(x) = \left\{ \begin{array}{ccc} x & \text{if} & x < 0 \\ 2-x & \text{if} & x > 0 \end{array} \right\}$ is

   Select all that apply.

   (a) odd
   (b) even
   (c) neither even nor odd
   (d) continuous on $[-\pi, \pi]$
   (e) discontinuous on $[-\pi, \pi]$

9. The Fourier series of the function $f(x) = \left\{ \begin{array}{r} x \text{ if } x < 0 \\ 2 - x \text{ if } x > 0 \end{array} \right\}$ on $[-2, 2]$

   Select all that apply.

   (a) contains only cosine terms
   (b) contains only sine terms
   (c) contains sine and cosine terms
   (d) contains a constant term
   (e) contains sine, cosine, and constant terms

10. The Fourier coefficients of the function $f(x) = x^2$ on $[-1, 1]$ are

    Select all that apply.

    (a) $a_0 = 2/3$
    (b) $a_n = 0$
    (c) $a_n = 4(-1)^n/(n^2\pi^2)$
    (d) $b_n = 0$
    (e) $b_n = 4(-1)^n/(n^2\pi^2)$

11. The Fourier series of the function $f(x) = x^2$ on $[-1, 1]$ is

    Select the correct answer.

    (a) $\sum_{n=1}^{\infty} 4(-1)^n \sin(n\pi x)/(n^2\pi^2) + \sum_{n=1}^{\infty} 4(-1)^n \cos(n\pi x)/(n^2\pi^2)$
    (b) $\sum_{n=1}^{\infty} 4(-1)^n \sin(n\pi x)/(n^2\pi^2)$
    (c) $\sum_{n=1}^{\infty} 4(-1)^n \cos(n\pi x)/(n^2\pi^2)$
    (d) $1/3 + \sum_{n=1}^{\infty} 4(-1)^n \sin(n\pi x)/(n^2\pi^2)$
    (e) $1/3 + \sum_{n=1}^{\infty} 4(-1)^n \cos(n\pi x)/(n^2\pi^2)$

12. The solution of the eigenvalue problem $y'' + \lambda y = 0$, $y(0) = 0$, $y(\pi) = 0$ is

    Select the correct answer.

    (a) $\lambda = n^2$, $y = \cos(nx)$, $n = 0, 1, 2, \ldots$
    (b) $\lambda = n^2$, $y = \sin(nx)$, $n = 1, 2, 3, \ldots$
    (c) $\lambda = n$, $y = \cos(nx)$, $n = 0, 1, 2, \ldots$
    (d) $\lambda = n$, $y = \sin(nx)$, $n = 1, 2, 3, \ldots$
    (e) $\lambda = n$, $y = \cos(nx) + \sin(nx)$, $n = 1, 2, 3, \ldots$

13. An example of a regular Sturm-Liouville problem is $y'' + \lambda y = 0$ with boundary conditions

    Select all that apply.

    (a) $y(0) + y'(1) = 0$, $y(1) = 0$
    (b) $y(0) = 0$, $y'(1) = 0$
    (c) $y(1) = 0$, $y'(1) = 0$
    (d) $y'(0) = 0$, $y(1) + y'(1) = 0$
    (e) $y$ is bounded on $[-1, 1]$

14. The problem $\frac{d}{dx}[r(x)y'] + (q(x) + \lambda p(x))y = 0$, $A_1 y(a) + B_1 y'(a) = 0$, $A_2 y(b) + B_2 y'(b) = 0$ is a regular Sturm-Liouville problem under certain conditions, including

    Select all that apply.

    (a) $p$, $q$, $r$ are piecewise continuous on $[a, b]$
    (b) $r(x) > 0$ and $p(x) > 0$ on $[a, b]$
    (c) $r(x) < 0$ and $p(x) > 0$ on $[a, b]$
    (d) $A_1 B_1 \neq 0$
    (e) $A_1^2 + B_1^2 \neq 0$

15. The problem $\frac{d}{dx}[r(x)y'] + (q(x) + \lambda p(x))y = 0$, $A_1 y(a) + B_1 y'(a) = 0$, $A_2 y(b) + B_2 y'(b) = 0$ is not a regular Sturm-Liouville problem under which of the following conditions.

    Select all that apply.

    (a) $r = 1/(x - a)$ on $(a, b)$
    (b) $q(x) = 0$ on $[a, b]$
    (c) $p(x) = x - a$ on $[a, b]$
    (d) $A_1 A_2 = 0$
    (e) $A_1^2 + B_1^2 = 0$

16. Which of the following differential equations are in self-adjoint form?

    Select all that apply.

    (a) $r(x)y'' + r'(x)y' + \lambda y = 0$
    (b) $y'' + y' + \lambda y = 0$
    (c) $(1 - x^2)y'' - 2xy' + \lambda y = 0$
    (d) $y'' + \lambda y = 0$
    (e) $x^2 y'' + xy' + (x^2 - n^2)y = 0$

17. The solution of the eigenvalue problem $y'' + \lambda y = 0$, $y(0) = 0$, $y'(1) = 0$ is

    Select the correct answer.

    (a) $\lambda = (n - 1/2)\pi$, $y = \cos((n - 1/2)\pi x) + \sin((n - 1/2)\pi x)$, $n = 1, 2, 3, \ldots$
    (b) $\lambda = (n - 1/2)\pi$, $y = \cos((n - 1/2)\pi x)$, $n = 1, 2, 3, \ldots$
    (c) $\lambda = (n - 1/2)\pi$, $y = \sin((n - 1/2)\pi x)$, $n = 1, 2, 3, \ldots$
    (d) $\lambda = (n - 1/2)^2\pi^2$, $y = \cos((n - 1/2)\pi x)$, $n = 1, 2, 3, \ldots$
    (e) $\lambda = (n - 1/2)^2\pi^2$, $y = \sin((n - 1/2)\pi x)$, $n = 1, 2, 3, \ldots$

18. The differential equation $(1 - x^2)y'' - 2xy' + \lambda y = 0$ is

    Select the correct answer.

    (a) Legendre's equation
    (b) Bessel's equation
    (c) the Fourier-Bessel
    (d) the hypergeometric
    (e) none of the above

19. The solution of the eigenvalue problem $(1 - x^2)y'' - 2xy' + \lambda y = 0$, where $y$ is bounded on $[-1, 1]$, is

    Select the correct answer.

    (a) $\lambda = n$, $y = P_n(x)$, $n = 1, 2, 3, \ldots$
    (b) $\lambda = n - 1$, $y = P_n(x)$, $n = 1, 2, 3, \ldots$
    (c) $\lambda = n + 1$, $y = P_n(x)$, $n = 1, 2, 3, \ldots$
    (d) $\lambda = n^2$, $y = P_n(x)$, $n = 1, 2, 3, \ldots$
    (e) $\lambda = n(n + 1)$, $y = P_n(x)$, $n = 1, 2, 3, \ldots$

20. Using the eigenfunctions of the previous problem, the Fourier-Legendre series for the function $f(x)$ is $\sum_{n=1}^{\infty} c_n P_n(x)$, where

    Select the correct answer.

    (a) $c_n = (2n + 1)\int_{-1}^{1} xf(x)P_n(x)dx$
    (b) $c_n = (2n + 1)\int_{-1}^{1} f(x)P_n(x)dx$
    (c) $c_n = (2n + 1)\int_{-1}^{1} f(x)P_n(x)dx/2$
    (d) $c_n = (2n - 1)\int_{-1}^{1} f(x)P_n(x)dx/2$
    (e) $c_n = (2n - 1)\int_{-1}^{1} xf(x)P_n(x)dx$

**ANSWER KEY**
***Zill Differential Equations 9e Chapter 11 Form D***

1. d
2. c
3. b, e
4. c, d, e
5. b
6. a
7. b, c
8. c, e
9. c
10. a, c, d
11. e
12. b
13. b, d
14. b, e
15. a, c, e
16. a, c, d
17. e
18. a
19. e
20. c

1. Calculate the inner product of $f_1(x) = x^2$ and $f_2(x) = x^3 - 1$ on the interval $[0, 1]$. Are the functions orthogonal on the interval?

2. Calculate the inner product of $f_1(x) = \cos x$ and $f_2(x) = \cos(2x)$ on the interval $[0, \pi]$. Are the functions orthogonal on the interval?

3. The Fourier Series of a function $f$ defined on $[-p, p]$ is

   $f(x) = a_0/2 + \sum_{n=1}^{\infty}(a_n \cos(n\pi x/p) + b_n \sin(n\pi x/p))$ where

   Select all that apply.

   (a) $a_0 = \int_{-p}^{p} f(x)dx/p$
   (b) $b_n = \int_{-p}^{p} f(x)\cos(n\pi x/p)dx/p$
   (c) $b_n = \int_{-p}^{p} f(x)\sin(n\pi x/p)dx/p$
   (d) $a_n = \int_{-p}^{p} f(x)\cos(n\pi x/p)dx/p$
   (e) $a_n = \int_{-p}^{p} f(x)\sin(n\pi x/p)dx/p$

4. In order to be assured by a theorem that the Fourier Series of $f$ on $[a, b]$ converges to $f$, which of the following conditions need to be satisfied?

   Select all that apply.

   (a) $f$ is continuous on $[a, b]$
   (b) $f'$ is continuous on $[a, b]$
   (c) $f$ is piecewise continuous on $[a, b]$
   (d) $f'$ is piecewise continuous on $[a, b]$
   (e) $f$ is integrable on $[a, b]$

5. Find the Fourier coefficients of $f(x) = x - x^2$ on $[-1, 1]$.

6. Find the Fourier series of $f(x) = x - x^2$ on $[-1, 1]$.

7. Let $f(x) = \cos(3x)$ if $0 \le x \le \pi$. Find the Fourier sine series of $f$.

8. Let $f(x) = \cos(3x)$ if $0 \le x \le \pi$. Find the Fourier cosine series of $f$.

9. The function $f(x) = \left\{ \begin{array}{lll} 1+x & \text{if} & x<0 \\ 1-x & \text{if} & x>0 \end{array} \right\}$ is

   Select all that apply.

   (a) odd
   (b) even
   (c) neither even nor odd
   (d) continuous on $[-\pi, \pi]$
   (e) discontinuous on $[-\pi, \pi]$

10. The Fourier series of the function $f(x) = \left\{ \begin{array}{lll} 1+x & \text{if} & x<0 \\ 1-x & \text{if} & x>0 \end{array} \right\}$ on $[-1, 1]$

    Select all that apply.

    (a) contains only cosine terms
    (b) contains only sine terms
    (c) contains sine and cosine terms
    (d) contains a constant term
    (e) contains sine, cosine, and constant terms

11. The solution of the eigenvalue problem $y'' + \lambda y = 0$, $y'(0) = 0$, $y'(\pi) = 0$ is

    Select the correct answer.

    (a) $\lambda = n^2$, $y = \cos(nx)$, $n = 0, 1, 2, \ldots$
    (b) $\lambda = n^2$, $y = \sin(nx)$, $n = 1, 2, 3, \ldots$
    (c) $\lambda = n$, $y = \cos(nx)$, $n = 0, 1, 2, \ldots$
    (d) $\lambda = n$, $y = \sin(nx)$, $n = 1, 2, 3, \ldots$
    (e) $\lambda = n$, $y = \cos(nx) + \sin(nx)$, $n = 1, 2, 3, \ldots$

12. The solution of the eigenvalue problem $y'' + \lambda y = 0$, $y'(0) = 0$, $y(1) = 0$ is

    Select the correct answer.

    (a) $\lambda = (n - 1/2)\pi$, $y = \cos((n - 1/2)\pi x) + \sin((n - 1/2)\pi x)$, $n = 1, 2, 3, \ldots$
    (b) $\lambda = (n - 1/2)\pi$, $y = \cos((n - 1/2)\pi x)$, $n = 1, 2, 3, \ldots$
    (c) $\lambda = (n - 1/2)\pi$, $y = \sin((n - 1/2)\pi x)$, $n = 1, 2, 3, \ldots$
    (d) $\lambda = (n - 1/2)^2\pi^2$, $y = \cos((n - 1/2)\pi x)$, $n = 1, 2, 3, \ldots$
    (e) $\lambda = (n - 1/2)^2\pi^2$, $y = \sin((n - 1/2)\pi x)$, $n = 1, 2, 3, \ldots$

13. The solution of the eigenvalue problem $rR'' + R' + r\lambda R = 0$, $R(0)$ is bounded, $R(4) = 0$ is ($J_0(z_n) = 0$)

    Select the correct answer.

    (a) $\lambda = z_n$, $y = J_0(\sqrt{\lambda} r)$, $n = 1, 2, 3, \ldots$
    (b) $\lambda = z_n/4$, $y = J_0(\sqrt{\lambda} r)$, $n = 1, 2, 3, \ldots$
    (c) $\lambda = z_n^2$, $y = J_0(\sqrt{\lambda} r)$, $n = 1, 2, 3, \ldots$
    (d) $\lambda = z_n^2/4$, $y = J_0(\sqrt{\lambda} r)$, $n = 1, 2, 3, \ldots$
    (e) $\lambda = z_n^2/16$, $y = J_0(\sqrt{\lambda} r)$, $n = 1, 2, 3, \ldots$

14. The differential equation $(1-x^2)y'' - 2xy' + \lambda y = 0$ is

    Select the correct answer.

    (a) Bessel's equation
    (b) Legendre's equation
    (c) the Fourier-Bessel
    (d) the hypergeometric
    (e) none of the above

15. The solution of the eigenvalue problem $(1-x^2)y'' - 2xy' + \lambda y = 0$, $y$ is bounded on $[-1, 1]$ is

    Select the correct answer.

    (a) $\lambda = n^2$, $y = P_n(x)$, $n = 1, 2, 3, \ldots$
    (b) $\lambda = n$, $y = P_n(x)$, $n = 1, 2, 3, \ldots$
    (c) $\lambda = n(n+1)$, $y = P_n(x)$, $n = 1, 2, 3, \ldots$
    (d) $\lambda = n - 1$, $y = P_n(x)$, $n = 1, 2, 3, \ldots$
    (e) $\lambda = n + 1$, $y = P_n(x)$, $n = 1, 2, 3, \ldots$

16. Using the eigenfunctions of the previous problem, the Fourier-Legendre series for the function $f(x)$ is $\sum_{n=1}^{\infty} c_n P_n(x)$, where

    Select the correct answer.

    (a) $c_n = (2n+1)\int_{-1}^{1} x f(x) P_n(x) dx$
    (b) $c_n = (2n+1)\int_{-1}^{1} f(x) P_n(x) dx/2$
    (c) $c_n = (2n+1)\int_{-1}^{1} f(x) P_n(x) dx$
    (d) $c_n = (2n-1)\int_{-1}^{1} f(x) P_n(x) dx/2$
    (e) $c_n = (2n-1)\int_{-1}^{1} x f(x) P_n(x) dx$

17. What is the name of the differential equation $x^2y'' + xy' + (\alpha^2x^2 - n^2)y = 0$?

18. What is the solution of the differential equation $x^2y'' + xy' + (\alpha^2x^2 - n^2)y = 0$ with conditions $y(0)$ is bounded, $y'(4) = 0$?

19. For the Sturm-Liouville problem of the previous problem, what is the weighting function?

20. Using the eigenfunctions of the previous two problems, what is the Fourier-Bessel expansion of a function $f(x)$?

**ANSWER KEY**
***Zill Differential Equations 9e Chapter 11 Form E***

1. $-1/6$, no
2. 0, yes
3. a, c, d
4. a, c, d, e
5. $a_0 = -2/3$, $a_n = 4(-1)^{n+1}/(n\pi)^2$, $b_n = 2(-1)^{n+1}/(n\pi)$
6. $f(x) \sim -1/3 + \sum_{n=1}^{\infty}[4(-1)^{n+1}\cos(n\pi x)/(n\pi)^2 + 2(-1)^{n+1}\sin(n\pi x)/(n\pi)]$
7. $f(x) \sim \sum_{n=1}^{\infty} 2n(1+(-1)^n)\sin(nx)/(\pi(n^2-9))$
8. $f(x) \sim \cos(3x)$
9. b, d
10. d
11. a
12. d
13. e
14. b
15. c
16. b
17. parameterized Bessel's differential equation of order n
18. $y = J_0(\alpha x)$ where $\alpha = z_n/4$, and $J_0'(z_n) = 0$
19. $w(x) = x$
20. $f(x) \sim \sum_{n=1}^{\infty} c_n J_0(\alpha x)$, where $c_n = \int_0^4 x f(x) J_0(\alpha x)dx / \int_0^4 x J_0^2(\alpha x)dx$

1. The norm of the function $f(x) = x - x^2$ on the interval $[0, 2]$ is
   Select the correct answer.
   (a) 0
   (b) 1
   (c) $1/\sqrt{6}$
   (d) $\sqrt{2/3}$
   (e) $\sqrt{16/15}$

2. The norm of the function $f(x) = x - 1$ on the interval $[0, 3]$ is
   Select the correct answer.
   (a) $3/2$
   (b) $\sqrt{3/2}$
   (c) $\sqrt{3}$
   (d) 3
   (e) 1

3. Is the set of functions $\{x^n\}$, $n = 0, 1, 2, \ldots$, orthogonal on the interval $[0, 1]$? Explain.

4. The function $f(x) = \left\{ \begin{array}{ccc} 3 & \text{if} & -2 < x < 0 \\ 1 & \text{if} & 0 < x < 2 \end{array} \right\}$ has a Fourier series on $[-2, 2]$ that converges at $x = 0$ to
   Select the correct answer.
   (a) 3
   (b) 2
   (c) 1
   (d) 0
   (e) unknown

5. The function $f(x) = \left\{ \begin{array}{ccc} 3 & \text{if} & -2 < x < 0 \\ 1 & \text{if} & 0 < x < 2 \end{array} \right\}$ has a Fourier series on $[-2, 2]$ that converges at $x = 1$ to
   Select the correct answer.
   (a) 0
   (b) 1
   (c) 2
   (d) 3
   (e) unknown

6. Let $f(x) = e^x$ if $0 \leq x \leq 2$. Find the Fourier sine series of $f$.

7. Let $f(x) = e^x$ if $0 \le x \le 2$. Find the Fourier cosine series of $f$.

8. Solve the eigenvalue problem $y'' + \lambda y = 0$, $y(0) = 0$, $y'(1) = 0$.

9. Using the eigenfunctions of the previous problem, what is the eigenfunction expansion of a function $f(x)$ on $[0, 1]$?

10. Using the eigenfunctions of the previous two problems, what is the eigenfunction expansion of the function $f(x) = x$ on $[0, 1]$?

11. Solve the eigenvalue problem $y'' + \lambda y = 0$, $y'(0) = 0$, $y'(\pi) = 0$.

12. Using the eigenfunctions of the previous problem, what is the eigenfunction expansion of a function $f(x)$ on $[0, \pi]$?

13. Using the eigenfunctions of the previous two problems, what is the eigenfunction expansion of the function $f(x) = x$ on $[0, \pi]$?

14. An example of a regular Sturm-Liouville problem is $y'' + \lambda y = 0$ with boundary conditions

    Select all that apply.

    (a) $y(0) = 0$, $y'(0) = 0$
    (b) $y'(0) = 0$, $y(1) + y'(1) = 0$
    (c) $y(0) + y(1) = 0$, $y'(1) = 0$
    (d) $y(0) = 0$, $y(1) = 0$
    (e) $y(0) = 0$, $y'(1) = 0$

15. The problem $\frac{d}{dx}[r(x)y'] + (q(x) + \lambda p(x))y = 0$, $A_1 y(a) + B_1 y'(a) = 0$, $A_2 y(b) + B_2 y'(b) = 0$ is a regular Sturm-Liouville problem under certain conditions, including

    Select all that apply.

    (a) $r(a) = 0$
    (b) $p(b) = 0$
    (c) $p$, $q$, $r$ are continuous on $[a, b]$
    (d) $r(x) > 0$ and $p(x) > 0$ on $[a, b]$
    (e) $A_1^2 + B_1^2 = 0$

16. The problem $\frac{d}{dx}[r(x)y'] + (q(x) + \lambda p(x))y = 0$, $A_1 y(a) + B_1 y'(a) = 0$, $A_2 y(b) + B_2 y'(b) = 0$ is not a regular Sturm-Liouville problem under which of the following conditions.

    Select all that apply.

    (a) $A_1 A_2 = 0$
    (b) $A_2^2 + B_1^2 = 0$
    (c) $r = 1$ on $[a, b]$
    (d) $p(x) = x - b$ on $[a, b]$
    (e) $q(x) = 1/(x - a)$ on $(a, b)$

17. Which of the following differential equations are in self-adjoint form?

    Select all that apply.

    (a) $(1-x^2)y'' - 2xy' + \lambda y = 0$
    (b) $r'(x)y'' + r(x)y' + \lambda y = 0$
    (c) $xy'' + y' + (x - n^2/x)y = 0$
    (d) $y'' + y' + \lambda y = 0$
    (e) $y'' + \lambda y = 0$

18. What is the name of the equation $(1-x^2)y'' - 2xy' + \lambda y = 0$?

19. Solve the eigenvalue problem $(1-x^2)y'' - 2xy' + \lambda y = 0$, $y$ is bounded on $[-1, 1]$.

20. Using the eigenfunctions of the previous problem, what is the Fourier-Legendre series expansion of a function $f(x)$?

**ANSWER KEY**
***Zill Differential Equations 9e Chapter 11 Form F***

1. e
2. c
3. No, for example $(1, x) = 1/2 \neq 0$
4. b
5. b
6. $f(x) \sim \sum_{n=1}^{\infty} 2n\pi(1 - e^2(-1)^n)\sin(n\pi x/2)/(4 + (n\pi)^2)$
7. $f(x) \sim (e^2 - 1)/2 - \sum_{n=1}^{\infty} 4(1 - e^2(-1)^n)\cos(n\pi x/2)/(4 + (n\pi)^2)$
8. $\lambda = (n - 1/2)^2\pi^2$, $y = \sin((n - 1/2)x)$, $n = 1, 2, 3, \ldots$
9. $f(x) \sim \sum_{n=1}^{\infty} c_n \sin((n - 1/2)\pi x)$, where $c_n = 2\int_0^1 f(x)\sin((n - 1/2)\pi x)dx$
10. $f(x) \sim \sum_{n=1}^{\infty} c_n \sin((n - 1/2)\pi x)$, where $c_n = 8(-1)^{n+1}/(\pi - 2n\pi)^2$
11. $\lambda = n^2$, $y = \cos(nx)$, $n = 0, 1, 2, \ldots$
12. $f(x) \sim \sum_{n=1}^{\infty} c_n \cos(nx)$, where $c_n = 2\int_0^{\pi} f(x)\cos(nx)dx/\pi$
13. $f(x) \sim \sum_{n=1}^{\infty} c_n \cos(nx)$, where $c_n = 2(-1 + (-1)^n)/(n^2\pi)$
14. b, d, e
15. c, d
16. d, e
17. a, c, e
18. Legendre's differential equation
19. $\lambda = n(n + 1)$, $y = P_n(x)$, $n = 1, 2, 3, \ldots$
20. $f(x) \sim \sum_{n=1}^{\infty} c_n P_n(x)$, where $c_n = \int_{-1}^{1} f(x)P_n(x)dx / \int_{-1}^{1} P_n(x)^2 dx$

1. Calculate the inner product of $f_1(x) = 1$ and $f_2(x) = x - x^2$ on the interval $[0, 3/2]$.

2. The functions $f_1(x) = \cos(2x)$ and $f_2(x) = \sin(2x)$ are orthogonal on the interval $[0, \pi]$. Form an orthonormal set from them.

3. The Fourier Series of a $f(x) = x^2$ defined on $[-p, p]$ is
   $f(x) = a_0/2 + \sum_{n=1}^{\infty}(a_n \cos(n\pi x/p) + b_n \sin(n\pi x/p))$ where
   Select all that apply.

   (a) $a_0 = 2/3$
   (b) $a_n = 0$
   (c) $a_n = \int_{-p}^{p} x^2 \cos(n\pi x/p)dx/p$
   (d) $b_n = 0$
   (e) $b_n = \int_{-p}^{p} x^2 \cos(n\pi x/p)dx/p$

4. In order to be assured by a theorem that the Fourier Series of $f$ on $[a, b]$ converges at $x$ to $(f(x+) + f(x-))/2$, which of the following conditions need to be satisfied?
   Select all that apply.

   (a) $f$ is continuous on $[a, b]$
   (b) $f'$ is continuous on $[a, b]$
   (c) $f$ is piecewise continuous on $[a, b]$
   (d) $f'$ is piecewise continuous on $[a, b]$

5. The function $f(x) = \left\{ \begin{array}{ccc} -1 - x & \text{if} & x < 0 \\ 1 - x & \text{if} & x > 0 \end{array} \right\}$ is
   Select all that apply.

   (a) odd
   (b) even
   (c) neither even nor odd
   (d) continuous on $[-\pi, \pi]$
   (e) discontinuous on $[-\pi, \pi]$

6. The Fourier series of the function $f(x) = \left\{ \begin{array}{ccc} -1 - x & \text{if} & x < 0 \\ 1 - x & \text{if} & x > 0 \end{array} \right\}$ on $[-2, 2]$
   Select all that apply.

   (a) contains only cosine terms
   (b) contains only sine terms
   (c) contains sine and cosine terms
   (d) contains a constant term
   (e) contains sine, cosine, and constant terms

7. Let $f(x) = x$ if $0 \le x \le 1$. Find the Fourier sine series of $f$.

8. Let $f(x) = x$ if $0 \le x \le 1$. Find the Fourier cosine series of $f$.

9. The solution of the eigenvalue problem $y'' + \lambda y = 0$, $y(0) = 0$, $y(2\pi) = 0$ is

   Select the correct answer.

   (a) $\lambda = n^2/4$, $y = \cos(nx/2)$, $n = 0, 1, 2, \ldots$
   (b) $\lambda = n^2/4$, $y = \sin(nx/2)$, $n = 1, 2, 3, \ldots$
   (c) $\lambda = n/2$, $y = \cos(nx/2)$, $n = 0, 1, 2, \ldots$
   (d) $\lambda = n/2$, $y = \sin(nx/2)$, $n = 1, 2, 3, \ldots$
   (e) $\lambda = n/2$, $y = \cos(nx/2) + \sin(nx/2)$, $n = 1, 2, 3, \ldots$

10. The solution of the eigenvalue problem $y'' + \lambda y = 0$, $y'(0) = 0$, $y'(2\pi) = 0$ is

    Select the correct answer.

    (a) $\lambda = n/2$, $y = \cos(nx/2) + \sin(nx/2)$, $n = 1, 2, 3, \ldots$
    (b) $\lambda = n/2$, $y = \cos(nx/2)$, $n = 0, 1, 2, \ldots$
    (c) $\lambda = n/2$, $y = \sin(nx/2)$, $n = 1, 2, 3, \ldots$
    (d) $\lambda = n^2/4$, $y = \cos(nx/2)$, $n = 0, 1, 2, \ldots$
    (e) $\lambda = n^2/4$, $y = \sin(nx/2)$, $n = 1, 2, 3, \ldots$

11. The solution of the eigenvalue problem $(1-x^2)y'' - 2xy' + \lambda y = 0$, where $y$ is bounded on $[-1, 1]$ is

    Select the correct answer.

    (a) $\lambda = n + 1$, $y = P_n(x)$, $n = 1, 2, 3, \ldots$
    (b) $\lambda = n$, $y = P_n(x)$, $n = 1, 2, 3, \ldots$
    (c) $\lambda = n^2$, $y = P_n(x)$, $n = 1, 2, 3, \ldots$
    (d) $\lambda = n(n-1)$, $y = P_n(x)$, $n = 1, 2, 3, \ldots$
    (e) $\lambda = n(n+1)$, $y = P_n(x)$, $n = 1, 2, 3, \ldots$

12. Write down the form of a Sturm-Liouville problem.

13. Consider the previous problem. Under what conditions is Sturm-Liouville problem a regular Sturm-Liouville problem? Is the problem $(1 - x^2)y'' - 2xy' + \lambda y = 0$, where $y$ is bounded on $[-1, 1]$, a regular Sturm-Liouville problem? Explain.

14. Write down the parameterized Bessel's differential equation of order $n$ in a form that makes it a Sturm-Liouville differential equation. Add boundary conditions at $x = 1$ and $x = 2$ to this Bessel's equation to make it a regular Sturm-Liouville problem.

15. In the previous problem, what is the solution?

16. The Legendre differential equation $\frac{d}{dx}[(1 - x^2)\frac{dy}{dx} + \lambda y = 0$ is in Sturm-Liouville form. Suppose that the solution also satisfies the condition that $y$ is bounded on $[-1, 1]$. What is the solution of this eigenvalue problem?

17. Use the eigenfunctions of the previous problem to write an eigenfunction expansion of a function $f(x)$.

18. In the previous problem, what conditions on $f$ would guarantee that the infinite series converges to the function on $[-1, 1]$?

19. The problem $y'' + \lambda y = 0$, $y(0) = 0$, $y(\pi) = 0$ is a regular Sturm-Liouville problem. Which of the following are eigenvalues of this problem?

    Select the correct answer.

    (a) $\lambda = 0$
    (b) $\lambda = 2\pi$
    (c) $\lambda = 3\pi$
    (d) $\lambda = 4$
    (e) $\lambda = 169$

20. The problem $y'' + \lambda y = 0$, $y(0) = 0$, $y(\pi) = 0$ is a regular Sturm-Liouville problem. Which of the following are eigenfunctions of this problem?

    Select all that apply.

    (a) $y = 0$
    (b) $y = \sin(3x)$
    (c) $y = \cos(3x)$
    (d) $y = \sin(3\pi x)$
    (e) $y = \sin(3\pi x)$

**ANSWER KEY**
***Zill Differential Equations 9e Chapter 11 Form G***

1. 0
2. $\{\sqrt{2/\pi}\cos(2x),\ \sqrt{2/\pi}\sin(2x)\}$
3. c, d
4. c, d
5. a, e
6. b
7. $f(x) \sim \sum_{n=1}^{\infty} 2(-1)^{n+1}\sin(n\pi x)/(n\pi)$
8. $f(x) \sim 1/2 + \sum_{n=1}^{\infty} 2((-1)^n - 1)\cos(n\pi x)/(n\pi)^2$
9. b
10. d
11. e
12. $\frac{d}{dx}[r(x)\frac{dy}{dx}] + (q(x) + \lambda p(x))y = 0,\ A_1 y(a) + B_1 y'(a) = 0,\ A_2 y(b) + B_2 y'(b) = 0$
13. $p$, $q$, $r$, $r'$ are continuous on $[a, b]$, $r > 0$ and $p > 0$ on $[a, b]$, $A_1^2 + B_1^2 \neq 0$, $A_2^2 + B_2^2 \neq 0$. No, because the boundary conditions are not of the correct form and $r(x)$ is zero at the endpoints.
14. $xy'' + y' + (\lambda x - n^2/x)y = 0,\ y(1) = 0,\ y(2) = 0$
15. $y = c_1 J_n(\sqrt{\lambda}x) + c_2 Y_n(\sqrt{\lambda}x)$, where $c_1$ and $c_2$ satisfy $c_1 J_n(\sqrt{\lambda}) + c_2 Y_n(\sqrt{\lambda}) = 0$, $c_1 J_n(2\sqrt{\lambda}) + c_2 Y_n(2\sqrt{\lambda}) = 0$
16. $\lambda = n(n + 1),\ y = P_n(x),\ n = 1, 2, 3, \ldots$
17. $f(x) \sim \sum_{n=1}^{\infty} c_n P_n(x)$, where $c_n = (2n + 1)\int_{-1}^{1} f(x)P_n(x)dx/2$
18. $f$ is continuous and $f$ and $f'$ are piecewise continuous on $(-1, 1)$
19. d, e
20. b

1. The norm of the function $f(x) = 1 - x^2$ on the interval $[0, 1]$ is

   Select the correct answer.

   (a) $\sqrt{1/15}$
   (b) $\sqrt{8/15}$
   (c) $\sqrt{16/15}$
   (d) $\sqrt{1/3}$
   (e) $\sqrt{2/3}$

2. The norm of the function $f(x) = x^3$ on the interval $[0, 1]$ is

   Select the correct answer.

   (a) $\sqrt{1/6}$
   (b) $\sqrt{1/4}$
   (c) $1/4$
   (d) $1/7$
   (e) $\sqrt{1/7}$

3. Is the set of functions $\{\sin(nx)\}$, $n = 1, 2, 3, \ldots$ orthogonal on the interval $[-\pi, \pi]$? Explain.

4. Is the set of functions $\{\sin(nx)\}$, $n = 1, 2, 3, \ldots$ complete on the interval $[-\pi, \pi]$? Explain.

5. The function $f(x) = \left\{ \begin{array}{ccc} x + \sin x & \text{if} & -5 \le x \le 0 \\ 3 + \cos(\pi x) & \text{if} & 0 < x \le 5 \end{array} \right\}$ has a Fourier series on $[-5, 5]$ that converges at $x = 0$ to

   Select the correct answer.

   (a) 0
   (b) 1
   (c) 2
   (d) 4

6. The function $f(x) = \left\{ \begin{array}{ccc} x + \sin x & \text{if} & -5 < x < 0 \\ 3 + \cos(\pi x) & \text{if} & 0 < x < 5 \end{array} \right\}$ has a Fourier series on $[-5, 5]$ that converges at $x = 3$ to

   Select the correct answer.

   (a) 1
   (b) 2
   (c) 3
   (d) 4
   (e) unknown

7. Let $f(x) = 1 - x$ if $0 \leq x \leq 1$. Find the Fourier sine series of $f$.

8. Let $f(x) = 1 - x$ if $0 \leq x \leq 1$. Find the Fourier cosine series of $f$.

9. Solve the eigenvalue problem $y'' + \lambda y = 0$, $y'(0) = 0$, $y(1) = 0$.

10. Solve the eigenvalue problem $y'' + \lambda y = 0$, $y(0) = 0$, $y(1) + y'(1) = 0$.

11. Solve the eigenvalue problem $rR'' + R' + r\lambda R = 0$, $R(0)$ is bounded, $R(1) = 0$.

12. The problem $\frac{d}{dx}[r(x)y'] + (q(x) + \lambda p(x))y = 0$, $A_1 y(a) + B_1 y'(a) = 0$, $A_2 y(b) + B_2 y'(b) = 0$ is a regular Sturm-Liouville problem under certain conditions, including

    Select all that apply.

    (a) $r(x) > 0$ and $p(x) < 0$ on $[a, b]$
    (b) $r(x) < 0$ and $p(x) > 0$ on $[a, b]$
    (c) $p$, $q$, $r$ are continuous on $[a, b]$
    (d) $r$ is piecewise smooth on $[a, b]$
    (e) $A_1^2 + B_1^2 \neq 0$

13. The problem $\frac{d}{dx}[r(x)y'] + (q(x) + \lambda p(x))y = 0$, $A_1 y(a) + B_1 y'(a) = 0$, $A_2 y(b) + B_2 y'(b) = 0$ is not a regular Sturm-Liouville problem under which of the following conditions.

    Select all that apply.

    (a) $q(x) = 0$ on $[a, b]$
    (b) $r = 1/(x - b)$ on $(a, b)$
    (c) $p(x) = x - a$ on $[a, b]$
    (d) $A_1^2 + B_1^2 = 0$
    (e) $A_1 A_2 = 0$

14. Which of the following differential equations are in self-adjoint form?

    Select all that apply.

    (a) $y'' + y' + \lambda y = 0$
    (b) $r(x)y'' + r'(x)y' + \lambda y = 0$
    (c) $(1 - x^2)y'' - 2xy' + \lambda y = 0$
    (d) $y'' + \lambda y = 0$
    (e) $xy'' + y' + (x - n^2/x)y = 0$

15. The parameterized Bessel differential equation $xy'' + y' + (\lambda x - m^2/x)y = 0$ is in Sturm-Liouville form. Suppose that the solution also satisfies the conditions that $y(0)$ is bounded and $y(1) = 0$. What is the solution of this eigenvalue problem?

16. Use the eigenfunctions of the previous problem to write an eigenfunction expansion of a function $f(x)$.

17. In the previous problem, what conditions on $f$ would guarantee that the infinite series converges to the function on $[0, 1]$?

18. Consider the Legendre functions $P_n(x)$. The value of $||P_n(x)||$ is

    Select the correct answer.

    (a) $2n + 1$
    (b) $(2n + 1)/2$
    (c) $2/(2n + 1)$
    (d) $\sqrt{2/(2n + 1)}$
    (e) $\sqrt{2n + 1}$

19. For a piecewise continuous function $f$, the Fourier-Legendre expansion is $\sum_{n=1}^{\infty} c_n P_n(x)$, where $c_n$ is

    Select the correct answer.

    (a) $2\int_{-1}^{1} f(x)P_n(x)dx/(2n + 1)$
    (b) $\sqrt{2/(2n + 1)}\int_{-1}^{1} f(x)P_n(x)dx$
    (c) $\sqrt{2n + 1}\int_{-1}^{1} f(x)P_n(x)dx$
    (d) $(2n + 1)\int_{-1}^{1} f(x)P_n(x)dx$
    (e) $(2n + 1)\int_{-1}^{1} f(x)P_n(x)dx/2$

20. In the previous problem, for the function $f(x) = \left\{ \begin{array}{ccc} 0 & \text{if} & -1 \le x \le 0 \\ 1 & \text{if} & 0 < x \le 1 \end{array} \right\}$, the first five coefficients are

    Select all that apply.

    (a) $c_0 = 1/2$
    (b) $c_1 = 3/4$
    (c) $c_2 = 2/3$
    (d) $c_3 = 0$
    (e) $c_4 = 0$

**ANSWER KEY**

***Zill Differential Equations 9e Chapter 11 Form H***

1. b
2. e
3. Yes, $\int_{-\pi}^{\pi} \sin(nx)\sin(mx)dx = 0$ if $n \neq m$.
4. No, for example, $(1, \sin(nx)) = 0$ for all integers $n$.
5. c
6. b
7. $f(x) \sim \sum_{n=1}^{\infty} 2\sin(n\pi x)/(n\pi)$
8. $f(x) \sim 1/2 + \sum_{n=1}^{\infty} 2(1-(-1)^n)\cos(n\pi x)/(n\pi)^2$
9. $\lambda = (n-1/2)^2\pi^2$, $y = \cos((n-1/2)\pi x)$, $n = 1, 2, 3, \ldots$
10. $\lambda = z_n^2$, $y = \sin(z_n x)$, where $z_n$ satisfies $\tan z_n = -z_n$
11. $\lambda = z_n$, $R = J_0(z_n r)$, where $J_0(z_n) = 0$
12. c, e
13. b, c, d
14. b, c, d, e
15. $\lambda = z_n^2$, $y = J_m(z_n x)$, $n = 1, 2, 3, \ldots$, where $J_m(z_n) = 0$
16. $f(x) = \sum_{n=1}^{\infty} c_n J_m(z_n x)$, where $c_n = \int_0^1 x f(x) J_m(z_n x)dx / \int_0^1 x J_m^2(z_n x)dx$
17. $f$ is continuous and piecewise smooth on $(0, 1)$
18. d
19. e
20. a, b, e