1. Write $\frac{\partial r}{\partial x}$ in terms of polar coordinates?

2. Write $\frac{\partial u}{\partial x}$, expressed in terms of polar coordinates?

3. Write down the Laplacian of $u(r, \theta)$ in polar coordinates.

4. Consider the problem $r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0$; $u(c, \theta) = f(\theta)$. Describe a physical situation having this as a mathematical model.

5. In the previous problem, separate variables using $u(r, \theta) = R(r)\Theta(\theta)$. What are the resulting differential equations and boundary conditions?

6. In the previous problem, what are the solutions of the eigenvalue problem. Is the eigenvalue problem a regular Sturm-Liouville problem?

7. In the three previous problems, what are the product solutions?

8. In the previous four problems, what is the infinite series solution of the problem $r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0$; $u(c, \theta) = f(\theta)$?

9. What is the solution of the problem $r^2R'' + rR' + \lambda R = 0$, $R(0) = 0$, $R(1) = 0$? Is this a regular Sturm-Liouville problem?

10. Write down the problem for the radial vibrations of a circular membrane of radius $c$, clamped along its circumference, with an initial displacement of $f(r)$ and an initial velocity of zero.

11. In the previous problem, separate variables with $u(r, t) = R(r)T(t)$. What are the resulting differential equations and boundary/initial conditions?

12. In the previous problem, what are the solutions of the eigenvalue problem? Is the problem a regular Sturm-Liouville problem?

13. In the previous three problems, what are the product solutions?

14. In the previous four problems, What is the infinite series solution of the problem for the radial vibrations of a circular membrane of radius $c$, clamped along its circumference, with an initial displacement of $f(r)$ and an initial velocity of zero?

15. Write down the relationships between Cartesian and spherical coordinates.

16. In the previous problem, what is $\frac{\partial r}{\partial x}$, where $r$ is the distance from the origin?

17. Write down the Laplacian of $u(r, \phi, \theta)$ in spherical coordinates.

18. The steady–state temperature distribution in a sphere of radius $c$ is given by Laplace's equation in spherical coordinates. Assume that the temperature on the boundary is a given function, $f(\theta)$. Separate variables, using $u(r, \theta) = R(r)\Theta(\theta)$. What are the resulting differential equations and boundary conditions for $R$ and $\Theta$?

19. In the previous problem, what are the solutions of the eigenvalue problem?

20. In the previous three problems, what is the infinite series solution of the original problem?

**ANSWER KEY**
***Zill Differential Equations 9e Chapter 13 Form A***

1. $\frac{\partial r}{\partial x} = \cos\theta$

2. $\frac{\partial u}{\partial x} = \cos\theta \frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}$

3. $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$

4. Steady–state temperature distribution in a circular plate of radius $c$ with specified temperature, $f$, along the boundary

5. $r^2R'' + rR' - \lambda R = 0$, $R(0)$ is bounded, $\Theta'' + \lambda\Theta = 0$, $\Theta(0) = \Theta(2\pi)$

6. $\lambda = n^2$, $\Theta = c_n\cos(n\theta) + d_n\sin(n\theta)$, $n = 0, 1, 2, \ldots$, no it is not regular

7. $u = c$ if $n = 0$, $u = r^n(c_n\cos(n\theta) + d_n\sin(n\theta))$ if $n = 1, 2, 3, \ldots$

8. $u = c_0 + \sum_{n=1}^{\infty} r^n(c_n\cos(n\theta) + d_n\sin(n\theta))$, where $c_0 = \int_0^{2\pi} f(\theta)d\theta/(2\pi)$,
   $c_n = \int_0^{2\pi} f(\theta)\cos(n\theta)d\theta/(c^n\pi)$,
   $d_n = \int_0^{2\pi} f(\theta)\sin(n\theta)d\theta/(c^n\pi)$

9. $R = 0$; no, it is not regular

10. $a^2(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}) = \frac{\partial^2 u}{\partial t^2}$; $u(0, t)$ is bounded, $u(c, t) = 0$; $u(r, 0) = f(r)$, $u_t(r, 0) = 0$

11. $rR'' + R' + \lambda rR = 0$; $R(0)$ is bounded, $R(c) = 0$; $T'' + \lambda a^2T = 0$; $T'(0) = 0$

12. $\lambda = z_n^2/c^2$; $R = J_0(z_nr/c)$, $n = 1, 2, 3, \ldots$, where $J_0(z_n) = 0$; no it is not regular

13. $u = J_0(z_nr/c)\cos(z_nat/c)$

14. $u = \sum_{n=1}^{\infty} a_nJ_0(z_nr/c)\cos(z_nat/c)$, where $a_n = 2\int_0^c rJ_0(z_nr/c)f(r)dr/(c^2J_1^2(z_n))$

15. $x = r\cos\phi\sin\theta$, $y = r\sin\phi\sin\theta$, $z = r\cos\theta$, $r^2 = x^2 + y^2 + z^2$, $\tan\phi = y/x$, $\cos\theta = z/\sqrt{x^2 + y^2 + z^2}$

16. $\frac{\partial r}{\partial x} = \cos\phi\sin\theta$

17. $\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta}$

18. $r^2R'' + 2rR' - \lambda R = 0$, $\sin\theta\Theta'' + \cos\theta\Theta' + \lambda\sin\theta\Theta = 0$, $\Theta$ is bounded for all $\theta$

19. $\lambda = n(n+1)$, $\Theta = P_n(\cos\theta)$, $n = 1, 2, 3, \ldots$

20. $u = \sum_{n=1}^{\infty} a_nr^nP_n(\cos\theta)$, where $a_n = (2n+1)\int_0^{\pi} f(\theta)P_n(\cos\theta)\sin\theta d\theta/(2c^n)$

1. Write $\frac{\partial r}{\partial y}$ in terms of polar coordinates?

2. Write $\frac{\partial u}{\partial y}$ in terms of polar coordinates?

3. Consider the problem $r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0$, $u(1,\theta) = 0$, $u(2,\theta) = f(\theta)$, $u(r,0) = 0$, $u(r,\pi/2) = 0$. Describe a physical situation having this as a mathematical model.

4. In the previous problem, separate variables using $u(r,\theta) = R(r)\Theta(\theta)$. What are the resulting differential equations and boundary conditions?

5. In the previous problem, what are the solutions of the eigenvalue problem. Is the eigenvalue problem a regular Sturm-Liouville problem?

6. In the three previous problems, what are the product solutions?

7. In the previous four problems, what is the infinite series solution of the problem $r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0$; $u(1,\theta) = 0$, $u(2,\theta) = f(\theta)$; $u(r,0) = 0$, $u(r,\pi/2) = 0$?

8. Write down the two dimensional heat equation in polar coordinates.

9. What is the solution of the problem $r^2R'' + rR' + \lambda R = 0$, $R(1) = 0$, $R(2) = 0$? Is this a regular Sturm-Liouville problem?

10. Write down the problem for the steady–state temperature in a circular cylinder of radius 1 and height 4, with a temperature of zero at $z = 0$ and at $r = 1$, and a constant temperature, $u_0$, at $z = 4$.

11. In the previous problem, separate variables with $u(r,z) = R(r)Z(z)$. What are the resulting differential equations and boundary conditions?

12. In the previous problem, what are the solutions of the eigenvalue problem? Is the problem a regular Sturm-Liouville problem?

13. In the previous three problems, what are the product solutions?

14. In the previous four problems, What is the infinite series solution of the problem for the steady–state temperature in a circular cylinder of radius 1 and height 4, with a temperature of zero at $z = 0$ and at $r = 1$, and a constant temperature, $u_0$, at $z = 4$?

15. Write down the relationships between Cartesian and spherical coordinates.

16. Write $\frac{\partial r}{\partial y}$ in terms of spherical coordinates?

17. Consider the steady–state temperature distribution, $u(r,\theta)$, in a uniform spherical mass of radius 5 with the temperature given as a function, $f(\theta)$ on the boundary. Write down the boundary value problem that models this situation.

18. In the previous problem, separate variables, using $u(r,\theta) = R(r)\Theta(\theta)$. What are the resulting differential equations and boundary conditions?

19. In the previous problem, what are the solutions of the eigenvalue problem?

20. In the previous three problems, what is the infinite series solution of the problem for the steady–state temperature distribution, $u(r,\theta)$, in a uniform spherical mass of radius 5 with the temperature given as a function, $f(\theta)$ on the boundary?

**ANSWER KEY**
***Zill Differential Equations 9e Chapter 13 Form B***

1. $\frac{\partial r}{\partial y} = \sin\theta$

2. $\frac{\partial u}{\partial y} = \sin\theta \frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}$

3. Steady–state temperature distribution in a quarter circular annulus of inner radius 1 and outer radius 2 with zero temperature at $r = 1$, $\theta = 0$, and $\theta = \pi/2$, and fixed temperature $f(\theta)$ at $r = 2$

4. $r^2R'' + rR' - \lambda R = 0$, $R(1) = 0$, $\Theta'' + \lambda\Theta = 0$, $\Theta(0) = 0$, $\Theta(\pi/2) = 0$

5. $\lambda = 4n^2$, $\Theta = \sin(2n\theta)$, $n = 1, 2, 3, \ldots$; yes, it is regular

6. $u = (r^{2n} - r^{-2n})\sin(2n\theta)$

7. $u = \sum_{n=1}^{\infty} c_n(r^{2n} - r^{-2n})\sin(2n\theta)$, where $c_n = 4\int_0^{\pi/2} f(\theta)\sin(2n\theta)d\theta/((2^{2n} - 2^{-2n})\pi)$

8. $k\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) = \frac{\partial u}{\partial t}$

9. $\lambda = (n\pi/\ln 2)^2$, $R = \sin(n\pi \ln r/\ln 2)$, $n = 1, 2, 3, \ldots$, yes, it is regular

10. $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0$, $u(1, z) = 0$, $u(r, 0) = 0$, $u(0, z)$ is bounded, $u(r, 4) = u_0$

11. $rR'' + R' + \lambda rR = 0$, $R(0)$ is bounded, $R(1) = 0$, $Z'' - \lambda Z = 0$, $Z(0) = 0$

12. $\lambda = z_n^2$, $R = J_0(z_n r)$, $n = 1, 2, 3, \ldots$, where $J_0(z_n) = 0$; no, it is not regular.

13. $u = J_0(z_n r)\sinh(z_n z)$

14. $u = \sum_{n=1}^{\infty} c_n J_0(z_n r)\sinh(z_n z)$, where $c_n = u_0 \int_0^1 rJ_0(z_n r)dr/(\sinh(4z_n)\int_0^1 rJ_0^2(z_n r)dr)$

15. $x = r\cos\phi\sin\theta$, $y = r\sin\phi\sin\theta$, $z = r\cos\theta$, $r^2 = x^2 + y^2 + z^2$, $\tan\phi = y/x$, $\cos\theta = z/\sqrt{x^2 + y^2 + z^2}$

16. $\frac{\partial r}{\partial y} = \sin\phi\sin\theta$

17. $\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} = 0$; $u(5, \theta) = f(\theta)$

18. $r^2R'' + 2rR' - \lambda R = 0$; $R(0)$ is bounded; $\sin(\theta)\Theta'' + \cos(\theta)\Theta' + \lambda\sin(\theta)\Theta = 0$; $\Theta$ is bounded everywhere

19. $\lambda = n(n + 1)$, $\Theta = P_n(\cos\theta)$, $n = 1, 2, 3, \ldots$

20. $u = \sum_{n=1}^{\infty} a_n r^n P_n(\cos\theta)$, where $a_n = (2n + 1)\int_0^{\pi} f(\theta)P_n(\cos\theta)\sin\theta d\theta/(5^n 2)$

1. In changing from Cartesian to polar coordinates, $\frac{\partial r}{\partial x}$ is

   Select the correct answer.

   (a) $\sin\theta$
   (b) $\cos\theta$
   (c) $\sin\theta/r$
   (d) $\cos\theta/r$
   (e) $-\sin\theta/r$

2. In changing from Cartesian to polar coordinates, $\frac{\partial u}{\partial x}$ is

   Select the correct answer.

   (a) $\sin\theta\frac{\partial u}{\partial r} + \cos\theta\frac{\partial u}{\partial \theta}/r$
   (b) $\sin\theta\frac{\partial u}{\partial r} - \cos\theta\frac{\partial u}{\partial \theta}/r$
   (c) $\cos\theta\frac{\partial u}{\partial r} - \sin\theta\frac{\partial u}{\partial \theta}/r$
   (d) $\cos\theta\frac{\partial u}{\partial r} + \cos\theta\frac{\partial u}{\partial \theta}/r$
   (e) $\sin\theta\frac{\partial u}{\partial r} + \cos\theta\frac{\partial u}{\partial \theta}$

3. The Laplacian in polar coordinates is

   Select the correct answer.

   (a) $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial^2 u}{\partial \theta^2}$
   (b) $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$
   (c) $\frac{\partial^2 u}{\partial r^2} - \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$
   (d) $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$
   (e) $\frac{\partial^2 u}{\partial r^2} - \frac{1}{r}\frac{\partial u}{\partial r} - \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$

4. Consider the steady–state temperature distribution in a circular disc of radius $c$ centered at the origin, with temperature given as a function, $f(\theta)$ on the boundary. The mathematical model of this situation is

   Select the correct answer.

   (a) $\frac{\partial^2 u}{\partial r^2} - \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,\ u(c,\theta) = f(\theta)$
   (b) $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,\ u(c,\theta) = f(\theta)$
   (c) $\frac{\partial^2 u}{\partial r^2} - \frac{1}{r}\frac{\partial u}{\partial r} - \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,\ u(c,\theta) = f(\theta)$
   (d) $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} = 0,\ u(c,\theta) = f(\theta)$
   (e) $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,\ u(c,\theta) = f(\theta)$

5. In the previous problem, separate variables using $u(r, \theta) = R(r)\Theta(\theta)$. The resulting problems are

   Select the correct answer.

   (a) $r^2R'' + rR' - \lambda R = 0$, $\Theta'' - \lambda\Theta = 0$, $\Theta(\theta) = \Theta(\theta + 2\pi)$
   (b) $r^2R'' + rR' + \lambda R = 0$, $\Theta'' + \lambda\Theta = 0$, $\Theta(\theta) = \Theta(\theta + 2\pi)$
   (c) $r^2R'' + rR' - \lambda R = 0$, $\Theta'' + \lambda\Theta = 0$, $\Theta(\theta) = \Theta(\theta + 2\pi)$
   (d) $r^2R'' + rR' - \lambda R = 0$, $\Theta'' + \lambda\Theta = 0$, $\Theta(\theta) = \Theta(\theta + \pi)$
   (e) $r^2R'' + rR' + \lambda R = 0$, $\Theta'' + \lambda\Theta = 0$, $\Theta(\theta) = \Theta(\theta + \pi)$

6. In the previous problem, the solution of the eigenvalue problem is

   Select the correct answer.

   (a) $\lambda = n$, $\Theta = c_n \cos(n\theta) + d_n \sin(n\theta)$, $n = 0, 1, 2, \ldots$
   (b) $\lambda = n^2$, $\Theta = c_n \cos(n\theta) + d_n \sin(n\theta)$, $n = 0, 1, 2, \ldots$
   (c) $\lambda = n^2$, $\Theta = c_n \cos(n\theta) + d_n \sin(n\theta)$, $n = 1, 2, 3, \ldots$
   (d) $\lambda = n^2$, $R = r^n$, $n = 0, 1, 2, \ldots$
   (e) $\lambda = n$, $R = r^n$, $n = 0, 1, 2, \ldots$

7. In the three previous problems, the product solutions are

   Select the correct answer.

   (a) $u_n = (a_n r^n + b_n r^{-n})(c_n \cos(n\theta) + d_n \sin(n\theta))$, $n = 0, 1, 2, \ldots$
   (b) $u_n = r^n(c_n \cos(n\theta) + d_n \sin(n\theta))$, $n = 1, 2, 3, \ldots$
   (c) $u_n = r^n(c_n \cos(n\theta) + d_n \sin(n\theta))$, $n = 0, 1, 2, \ldots$
   (d) $u_n = r^n(c_n e^{n\theta} + d_n e^{-n\theta})$, $n = 0, 1, 2, \ldots$
   (e) $u_n = r^n(c_n e^{n\theta} + d_n e^{-n\theta})$, $n = 1, 2, 3, \ldots$

8. In the previous four problems, the infinite series solution of the original problem is $u = A_0 + \sum_{n=1}^{\infty} r^n(A_n \cos(n\theta) + B_n \sin(n\theta))$ where

   Select all that apply.

   (a) $A_0 = \int_0^{2\pi} f(\theta)d\theta/(2\pi)$
   (b) $A_n = \int_0^{2\pi} f(\theta)\sin(n\theta)d\theta/(c^n\pi)$
   (c) $A_n = \int_0^{2\pi} f(\theta)\cos(n\theta)d\theta/(c^n\pi)$
   (d) $B_n = \int_0^{2\pi} f(\theta)\sin(n\theta)d\theta/(c^n\pi)$
   (e) $B_n = \int_0^{2\pi} f(\theta)\cos(n\theta)d\theta/(c^n\pi)$

9. The solution of $r^2R'' + rR' + \lambda R = 0,\ R(0) = 0,\ R(1) = 0$ is

   Select the correct answer.

   (a) $\lambda = n\pi,\ R = \sin(n\pi \ln r)$
   (b) $\lambda = (n\pi)^2,\ R = \sin(n\pi \ln r)$
   (c) $\lambda = (n\pi)^2,\ R = \sin(n\pi \ln r) + \cos(n\pi \ln r)$
   (d) $\lambda = n\pi,\ R = \sin(n\pi \ln r) - \cos(n\pi \ln r)$
   (e) none of the above

10. Consider the steady–state temperature distribution in a circular cylinder of radius 2 and height 3, with zero temperature at $r = 2$ and at $z = 3$ and a temperature of 10 at $z = 0$. The mathematical model for this problem is

    Select the correct answer.

    (a) $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{\partial^2 u}{\partial z^2} = 0,\ u(2, z) = 0,\ u(r, 3) = 0,\ u(r, 0) = 10$
    (b) $\frac{\partial^2 u}{\partial r^2} - \frac{1}{r}\frac{\partial u}{\partial r} - \frac{\partial^2 u}{\partial z^2} = 0,\ u(2, z) = 0,\ u(r, 3) = 0,\ u(r, 0) = 10$
    (c) $\frac{\partial^2 u}{\partial r^2} - \frac{1}{r^2}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0,\ u(2, z) = 0,\ u(r, 3) = 0,\ u(r, 0) = 10$
    (d) $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0,\ u(2, z) = 0,\ u(r, 3) = 0,\ u(r, 0) = 10$
    (e) $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0,\ u(2, z) = 0,\ u(r, 3) = 0,\ u(r, 0) = 10$

11. In the previous problem, after separating variables, the resulting problems are

    Select the correct answer.

    (a) $rR'' - R' + \lambda rR = 0$, $R(0)$ is bounded, $R(2) = 0,\ Z'' + \lambda Z = 0,\ Z(3) = 0$
    (b) $rR'' + R' - \lambda rR = 0$, $R(0)$ is bounded, $R(2) = 0,\ Z'' - \lambda Z = 0,\ Z(3) = 0$
    (c) $rR'' - R' - \lambda rR = 0$, $R(0)$ is bounded, $R(2) = 0,\ Z'' - \lambda Z = 0,\ Z(3) = 0$
    (d) $rR'' + R' + \lambda rR = 0$, $R(0)$ is bounded, $R(2) = 0,\ Z'' - \lambda Z = 0,\ Z(3) = 0$
    (e) $rR'' + R' + \lambda rR = 0$, $R(0)$ is bounded, $R(2) = 0,\ Z'' + \lambda Z = 0,\ Z(3) = 0$

12. In the previous problem, the solution of the eigenvalue problem is

    Select the correct answer.

    (a) $\lambda = z_n^2/4$, where $J_0(z_n) = 0$; $R = J_0(z_n r/2)$
    (b) $\lambda = z_n^2$, where $J_0(z_n) = 0$; $R = J_0(z_n r)$
    (c) $\lambda = n\pi/9,\ Z = \cos(n\pi z/3)$
    (d) $\lambda = n^2\pi^2/9,\ Z = \cos(n\pi z/3)$
    (e) $\lambda = n^2\pi^2/9,\ Z = \sin(n\pi z/3)$

13. In the previous three problems, the product solutions are

    Select the correct answer.

    (a) $u = J_0(n\pi r/3)\cos(n\pi z/3)$
    (b) $u = J_0(n^2\pi^2 r/9)\cos(n\pi z/3)$
    (c) $u = J_0(n^2\pi^2 r/9)\sin(n\pi z/3)$
    (d) $u = J_0(z_n r)\cosh(z_n(z-3))$
    (e) $u = J_0(z_n r/2)\sinh(z_n(z-3)/2)$

14. In the previous four problems, the infinite series solution is (for certain values of the constants)

    Select the correct answer.

    (a) $u = \sum_{n=1}^{\infty} c_n J_0(n\pi r/3)\cos(n\pi z/3)$
    (b) $u = \sum_{n=1}^{\infty} c_n J_0(n^2\pi^2 r/9)\cos(n\pi z/3)$
    (c) $u = \sum_{n=1}^{\infty} c_n J_0(n^2\pi^2 r/9)\sin(n\pi z/3)$
    (d) $u = \sum_{n=1}^{\infty} c_n J_0(z_n r/2)\sinh(z_n(z-3)/2)$
    (e) $u = \sum_{n=1}^{\infty} c_n J_0(z_n r)\cosh(z_n(z-3))$

15. The relationships between Cartesian and spherical coordinates are

    Select all that apply.

    (a) $r^2 = x^2 + y^2$
    (b) $\tan\phi = y/x$
    (c) $x = r\cos\phi\sin\theta$
    (d) $y = r\cos\phi\cos\theta$
    (e) $z = r\cos\theta$

16. When changing from Cartesian to spherical coordinates, $\frac{\partial r}{\partial z} =$

    Select the correct answer.

    (a) $\cos\phi$
    (b) $\cos\theta$
    (c) $\tan\theta$
    (d) $\sin\phi$
    (e) $\sin\theta$

17. When changing from Cartesian to spherical coordinates, $\frac{\partial \phi}{\partial x} =$

    Select the correct answer.

    (a) $-\sin\phi/(r\sin\theta)$
    (b) $\sin\phi/(r\sin\theta)$
    (c) $-\cos\phi/(r\sin\theta)$
    (d) $\cos\phi/(r\cos\theta)$
    (e) $\sin\phi/(r\cos\theta)$

18. In the problem $\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} = 0$, separate variables, using $u(r,\theta) = R(r)\Theta(\theta)$. The resulting problems for $R$ and $\Theta$ are

    Select the correct answer.

    (a) $r^2R'' + 2rR' + \lambda R = 0$, $R(0)$ is bounded; $\sin(\theta)\Theta'' + \cos(\theta)\Theta' + \lambda\sin(\theta)\Theta = 0$, $\Theta$ is bounded on $[0,\pi]$.
    (b) $r^2R'' + 2rR' - \lambda R = 0$, $R(0)$ is bounded; $\sin(\theta)\Theta'' + \cos(\theta)\Theta' + \lambda\sin(\theta)\Theta = 0$, $\Theta$ is bounded on $[0,\pi]$.
    (c) $r^2R'' + 2rR' - \lambda R = 0$, $R(0)$ is bounded; $\sin(\theta)\Theta'' + \cos(\theta)\Theta' - \lambda\sin(\theta)\Theta = 0$, $\Theta$ is bounded on $[0,\pi]$.
    (d) $r^2R'' - 2rR' - \lambda R = 0$, $R(0)$ is bounded; $\sin(\theta)\Theta'' + \cos(\theta)\Theta' + \lambda\sin(\theta)\Theta = 0$, $\Theta$ is bounded on $[0,\pi]$.
    (e) $r^2R'' + 2rR' - \lambda R = 0$, $R(0)$ is bounded; $\sin(\theta)\Theta'' - \cos(\theta)\Theta' + \lambda\sin(\theta)\Theta = 0$, $\Theta$ is bounded on $[0,\pi]$.

19. In the previous problem, the solution of the eigenvalue problem is

    Select the correct answer.

    (a) $\lambda = n(n+1)$, $R = r^n$, $n = 1, 2, 3, \ldots$
    (b) $\lambda = n(n-1)$, $R = r^{-n}$, $n = 1, 2, 3, \ldots$
    (c) $\lambda = n(n-1)$, $\Theta = P_n(\cos\theta)$, $n = 1, 2, 3, \ldots$
    (d) $\lambda = n(n+1)$, $\Theta = P_n(\cos\theta)$, $n = 1, 2, 3, \ldots$
    (e) $\lambda = n(n+1)$, $\Theta = P_n(\sin\theta)$, $n = 1, 2, 3, \ldots$

20. In the previous two problems, the product solutions are

    Select the correct answer.

    (a) $u = r^n P_n(\cos\theta)$
    (b) $u = r^{-n} P_n(\cos\theta)$
    (c) $u = r^n P_n(\sin\theta)$
    (d) $u = r^{-n} P_n(\sin\theta)$
    (e) none of the above

**ANSWER KEY**
***Zill Differential Equations 9e Chapter 13 Form C***

1. b
2. c
3. b
4. e
5. c
6. b
7. c
8. a, c, d
9. e
10. d
11. d
12. a
13. e
14. d
15. b, c, e
16. b
17. a
18. b
19. d
20. a

1. In changing from Cartesian to polar coordinates, $\frac{\partial r}{\partial y}$ is

   Select the correct answer.

   (a) $\sin\theta$
   (b) $\cos\theta$
   (c) $\sin\theta/r$
   (d) $\cos\theta/r$
   (e) $-\sin\theta/r$

2. In changing from Cartesian to polar coordinates, $\frac{\partial u}{\partial y}$ is

   Select the correct answer.

   (a) $\cos\theta\frac{\partial u}{\partial r} - \sin\theta\frac{\partial u}{\partial \theta}/r$
   (b) $\cos\theta\frac{\partial u}{\partial r} + \cos\theta\frac{\partial u}{\partial \theta}/r$
   (c) $\sin\theta\frac{\partial u}{\partial r} + \cos\theta\frac{\partial u}{\partial \theta}$
   (d) $\sin\theta\frac{\partial u}{\partial r} + \cos\theta\frac{\partial u}{\partial \theta}/r$
   (e) $\sin\theta\frac{\partial u}{\partial r} - \cos\theta\frac{\partial u}{\partial \theta}/r$

3. Consider the steady–state temperature distribution in a half circular disc of radius $c$ centered at the origin, with temperature given as a function, $f(\theta)$ on the boundary $r = c$ and zero on the boundaries $\theta = 0$ and $\theta = \pi$. The mathematical model of this situation is

   Select the correct answer.

   (a) $\frac{\partial^2 u}{\partial r^2} - \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,\ u(c,\theta) = f(\theta),\ u(r,0) = 0,\ u(r,\pi) = 0$
   (b) $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,\ u(c,\theta) = f(\theta),\ u(r,0) = 0,\ u(r,\pi) = 0$
   (c) $\frac{\partial^2 u}{\partial r^2} - \frac{1}{r}\frac{\partial u}{\partial r} - \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,\ u(c,\theta) = f(\theta),\ u(r,0) = 0,\ u(r,\pi) = 0$
   (d) $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} = 0,\ u(c,\theta) = f(\theta),\ u(r,0) = 0,\ u(r,\pi) = 0$
   (e) $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,\ u(c,\theta) = f(\theta),\ u(r,0) = 0,\ u(r,\pi) = 0$

4. In the previous problem, separate variables using $u(r,\theta) = R(r)\Theta(\theta)$. The resulting problems are

   Select the correct answer.

   (a) $R'' + rR' - r^2\lambda R = 0,\ \Theta'' + \lambda\Theta = 0,\ \Theta(0) = 0,\ \Theta(\pi) = 0$
   (b) $R'' + rR' + r^2\lambda R = 0,\ \Theta'' + \lambda\Theta = 0,\ \Theta(0) = 0,\ \Theta(\pi) = 0$
   (c) $r^2R'' + rR' - \lambda R = 0,\ \Theta'' - \lambda\Theta = 0,\ \Theta(0) = 0,\ \Theta(\pi) = 0$
   (d) $r^2R'' + rR' + \lambda R = 0,\ \Theta'' + \lambda\Theta = 0,\ \Theta(0) = 0,\ \Theta(\pi) = 0$
   (e) $r^2R'' + rR' - \lambda R = 0,\ \Theta'' + \lambda\Theta = 0,\ \Theta(0) = 0,\ \Theta(\pi) = 0$

5. In the previous problem, the solution of the eigenvalue problem is

   Select the correct answer.

   (a) $\lambda = n^2$, $R = r^n$, $n = 0, 1, 2, \ldots$
   (b) $\lambda = n$, $R = r^n$, $n = 0, 1, 2, \ldots$
   (c) $\lambda = n$, $\Theta = c_n \cos(n\theta) + d_n \sin(n\theta)$, $n = 0, 1, 2, \ldots$
   (d) $\lambda = n^2$, $\Theta = d_n \sin(n\theta)$, $n = 1, 2, 3, \ldots$
   (e) $\lambda = n^2$, $\Theta = c_n \cos(n\theta)$, $n = 0, 1, 2, \ldots$

6. In the three previous problems, the product solutions are

   Select the correct answer.

   (a) $u_n = r^n(c_n e^{n\theta} + d_n e^{-n\theta})$, $n = 0, 1, 2, \ldots$
   (b) $u_n = r^n(c_n e^{n\theta} + d_n e^{-n\theta})$, $n = 1, 2, 3, \ldots$
   (c) $u_n = r^n(c_n \cos(n\theta) + d_n \sin(n\theta))$, $n = 0, 1, 2, \ldots$
   (d) $u_n = r^n \cos(n\theta)$, $n = 0, 1, 2, \ldots$
   (e) $u_n = r^n \sin(n\theta)$, $n = 1, 2, 3, \ldots$

7. In the previous four problems, the infinite series solution of the original problem is $u = \sum_{n=1}^{\infty} r^n(A_n \cos(n\theta) + B_n \sin(n\theta))$ where

   Select all that apply.

   (a) $A_n = \int_0^{2\pi} f(\theta)\cos(n\theta)d\theta/(c^n\pi)$
   (b) $B_n = \int_0^{2\pi} f(\theta)\sin(n\theta)d\theta/(c^n\pi)$
   (c) $B_n = \int_0^{2\pi} f(\theta)\cos(n\theta)d\theta/(c^n\pi)$
   (d) $A_n = 0$
   (e) $B_n = 0$

8. The two dimensional wave equation in polar coordinates is

   Select the correct answer.

   (a) $a^2\left(\frac{\partial^2 u}{\partial r^2} - \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) = \frac{\partial^2 u}{\partial t^2}$
   (b) $a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) = \frac{\partial^2 u}{\partial t^2}$
   (c) $a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) = \frac{\partial^2 u}{\partial t^2}$
   (d) $a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial^2 u}{\partial \theta^2}\right) = \frac{\partial^2 u}{\partial t^2}$
   (e) $a^2\left(\frac{\partial^2 u}{\partial r^2} - \frac{1}{r^2}\frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial^2 u}{\partial \theta^2}\right) = \frac{\partial^2 u}{\partial t^2}$

9. The solution of $r^2R'' + rR' + \lambda R = 0,\ R(1) = 0,\ R(2) = 0$ is

   Select the correct answer.

   (a) $\lambda = n\pi/\ln 2,\ R = \sin(n\pi \ln r/\ln 2)$
   (b) $\lambda = (n\pi/\ln 2)^2,\ R = \sin(n\pi \ln r/\ln 2)$
   (c) $\lambda = (n\pi/\ln 2)^2,\ R = \cos(n\pi \ln r/\ln 2)$
   (d) $\lambda = n\pi/\ln 2,\ R = \cos(n\pi \ln r/\ln 2)$
   (e) none of the above

10. Consider the vibrations of a circular membrane of radius 2, clamped along the circumference, with an initial displacement of $1 - \sin(\pi r/2)$ and an initial velocity of zero. The mathematical model for this situation is

    Select the correct answer.

    (a) $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial t^2},\ u(2,t) = 0,\ u(r,0) = 1 - \sin(\pi r/2),\ u_t(r,0) = 0$
    (b) $\frac{\partial^2 u}{\partial r^2} - \frac{1}{r}\frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial t^2},\ u(2,t) = 0,\ u(r,0) = 1 - \sin(\pi r/2),\ u_t(r,0) = 0$
    (c) $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = -\frac{\partial^2 u}{\partial t^2},\ u(2,t) = 0,\ u(r,0) = 1 - \sin(\pi r/2),\ u_t(r,0) = 0$
    (d) $\frac{\partial^2 u}{\partial r^2} - \frac{1}{r^2}\frac{\partial u}{\partial r} = -\frac{\partial^2 u}{\partial t^2},\ u(2,t) = 0,\ u(r,0) = 1 - \sin(\pi r/2),\ u_t(r,0) = 0$
    (e) $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial t^2},\ u(2,t) = 0,\ u(r,0) = 1 - \sin(\pi r/2),\ u_t(r,0) = 0$

11. In the previous problem, separate variables using $u(r,t) = R(r)T(t)$. The resulting problems are

    Select the correct answer.

    (a) $rR'' + R' + r\lambda R = 0$, $R(0)$ is bounded, $R(2) = 0,\ T'' - \lambda T = 0,\ T'(0) = 0$
    (b) $rR'' + R' + r\lambda R = 0$, $R(0)$ is bounded, $R(2) = 0,\ T'' + \lambda T = 0,\ T'(0) = 0$
    (c) $rR'' + R' - r\lambda R = 0$, $R(0)$ is bounded, $R(2) = 0,\ T'' + \lambda T = 0,\ T'(0) = 0$
    (d) $rR'' + R' + \lambda R = 0$, $R(0)$ is bounded, $R(2) = 0,\ T'' + \lambda T = 0,\ T'(0) = 0$
    (e) $rR'' + R' + \lambda R = 0$, $R(0)$ is bounded, $R(2) = 0,\ T'' - \lambda T = 0,\ T'(0) = 0$

12. In the previous problem, the solution of the eigenvalue problem is

    Select the correct answer.

    (a) $\lambda = n\pi,\ T = a_n \cos(n\pi t) + b_n \sin(n\pi t)$
    (b) $\lambda = n^2\pi^2,\ T = a_n \cos(n\pi t) + b_n \sin(n\pi t)$
    (c) $\lambda = z_n^2/4$, where $J_0(z_n) = 0,\ R = J_0(z_n r/2)$
    (d) $\lambda = z_n/2$, where $J_0(z_n) = 0,\ R = J_0(z_n r/2)$
    (e) $\lambda = z_n^2$, where $J_0(z_n) = 0,\ R = J_0(z_n r)$

13. In the previous three problems, the product solutions are

    Select the correct answer.

    (a) $u = J_0(z_n r/2)(a_n \cos(z_n t/2) + b_n \sin(z_n t/2))$
    (b) $u = J_0(z_n r/2)a_n \cos(z_n t/2)$
    (c) $u = J_0(z_n r/2)b_n \sin(z_n t/2)$
    (d) $u = J_0(n\pi r)b_n \sin(n\pi t)$
    (e) $u = J_0(n\pi r)a_n \cos(n\pi t)$

14. In the previous four problems, the infinite series solution of the original problem is (for certain values of the constants $a_n$ and $b_n$)

    Select the correct answer.

    (a) $u = \sum_{n=1}^{\infty} a_n J_0(z_n r/2) \cos(z_n t/2)$
    (b) $u = \sum_{n=1}^{\infty} b_n J_0(z_n r/2) \sin(z_n t/2)$
    (c) $u = \sum_{n=1}^{\infty} b_n J_0(n\pi r) \sin(n\pi t)$
    (d) $u = \sum_{n=1}^{\infty} a_n J_0(n\pi r) \cos(n\pi t)$
    (e) $u = \sum_{n=1}^{\infty} J_0(z_n r/2)(a_n \cos(n\pi t) + b_n \sin(n\pi t))$

15. The equations relating Cartesian and spherical coordinates include

    Select all that apply.

    (a) $y = r \cos\phi \cos\theta$
    (b) $x = r \cos\phi \sin\theta$
    (c) $r = x^2 + y^2 + z^2$
    (d) $z = r \cos\theta$
    (e) $\tan\phi = y/x$

16. In changing from Cartesian to spherical coordinates, $\frac{\partial r}{\partial x}$ becomes

    Select the correct answer.

    (a) $\sin\phi \sin\theta$
    (b) $\sin\phi \cos\theta$
    (c) $\cos\phi \sin\theta$
    (d) $\cos\phi \cos\theta$
    (e) $\sin\phi$

17. In separating variables, using $u(r,\theta) = R(r)\Theta(\theta)$, in the equation $\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} = 0$, the resulting equation for $R$ is

    Select the correct answer.

    (a) $r^2R'' + 2rR' - \lambda R = 0$
    (b) $r^2R'' - 2rR' - \lambda R = 0$
    (c) $rR'' + 2rR' - \lambda rR = 0$
    (d) $rR'' + 2R' - \lambda rR = 0$
    (e) $r^2R'' - 2rR' - \lambda rR = 0$

18. Using the separation of the previous problem, the equation for $\Theta$ becomes

    Select the correct answer.

    (a) $\Theta'' - \lambda\Theta = 0$
    (b) $\Theta'' + \lambda\Theta = 0$
    (c) $\sin(\theta)\Theta'' + \cos(\theta)\Theta' + \lambda\sin(\theta)\Theta = 0$
    (d) $\cos(\theta)\Theta'' - \sin(\theta)\Theta' + \lambda\sin(\theta)\Theta = 0$
    (e) $\sin(\theta)\Theta'' - \cos(\theta)\Theta' + \lambda\sin(\theta)\Theta = 0$

19. In the previous problem, if we also require that $\Theta$ be bounded everywhere, the solution of the eigenvalue problem is

    Select the correct answer.

    (a) $\lambda = n(n+1)$, $\Theta = P_n(\sin\theta)$, $n = 1, 2, 3, \ldots$
    (b) $\lambda = n(n-1)$, $\Theta = \sin(n\theta)$, $n = 1, 2, 3, \ldots$
    (c) $\lambda = n(n-1)$, $\Theta = \cos(n\theta)$, $n = 1, 2, 3, \ldots$
    (d) $\lambda = n(n+1)$, $\Theta = P_n(\cos\theta)$, $n = 1, 2, 3, \ldots$
    (e) $\lambda = n(n-1)$, $\Theta = P_n(\sin\theta)$, $n = 1, 2, 3, \ldots$

20. In the previous three problems, the solution for $R$ is

    Select the correct answer.

    (a) $R = r^n$
    (b) $R = r^{-n-1}$
    (c) $R = c_1r^n + c_2r^{-n-1}$
    (d) $R = c_1r^n + c_2r^{n+1}$
    (e) $R = c_1r^n + c_2r^{n-1}$

**ANSWER KEY**

***Zill Differential Equations 9e Chapter 13 Form D***

1. a
2. d
3. e
4. e
5. d
6. e
7. b, d
8. c
9. b
10. a
11. b
12. c
13. b
14. a
15. b, d, e
16. c
17. a
18. c
19. d
20. c

1. Write $\frac{\partial \theta}{\partial x}$ in terms of polar coordinates?

2. Write $\frac{\partial u}{\partial y}$ in terms of polar coordinates?

3. Write down the Laplacian of $u(r, \theta)$ in polar coordinates.

4. Consider the steady–state temperature distribution in a half circular annular disc with $1 \le r \le 2$, $0 \le \theta \le \pi$, with zero temperature on the curved boundaries $r = 1$ and $r = 2$, zero temperature on the boundary $\theta = 0$, and a given temperature, $f(r)$, on the boundary $\theta = \pi$. The mathematical model of this situation is

   Select the correct answer.

   (a) $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$, $u(1, \theta) = 0$, $u(2, \theta) = 0$, $u(r, 0) = 0$, $u(r, \pi) = f(r)$
   (b) $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} = 0$, $u(1, \theta) = 0$, $u(2, \theta) = 0$, $u(r, 0) = 0$, $u(r, \pi) = f(r)$
   (c) $\frac{\partial^2 u}{\partial r^2} - \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$, $u(1, \theta) = 0$, $u(2, \theta) = 0$, $u(r, 0) = 0$, $u(r, \pi) = f(r)$
   (d) $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$, $u(1, \theta) = 0$, $u(2, \theta) = 0$, $u(r, 0) = 0$, $u(r, \pi) = f(r)$
   (e) $\frac{\partial^2 u}{\partial r^2} - \frac{1}{r}\frac{\partial u}{\partial r} - \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$, $u(1, \theta) = 0$, $u(2, \theta) = 0$, $u(r, 0) = 0$, $u(r, \pi) = f(r)$

5. In the previous problem, separate variables using $u(r, \theta) = R(r)\Theta(\theta)$. The resulting problems are

   Select the correct answer.

   (a) $r^2 R'' + rR' - \lambda R = 0$, $R(1) = 0$, $R(2) = 0$, $\Theta'' - \lambda\Theta = 0$, $\Theta(0) = 0$
   (b) $r^2 R'' + rR' + \lambda R = 0$, $R(1) = 0$, $R(2) = 0$, $\Theta'' - \lambda\Theta = 0$, $\Theta(0) = 0$
   (c) $r^2 R'' + rR' + \lambda R = 0$, $R(1) = 0$, $\Theta'' + \lambda\Theta = 0$, $\Theta(0) = 0$, $\Theta(\pi) = 0$
   (d) $R'' + rR' - r^2\lambda R = 0$, $R(1) = 0$, $R(2) = 0$, $\Theta'' + \lambda\Theta = 0$, $\Theta(0) = 0$
   (e) $R'' + rR' + r^2\lambda R = 0$, $R(1) = 0$, $\Theta'' + \lambda\Theta = 0$, $\Theta(0) = 0$, $\Theta(\pi) = 0$

6. In the previous problem, the solution of the eigenvalue problem is

   Select the correct answer.

   (a) $\lambda = n$, $\Theta = c_n \cos(n\theta) + d_n \sin(n\theta)$, $n = 0, 1, 2, \ldots$
   (b) $\lambda = n^2$, $\Theta = d_n \sin(n\theta)$, $n = 1, 2, 3, \ldots$
   (c) $\lambda = n^2$, $\Theta = c_n \cos(n\theta)$, $n = 0, 1, 2, \ldots$
   (d) $\lambda = (n\pi/\ln 2)^2$, $R = \sin(n\pi \ln r/\ln 2)$, $n = 1, 2, 3, \ldots$
   (e) $\lambda = (n\pi/\ln 2)$, $R = \sin(n\pi \ln r/\ln 2)$, $n = 1, 2, 3, \ldots$

7. In the three previous problems, the product solutions are

   Select the correct answer.

   (a) $u_n = r^n(c_n \cos(n\theta) + d_n \sin(n\theta))$, $n = 0, 1, 2, \ldots$
   (b) $u_n = r^n(d_n \sin(n\theta))$, $n = 1, 2, 3, \ldots$
   (c) $u_n = r^n(c_n \cos(n\theta))$, $n = 0, 1, 2, \ldots$
   (d) $u_n = \sin(n\pi \ln r/\ln 2)\cosh(n\pi\theta/\ln 2)$, $n = 1, 2, 3, \ldots$
   (e) $u_n = \sin(n\pi \ln r/\ln 2)\sinh(n\pi\theta/\ln 2)$, $n = 1, 2, 3, \ldots$

8. In the results of previous four problems, the infinite series solution of the problem for the steady–state temperature distribution in a half circular annular disc with $1 \le r \le 2$, $0 \le \theta \le \pi$, with zero temperature on the curved boundaries $r = 1$ and $r = 2$, zero temperature on the boundary $\theta = 0$, and a given temperature, $f(r)$, on the boundary $\theta = \pi$ is (for certain values of the constants $c_n$ and $d_n$)

   Select all that apply.

   (a) $u = \sum_{n=1}^{\infty} c_n r^n(c_n \cos(n\theta) + d_n \sin(n\theta))$
   (b) $u = \sum_{n=1}^{\infty} c_n r^n(d_n \sin(n\theta))$
   (c) $u = \sum_{n=1}^{\infty} c_n r^n(c_n \cos(n\theta))$
   (d) $u = \sum_{n=1}^{\infty} d_n \sin(n\pi \ln r/\ln 2)\cosh(n\pi\theta/\ln 2)$
   (e) $u = \sum_{n=1}^{\infty} d_n \sin(n\pi \ln r/\ln 2)\sinh(n\pi\theta/\ln 2)$

9. What is the solution of the problem $\Theta'' + \lambda\Theta = 0$, $\Theta(\theta) = \Theta(\theta + 2\pi)$? Is this a regular Sturm-Liouville problem?

10. Write down the relationships between Cartesian and spherical coordinates.

11. For spherical coordinates, what is $\frac{\partial r}{\partial x}$

12. For spherical coordinates, what is $\frac{\partial \phi}{\partial z}$?

13. The mathematical model for steady–state temperature distribution, $u(r, z)$, in a cylinder of radius 3 and height 5, with zero temperature at $z = 0$ and $r = 3$ and with a given constant temperature $u_0$ on the side $z = 5$ is

    Select the correct answer.

    (a) $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0$, $u(r, 0) = 0$, $u(3, z) = 0$, $u(r, 5) = u_0$
    (b) $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0$, $u(r, 0) = 0$, $u(3, z) = 0$, $u(r, 5) = u_0$
    (c) $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial^2 u}{\partial z^2} = 0$, $u(r, 0) = 0$, $u(3, z) = 0$, $u(r, 5) = u_0$
    (d) $\frac{\partial^2 u}{\partial r^2} - \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial z^2} = 0$, $u(r, 0) = 0$, $u(3, z) = 0$, $u(r, 5) = u_0$
    (e) $\frac{\partial^2 u}{\partial r^2} - \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0$, $u(r, 0) = 0$, $u(3, z) = 0$, $u(r, 5) = u_0$

14. In the previous problem, separate variables, using $u(r,z) = R(r)Z(z)$. The resulting problems for $R$ and $Z$ are (including the condition that $R(0)$ is bounded)

    Select the correct answer.

    (a) $rR'' + R' - \lambda R = 0,\ R(3) = 0,\ Z'' - \lambda Z = 0,\ Z(0) = 0,\ Z(5) = u_0$
    (b) $rR'' + R' - \lambda R = 0,\ R(3) = 0,\ Z'' - \lambda Z = 0,\ Z(0) = 0,$
    (c) $rR'' + R' + \lambda rR = 0,\ R(3) = 0,\ Z'' - \lambda Z = 0,\ Z(0) = 0,$
    (d) $rR'' + R' + \lambda rR = 0,\ R(3) = 0,\ Z'' + \lambda Z = 0,\ Z(0) = 0,$
    (e) $rR'' + R' - \lambda rR = 0,\ R(3) = 0,\ Z'' + \lambda Z = 0,\ Z(0) = 0,\ Z(5) = u_0$

15. In the previous problem, the solution of the eigenvalue problem is

    Select the correct answer.

    (a) $\lambda = (n\pi/5)^2,\ Z = \cos(n\pi z/5),\ n = 1, 2, 3, \ldots$
    (b) $\lambda = (n\pi/5)^2,\ Z = \sin(n\pi z/5),\ n = 1, 2, 3, \ldots$
    (c) $\lambda = n\pi/5,\ Z = \sin(n\pi z/5),\ n = 1, 2, 3, \ldots$
    (d) $\lambda = z_n/3,\ R = J_0(z_n r/3),\ n = 1, 2, 3, \ldots,$ where $J_0(z_n) = 0$
    (e) $\lambda = (z_n/3)^2,\ R = J_0(z_n r/3),\ n = 1, 2, 3, \ldots,$ where $J_0(z_n) = 0$

16. In the previous three problems, the product solutions are

    Select the correct answer.

    (a) $u = J_0(z_n r/3)\cosh(z_n z/3)$
    (b) $u = J_0(z_n r/3)\sinh(z_n z/3)$
    (c) $u = J_0(n\pi r/5)\cos(n\pi z/5)$
    (d) $u = J_0(n\pi r/5)\sin(n\pi z/5)$
    (e) $u = J_0(n\pi r/3)\cos(n\pi z/5)$

17. Write down the infinite series solution for the previous four problems.

18. Write down the Laplacian of a function $u(r,\phi,\theta)$ in spherical coordinates.

19. If the function $u$ in spherical coordinates is independent of $\theta$, what are the equations resulting from the separation of variables, using $u(r,\phi) = R(r)\Phi(\phi)$?

20. If the function $u$ in spherical coordinates is independent of $\phi$, what are the equations resulting from the separation of variables, using $u(r,\theta) = R(r)\Phi(\theta)$?

**ANSWER KEY**

***Zill Differential Equations 9e Chapter 13 Form E***

1. $\frac{\partial\theta}{\partial x} = -\sin\theta/r$
2. $\frac{\partial u}{\partial y} = \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}$
3. $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}$
4. a
5. b
6. d
7. e
8. e
9. $\lambda = n^2$, $\Theta = c_1\cos(n\theta) + c_2\sin(n\theta)$, $n = 0, 1, 2, \ldots$; no, it is not regular
10. $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, $r^2 = x^2 + y^2 + z^2$, $\tan\phi = y/x$, $\cos\theta = z/\sqrt{x^2 + y^2 + z^2}$
11. $\frac{\partial r}{\partial x} = \sin\theta\cos\phi$
12. $\frac{\partial\phi}{\partial z} = 0$
13. a
14. c
15. e
16. b
17. $u = \sum_{n=1}^{\infty} c_n J_0(z_n r/3)\sinh(z_n z/3)$, where $c_n = u_0\int_0^3 rJ_0(z_n r/3)dr/(sinh(5z_n/3)\int_0^3 rJ_0{}^2(z_n r/3)dr)$
18. $u_{rr} + 2u_r/r + u_{\phi\phi}/(r^2\sin^2\theta) + u_{\theta\theta}/r^2 + \cot\theta u_\theta/r^2$
19. Not possible, because of the $1/\sin^2\theta$ term multiplying $u_{\phi\phi}$
20. $r^2R'' + 2rR' - \lambda R = 0$, $\sin\theta\Theta'' + \cos\theta\Theta' + \lambda\sin\theta\Theta = 0$

1. In changing from Cartesian to polar coordinates, $\frac{\partial\theta}{\partial y}$ is

   Select the correct answer.

   (a) $\sin\theta$
   (b) $\cos\theta$
   (c) $\cos\theta/r$
   (d) $\sin\theta/r$
   (e) $-\sin\theta/r$

2. In changing from Cartesian to polar coordinates, $\frac{\partial u}{\partial y}$ is

   Select the correct answer.

   (a) $\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}$
   (b) $\cos\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}$
   (c) $\sin\theta\frac{\partial u}{\partial r} + \cos\theta\frac{\partial u}{\partial\theta}$
   (d) $\sin\theta\frac{\partial u}{\partial r} - \frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}$
   (e) $\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}$

3. The Laplacian in polar coordinates is

   Select the correct answer.

   (a) $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial^2 u}{\partial\theta^2}$
   (b) $\frac{\partial^2 u}{\partial r^2} - \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}$
   (c) $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}$
   (d) $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}$
   (e) $\frac{\partial^2 u}{\partial r^2} - \frac{1}{r}\frac{\partial u}{\partial r} - \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}$

4. Consider the problem $r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial\theta^2} = 0$; $u(c,\theta) = f(\theta)$, $u(r,0) = 0$, $u(r,\pi) = 0$. Describe a physical situation having this as a mathematical model.

5. In the previous problem, separate variables using $u(r,\theta) = R(r)\Theta(\theta)$. What are the resulting differential equations and boundary conditions?

6. In the previous problem, what are the solutions of the eigenvalue problem? Is the eigenvalue problem a regular Sturm-Liouville problem?

7. In the three previous problems, what are the product solutions?

8. In the previous four problems, what is the infinite series solution of the problem $r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial\theta^2} = 0$; $u(c,\theta) = f(\theta)$, $u(r,0) = 0$, $u(r,\pi) = 0$?

9. The two dimensional heat equation in polar coordinates is

   Select the correct answer.

   (a) $k\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) = \frac{\partial u}{\partial t}$
   (b) $k\left(\frac{\partial^2 u}{\partial r^2} - \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) = \frac{\partial u}{\partial t}$
   (c) $k\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) = \frac{\partial u}{\partial t}$
   (d) $k\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial^2 u}{\partial \theta^2}\right) = \frac{\partial u}{\partial t}$
   (e) $k\left(\frac{\partial^2 u}{\partial r^2} - \frac{1}{r^2}\frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial^2 u}{\partial \theta^2}\right) = \frac{\partial u}{\partial t}$

10. The solution of $r^2R'' + rR' + \lambda R = 0,\ R(0) = 0,\ R(1) = 0$ is

    Select the correct answer.

    (a) $\lambda = n\pi,\ R = \sin(n\pi \ln r)$
    (b) $\lambda = (n\pi)^2,\ R = \sin(n\pi \ln r)$
    (c) $\lambda = (n\pi)^2,\ R = \sin(n\pi \ln r) + \cos(n\pi \ln r)$
    (d) $\lambda = n\pi,\ R = \sin(n\pi \ln r) - \cos(n\pi \ln r)$
    (e) none of the above

11. Write down the relationships between Cartesian and spherical coordinates.

12. In changing from Cartesian to spherical coordinates, what is $\frac{\partial \theta}{\partial x}$?

13. In changing from Cartesian to spherical coordinates, what is $\frac{\partial \phi}{\partial x}$?

14. The two-dimensional wave equation in polar coordinates is

    Select the correct answer

    (a) $a^2\left(\frac{\partial^2 u}{\partial r^2} - \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\{\partial^2 u}{\partial \theta^2}\right) = \frac{\partial^2 u}{\partial t^2}$
    (b) $a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) = \frac{\partial^2 u}{\partial t^2}$
    (c) $a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial^2 u}{\partial \theta^2}\right) = \frac{\partial^2 u}{\partial t^2}$
    (d) $a^2\left(\frac{\partial^2 u}{\partial r^2} - \frac{1}{r^2}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) = \frac{\partial^2 u}{\partial t^2}$
    (e) $a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) = \frac{\partial^2 u}{\partial t^2}$

15. Consider the wave equation from the previous problem, but consider the case where $u$ is independent of $\theta$. Separate variables, using $u(r,t) = R(r)T(t)$. The resulting equations for $R$ and $T$ are

    Select the correct answer

    (a) $r^2R'' + rR' + \lambda R = 0,\ T'' + a^2\lambda T = 0$
    (b) $r^2R'' + rR' - \lambda R = 0,\ T'' + a^2\lambda T = 0$
    (c) $rR'' + R' - \lambda rR = 0,\ T'' + a^2\lambda T = 0$
    (d) $rR'' + R' + \lambda rR = 0,\ T'' + a^2\lambda T = 0$
    (e) $rR'' + R' + \lambda rR = 0,\ T'' - a^2\lambda T = 0$

16. In the previous problem, assume that $R$ satisfies the boundary conditions $R(0)$ is bounded, $R(1) = 0$. The solution of the resulting eigenvalue problem is

    Select the correct answer

    (a) $\lambda = z_n$, $R = J_0(z_n r)$, $n = 1, 2, 3, \ldots$, where $J_0(z_n) = 0$
    (b) $\lambda = z_n^2$, $R = J_0(z_n r)$, $n = 1, 2, 3, \ldots$, where $J_0(z_n) = 0$
    (c) $\lambda = (n\pi)^2$, $R = J_0(n\pi r)$, $n = 1, 2, 3, \ldots$
    (d) $\lambda = n\pi$, $R = J_0(n\pi r)$, $n = 1, 2, 3, \ldots$
    (e) $\lambda = n^2$, $R = J_0(nr)$, $n = 1, 2, 3, \ldots$

17. In the previous two problems, the product solutions are

    Select the correct answer

    (a) $u = J_0(z_n r)\sin(az_n t)$
    (b) $u = J_0(z_n r)\cos(az_n t)$
    (c) $u = J_0(z_n r)(a_n \cos(az_n t) + b_n \sin(az_n t))$
    (d) $u = J_0(n\pi r)(a_n \cos(an\pi t) + b_n \sin(an\pi t))$
    (e) $u = J_0(nr)(a_n \cos(ant) + b_n \sin(ant))$

18. Write down the three-dimensional wave equation for $u(r, \phi, \theta, t)$ in spherical coordinates.

19. In the previous problem, assume that the function $u$ is independent of $\phi$ and $\theta$. What are the equations that result from separation of variables using $u(r, t) = R(r)T(t)$?

20. What is the solution of the eigenvalue problem $rR'' + R' + \lambda rR = 0$, $R(0)$ is bounded, $R(1) = 0$?

**ANSWER KEY**
***Zill Differential Equations 9e Chapter 13 Form F***

1. c
2. e
3. d
4. Steady–state temperature distribution in a half circular plate of radius $c$, with zero temperature on the diameter and a given temperature, $f(\theta)$ on the curved side
5. $r^2R'' + rR' - \lambda R = 0$, $R(0)$ is bounded, $\Theta'' + \lambda\Theta = 0$, $\Theta(0) = 0$, $\Theta(\pi) = 0$
6. $\lambda = n^2$, $\Theta = \sin(n\theta)$, $n = 1, 2, 3, \ldots$; yes, it is regular
7. $u = r^n \sin(n\theta)$
8. $u = \sum_{n=1}^{\infty} a_n r^n \sin(n\theta)$, where $a_n = 2\int_0^{\pi} f(\theta)\sin(n\theta)d\theta/(c^n\pi)$
9. a
10. e
11. $x = r\cos\phi\sin\theta$, $y = r\sin\phi\sin\theta$, $z = r\cos\theta$, $r^2 = x^2 + y^2 + z^2$, $\tan\phi = y/x$, $\cos\theta = z/\sqrt{x^2 + y^2 + z^2}$
12. $\frac{\partial\theta}{\partial x} = \cos\phi\cos\theta/r$
13. $\frac{\partial\phi}{\partial x} = -\sin\phi/(r\sin\theta)$
14. e
15. d
16. b
17. c
18. $a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial\theta}\right) = \frac{\partial^2 u}{\partial t^2}$
19. $rR'' + 2R' + \lambda rR = 0$, $T'' + a^2\lambda T = 0$
20. $\lambda = z_n^2$, $R = J_0(z_n r)$, $n = 1, 2, 3, \ldots$, where $J_0(z_n) = 0$

1. Write $\frac{\partial \theta}{\partial y}$ in terms of polar coordinates?

2. Write $\frac{\partial u}{\partial x}$ in terms of polar coordinates?

3. Consider the steady–state temperature distribution in a quarter of a circular disc of radius $c$ centered at the origin, with temperature given as a function, $f(\theta)$ on the boundary $r = c$ and zero on the boundaries $\theta = 0$ and $\theta = \pi/2$. The mathematical model of this situation is

   Select the correct answer.

   (a) $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} = 0,\ u(c,\theta) = f(\theta),\ u(r,0) = 0,\ u(r,\pi/2) = 0$
   (b) $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,\ u(c,\theta) = f(\theta),\ u(r,0) = 0,\ u(r,\pi/2) = 0$
   (c) $\frac{\partial^2 u}{\partial r^2} - \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,\ u(c,\theta) = f(\theta),\ u(r,0) = 0,\ u(r,\pi/2) = 0$
   (d) $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,\ u(c,\theta) = f(\theta),\ u(r,0) = 0,\ u(r,\pi/2) = 0$
   (e) $\frac{\partial^2 u}{\partial r^2} - \frac{1}{r}\frac{\partial u}{\partial r} - \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,\ u(c,\theta) = f(\theta),\ u(r,0) = 0,\ u(r,\pi/2) = 0$

4. In the previous problem, separate variables using $u(r,\theta) = R(r)\Theta(\theta)$. The resulting problems are (including the conditions: $R(0)$ is bounded, $\Theta(0) = 0$, $\Theta(\pi/2) = 0$)

   Select the correct answer.

   (a) $R'' + rR' - r^2\lambda R = 0,\ \Theta'' + \lambda\Theta = 0$
   (b) $R'' + rR' + r^2\lambda R = 0,\ \Theta'' + \lambda\Theta = 0$
   (c) $r^2R'' + rR' - \lambda R = 0,\ \Theta'' - \lambda\Theta = 0$
   (d) $r^2R'' + rR' - \lambda R = 0,\ \Theta'' + \lambda\Theta = 0$
   (e) $r^2R'' + rR' + \lambda R = 0,\ \Theta'' + \lambda\Theta = 0$

5. In the previous problem, the solution of the eigenvalue problem is

   Select the correct answer.

   (a) $\lambda = 2n,\ \Theta = c_n\cos(2n\theta) + d_n\sin(2n\theta),\ n = 0, 1, 2, \ldots$
   (b) $\lambda = 4n^2,\ \Theta = d_n\sin(2n\theta),\ n = 1, 2, 3, \ldots$
   (c) $\lambda = 4n^2,\ \Theta = c_n\cos(2n\theta),\ n = 0, 1, 2, \ldots$
   (d) $\lambda = 4n^2,\ R = r^{2n},\ n = 0, 1, 2, \ldots$
   (e) $\lambda = 2n,\ R = r^{2n},\ n = 0, 1, 2, \ldots$

6. In the three previous problems, the product solutions are

   Select the correct answer.

   (a) $u_n = r^{2n}(c_ne^{2n\theta} + d_ne^{-2n\theta}),\ n = 0, 1, 2, \ldots$
   (b) $u_n = r^{2n}(c_ne^{2n\theta} + d_ne^{-2n\theta}),\ n = 1, 2, 3, \ldots$
   (c) $u_n = r^{2n}(c_n\cos(2n\theta) + d_n\sin(2n\theta)),\ n = 0, 1, 2, \ldots$
   (d) $u_n = r^{2n}\cos(2n\theta),\ n = 0, 1, 2, \ldots$
   (e) $u_n = r^{2n}\sin(2n\theta),\ n = 1, 2, 3, \ldots$

7. In the previous four problems, the infinite series solution of the original problem is $u = \sum_{n=1}^{\infty} r^{2n}(A_n \cos(2n\theta) + B_n \sin(2n\theta))$ where

   Select all that apply.

   (a) $A_n = 0$
   (b) $B_n = 0$
   (c) $B_n = 4\int_0^{\pi/2} f(\theta)\sin(2n\theta)d\theta/(c^{2n}\pi)$
   (d) $B_n = 4\int_0^{\pi/2} f(\theta)\cos(2n\theta)d\theta/(c^{2n}\pi)$
   (e) $A_n = 4\int_0^{\pi/2} f(\theta)\cos(2n\theta)d\theta/(c^{2n}\pi)$

8. Write down the two dimensional heat equation in polar coordinates.

9. What is the solution of the problem $\Theta'' + \lambda\Theta = 0$, $\Theta(0) = 0$, $\Theta(\pi) = 0$? Is this a regular Sturm-Liouville problem?

10. Write down the relationships between Cartesian and spherical coordinates.

11. In changing from Cartesian to spherical coordinates, what is $\frac{\partial \theta}{\partial y}$?

12. In changing from Cartesian to spherical coordinates, what is $\frac{\partial \phi}{\partial y}$?

13. Consider the equation $\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} = 0$. This might represent

    Select the correct answer

    (a) steady–state temperature in a circular plate
    (b) steady–state temperature in a circular cylinder
    (c) steady–state temperature in a sphere
    (d) a vibrating circular cylinder
    (e) none of the above

14. In the previous problem, separation of variables using $u(r,\theta) = R(r)\Theta(\theta)$ results in

    Select the correct answer

    (a) $r^2R'' + 2rR' + \lambda R = 0$, $\sin(\theta)\Theta'' + \cos(\theta)\Theta' + \lambda\sin(\theta)\Theta = 0$
    (b) $r^2R'' + 2rR' - \lambda R = 0$, $\sin(\theta)\Theta'' + \cos(\theta)\Theta' + \lambda\sin(\theta)\Theta = 0$
    (c) $r^2R'' - 2rR' + \lambda R = 0$, $\sin(\theta)\Theta'' + \cos(\theta)\Theta' + \lambda\sin(\theta)\Theta = 0$
    (d) $r^2R'' - 2rR' - \lambda R = 0$, $\sin(\theta)\Theta'' + \cos(\theta)\Theta' + \lambda\sin(\theta)\Theta = 0$
    (e) $rR'' + 2R' - \lambda rR = 0$, $\sin(\theta)\Theta'' + \cos(\theta)\Theta' + \lambda\sin(\theta)\Theta = 0$

15. In the previous problem, the restriction on $\Theta$ is that $\Theta$ is bounded everywhere. The solution of this eigenvalue problem is

    Select the correct answer

    (a) $\lambda = n^2\pi^2$, $\Theta = \sin(n\pi\theta)$, $n = 1, 2, 3, \ldots$
    (b) $\lambda = n^2$, $\Theta = \sin(n\theta)$, $n = 1, 2, 3, \ldots$
    (c) $\lambda = n$, $\Theta = \sin(n\theta)$, $n = 1, 2, 3, \ldots$
    (d) $\lambda = n(n+1)$, $\Theta = P_n(\cos\theta)$, $n = 1, 2, 3, \ldots$
    (e) $\lambda = n^2$, $\Theta = P_n(\cos\theta)$, $n = 1, 2, 3, \ldots$

16. Consider the problem $\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial t^2}$, $u(0,t) = 0$, $u(2,t) = 0$, $u(r,0) = f(r)$, $\frac{\partial u}{\partial t}(r,0) = 0$. Verify that the left hand side can be written as $\frac{1}{r}\frac{\partial^2(ru)}{\partial r^2}$, and simplify by using the substitution $v = ru$.

17. In the previous problem, separate variables using $v(r,t) = R(r)T(t)$. What are the resulting problems for $R$ and $T$?

18. In the previous problem, what are the solutions of the eigenvalue problem?

19. In the previous three problems, what are the product solutions?

20. In the previous four problems, what is the infinite series solution of the problem $\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial t^2}$, $u(0,t) = 0$, $u(2,t) = 0$, $u(r,0) = f(r)$, $\frac{\partial u}{\partial t}(r,0) = 0$?

1. $\frac{\partial \theta}{\partial y} = \cos\theta / r$

2. $\frac{\partial u}{\partial x} = \cos\theta \frac{\partial u}{\partial r} - \sin\theta \frac{\partial u}{\partial \theta} / r$

3. b

4. d

5. b

6. e

7. a, c

8. $k\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) = \frac{\partial u}{\partial t}$

9. $\lambda = n^2$, $\Theta = \sin(n\theta)$, $n = 1, 2, 3, \ldots$; yes, it is regular

10. $x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta,\ r^2 = x^2 + y^2 + z^2,\ \tan\phi = y/x,$ $\cos\theta = z/\sqrt{x^2 + y^2 + z^2}$

11. $\frac{\partial \theta}{\partial y} = \cos\theta\sin\phi / r$

12. $\frac{\partial \phi}{\partial y} = \cos\phi/(r\sin\theta)$

13. c

14. b

15. d

16. $\frac{1}{r}\frac{\partial^2}{\partial r^2}(ru) = \frac{1}{r}\frac{\partial}{\partial r}(u + r\frac{\partial u}{\partial r}) = \frac{1}{r}\left(\frac{\partial u}{\partial r} + \frac{\partial u}{\partial r} + r\frac{\partial^2 u}{\partial r^2}\right) = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r},\ \frac{\partial^2 v}{\partial r^2} = \frac{\partial^2 v}{\partial t^2}$

17. $R'' + \lambda R = 0,\ R(0) = 0,\ R(2) = 0,\ T'' + \lambda T = 0,\ T'(0) = 0$

18. $\lambda = (n\pi/2)^2,\ R = \sin(n\pi r/2),\ n = 1, 2, 3, \ldots$

19. $u = \sin(n\pi r/2)\cos(n\pi t/2)$

20. $u = \sum_{n=1}^{\infty} c_n \sin(n\pi r/2)\cos(n\pi t/2),$ where $c_n = \int_0^2 f(r)\sin(n\pi r/2)dr$

1. In changing from Cartesian to polar coordinates, $\frac{\partial \theta}{\partial x}$ is

   Select the correct answer.

   (a) $\sin\theta$
   (b) $\cos\theta$
   (c) $\cos\theta/r$
   (d) $\sin\theta/r$
   (e) $-\sin\theta/r$

2. In changing from Cartesian to polar coordinates, $\frac{\partial u}{\partial x}$ is

   Select the correct answer.

   (a) $\cos\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}$
   (b) $\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}$
   (c) $\sin\theta\frac{\partial u}{\partial r} + \cos\theta\frac{\partial u}{\partial \theta}$
   (d) $\sin\theta\frac{\partial u}{\partial r} - \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}$
   (e) $\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}$

3. Consider the problem $r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0$, $u(1,\theta) = 0$, $u(2,\theta) = 0$, $u(r,0) = 0$, $u(r,\pi/2) = f(r)$. Describe a physical situation having this as a mathematical model.

4. In the previous problem, separate variables using $u(r,\theta) = R(r)\Theta(\theta)$. What are the resulting problems?

5. In the previous problem, what are the solutions of the eigenvalue problem. Is the eigenvalue problem a regular Sturm-Liouville problem?

6. In the three previous problems, what are the product solutions?

7. In the previous four problems, what is the infinite series solution of the original problem?

8. The solution of $r^2R'' + rR' + \lambda R = 0$, $R(1) = 0$, $R(3) = 0$ is

   Select the correct answer.

   (a) $\lambda = n\pi/\ln 3$, $R = \cos(n\pi \ln r/\ln 3)$
   (b) $\lambda = n\pi/\ln 3$, $R = \sin(n\pi \ln r/\ln 3)$
   (c) $\lambda = (n\pi/\ln 3)^2$, $R = \sin(n\pi \ln r/\ln 3)$
   (d) $\lambda = (n\pi/\ln 3)^2$, $R = \cos(n\pi \ln r/\ln 3)$
   (e) none of the above

9. Write down the relationships between Cartesian and spherical coordinates.

10. Write $\frac{\partial \theta}{\partial z}$ in terms of spherical coordinates?

11. Write $\frac{\partial \phi}{\partial z}$ in terms of spherical coordinates?

12. Consider the steady–state temperature in a circular cylinder of radius 2 and height 3, with zero temperature at $r = 2$, temperature of 20 at $z = 3$, and which is insulated at $z = 0$. The mathematical model for the temperature, $u(r, z)$, is (including: $u(0, z)$ is bounded, $u(2, t) = 0$, $u(r, 3) = 20$)

    Select the correct answer

    (a) $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0$, $\frac{\partial u}{\partial z}(r, 0) = 0$
    (b) $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial z^2}$, $\frac{\partial u}{\partial z}(r, 0) = 0$
    (c) $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial z^2}$, $u(r, 0) = 0$
    (d) $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0$, $u(r, 0) = 0$
    (e) $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial z^2}$, $\frac{\partial u}{\partial z}(r, 0) = 0$

13. In the previous problem, after separating variables using $u(r, z) = R(r)Z(z)$, the resulting problems are

    Select the correct answer

    (a) $rR'' + R' - \lambda rR = 0$, $R(0)$ is bounded, $R(2) = 0$, $Z'' + \lambda Z = 0$, $Z'(0) = 0$, $Z(3) = 0$
    (b) $r^2R'' + rR' - \lambda R = 0$, $R(0)$ is bounded, $R(2) = 0$, $Z'' + \lambda Z = 0$, $Z'(0) = 0$, $Z(3) = 0$
    (c) $r^2R'' + rR' + \lambda R = 0$, $R(0)$ is bounded, $R(2) = 0$, $Z'' + \lambda Z = 0$, $Z(3) = 0$
    (d) $rR'' + R' + \lambda rR = 0$, $R(0)$ is bounded, $R(2) = 0$, $Z'' - \lambda Z = 0$, $Z'(0) = 0$
    (e) $rR'' + R' + \lambda rR = 0$, $R(0)$ is bounded, $R(2) = 0$, $Z'' + \lambda Z = 0$, $Z'(0) = 0$, $Z(3) = 0$

14. The solution of the eigenvalue problem of the previous problem is

    Select the correct answer

    (a) $\lambda = n\pi/3$, $Z = \cos(n\pi z/3)$, $n = 1, 2, 3, \ldots$
    (b) $\lambda = n\pi/3$, $Z = \sin(n\pi z/3)$, $n = 1, 2, 3, \ldots$
    (c) $\lambda = z_n/2$, $R = J_0(z_n r/2)$, where $J_0(z_n) = 0$, $n = 1, 2, 3, \ldots$
    (d) $\lambda = z_n^2$, $R = J_0(z_n r)$, where $J_0(z_n) = 0$, $n = 1, 2, 3, \ldots$
    (e) $\lambda = z_n^2/4$, $R = J_0(z_n r/2)$, where $J_0(z_n) = 0$, $n = 1, 2, 3, \ldots$

15. Product solutions of the previous three problems are

    Select the correct answer

    (a) $u = J_0(z_n r/2)\sinh(z_n z/2)$
    (b) $u = J_0(z_n r/2)\cosh(z_n z/2)$
    (c) $u = J_0(n^2\pi^2 r/9)\cos(n\pi z/3)$
    (d) $u = J_0(n\pi r/3)\sin(n\pi z/3)$
    (e) $u = J_0(n\pi r/3)\cos(n\pi z/3)$

16. The infinite series solution of the previous four problems is (for certain values of the constants $c_n$)

    Select the correct answer

    (a) $u = \sum_{n=1}^{\infty} c_n J_0((n\pi/3)^2 r)\cos(n\pi z/3)$
    (b) $u = \sum_{n=1}^{\infty} c_n J_0(z_n r/2)\sinh(z_n z/2)$
    (c) $u = \sum_{n=1}^{\infty} c_n J_0(z_n r/2)\cosh(z_n z/2)$
    (d) $u = \sum_{n=1}^{\infty} c_n J_0(n\pi r/3)\sin(n\pi z/3)$
    (e) $u = \sum_{n=1}^{\infty} c_n J_0(n\pi r/3)\cos(n\pi z/3)$

17. Write down the Laplacian of $u$ in cylindrical coordinates.

18. Write down the Laplacian of $u$ in spherical coordinates.

19. What difficulty would you encounter if you were to try to solve Laplace's equation in spherical coordinates for a function which depends only on $r$ and $\phi$ and is independent of $\theta$?

20. What is the solution of $\sin(\theta)\Theta'' + \cos(\theta)\Theta' + \lambda\sin(\theta)\Theta = 0$, $\Theta$ is bounded on $[0, \pi]$?

**ANSWER KEY**

***Zill Differential Equations 9e Chapter 13 Form H***

1. e

2. b

3. Steady–state temperature distribution in a quarter circular annulus between $r = 1$ and $r = 2$, with zero temperature on $r = 1$, $r = 2$, and $\theta = 0$, and temperature of $f$ on $\theta = \pi/2$

4. $r^2R'' + rR' + \lambda R = 0$, $R(1) = 0$, $R(2) = 0$, $\Theta'' - \lambda\Theta = 0$, $\Theta(0) = 0$

5. $\lambda = (n\pi/\ln 2)^2$, $R = \sin(n\pi \ln r/\ln 2)$, $n = 1, 2, 3, \ldots$

6. $u = \sin(n\pi \ln r/\ln 2)\sinh(n\pi\theta/\ln 2)$

7. $u = \sum_{n=1}^{\infty} c_n \sin(n\pi \ln r/\ln 2)\sinh(n\pi\theta/\ln 2)$, where
$c_n = \int_1^2 rf(r)\sin(n\pi \ln r/\ln 2)dr/(\sinh(n\pi^2/(2\ln 2))\int_1^2 r\sin^2(n\pi \ln r/\ln 2)dr)$

8. c

9. $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, $r^2 = x^2 + y^2 + z^2$, $\tan\phi = y/x$, $\cos\theta = z/\sqrt{x^2 + y^2 + z^2}$

10. $\frac{\partial\theta}{\partial z} = -\sin\theta/r$

11. $\frac{\partial\phi}{\partial z} = 0$

12. a

13. d

14. e

15. b

16. c

17. $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\partial^2 u}{\partial z^2}$

18. $\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial\theta}$

19. $\theta$ appears in the coefficient of $\frac{\partial^2 u}{\partial\phi^2}$

20. $\lambda = n(n+1)$, $\Theta = P_n(\cos\theta)$, $n = 1, 2, 3, \ldots$