1. Define $\text{erf}(x)$.

2. What is $\lim_{x\to\infty} \text{erf}(x)$?

3. What is $\mathcal{L}\{\text{erf}(\sqrt{t})\}$?

4. What is $\mathcal{L}\{e^t \text{erf}(\sqrt{t})\}$?

5. What is $\mathcal{L}\left\{\frac{\partial u(x,t)}{\partial t}\right\}$?

6. Consider the wave equation problem $a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$ with initial and boundary conditions $u(0,t) = 0$, $u(1,t) = 0$, $u(x,0) = f(x)$, $\frac{\partial u}{\partial t}(x,0) = 0$. Describe a physical problem with this as a mathematical model.

7. In the previous problem, apply the Laplace transform to the differential equation. What is the resulting problem for $U(x,s) = \mathcal{L}\{u(x,t)\}$?

8. In the previous problem, if $f(x) = \sin(\pi x)$, what is the solution for $U(x,s)$?

9. In the previous three problems, what is the solution for $u(x,t)$?

10. Define the Fourier integral of a function f.

11. Give conditions on the function f that guarantee that the Fourier integral of f converges.

12. In the previous problem, what additional conditions would guarantee that the Fourier integral of f converges to $f(x)$?

13. Let $f(x) = \left\{\begin{array}{lc} 0 & \text{if} \quad x < -1 \\ 1 & \text{if} \quad -1 < x < 1 \\ 0 & \text{if} \quad x > 1 \end{array}\right\}$. Find the Fourier integral representation of f.

14. For the function defined in the previous problem, find the Fourier cosine integral representation.

15. For the function defined in the previous two problems, find the Fourier sine integral representation.

16. Write down the Fourier transform pairs for a function f.

17. Consider the heat equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ on $-\infty < x < \infty$, subject to the condition $u(x,0) = \left\{\begin{array}{lc} 0 & \text{if} \quad x < -1 \\ 1 & \text{if} \quad -1 < x < 1 \\ 0 & \text{if} \quad x > 1 \end{array}\right\}$. Apply a Fourier transform, and write down the transformed equation for the Fourier transform $U(\alpha,t)$.

18. In the previous problem, what is the transformed initial condition?

19. In the previous two problems, what is the solution for the Fourier transform, $U(\alpha,t)$?

20. In the previous three problems, what is the solution for $u(x,t)$?

ANSWER KEY
Zill Differential Equations 9e Chapter 14 Form A

1. $\text{erf}(x) = 2\int_0^x e^{-u^2}\,du/\sqrt{\pi}$
2. 1
3. $1/(s\sqrt{s+1})$
4. $1/(\sqrt{s}(s-1))$
5. $sU(x,s) - u(x,0)$
6. Vibrations of a string, tightly stretched between $x = 0$ and $x = 1$, with initial position $f(x)$ and zero initial velocity
7. $a^2U_{xx} = s^2U - sf(x)$, $U(0,s) = 0$, $U(1,s) = 0$
8. $U = [s/(s^2 + a^2\pi^2)]\sin(\pi x)$
9. $u = \cos(a\pi t)\sin(\pi x)$
10. $f(x) = \int_0^\infty [(\int_{-\infty}^\infty f(t)\cos(\alpha t)\,dt)\cos(\alpha x) + (\int_{-\infty}^\infty f(t)\sin\alpha t\,dt)\sin(\alpha x)]\,d\alpha/\pi$
11. f and f' are piecewise continuous on every finite interval and f is absolutely integrable on $(-\infty, \infty)$
12. f is continuous on $(-\infty, \infty)$
13. $f(x) = 2\int_0^\infty [\sin\alpha\cos(\alpha x)/\alpha]\,d\alpha/\pi$
14. $f(x) = 2\int_0^\infty [\sin\alpha\cos(\alpha x)/\alpha]\,d\alpha/\pi$
15. $f(x) = 2\int_0^\infty [(1-\cos\alpha)\sin(\alpha x)/\alpha]\,d\alpha/\pi$ on $[0, \infty)$
16. $\mathcal{F}\{f(x)\} = \int_{-\infty}^\infty f(x)e^{i\alpha x}\,dx = F(\alpha)$, $\mathcal{F}^{-1}\{F(\alpha)\} = \int_{-\infty}^\infty F(\alpha)e^{-i\alpha x}\,d\alpha/(2\pi) = f(x)$
17. $-\alpha^2U = U_t$
18. $U(\alpha, 0) = 2\sin\alpha/\alpha$
19. $U = 2\sin\alpha e^{-\alpha^2 t}/\alpha$
20. $u = \int_{-\infty}^\infty [\sin(\alpha)e^{-\alpha^2 t}e^{-i\alpha x}/\alpha]\,d\alpha/\pi$

1. Define $\text{erfc}(x)$.

2. What is $\lim_{x\to\infty} \text{erfc}(x)$?

3. What is $\mathcal{L}\{\text{erfc}(\sqrt{t})\}$?

4. What is $\mathcal{L}\{e^t \text{erfc}(\sqrt{t})\}$?

5. Consider the heat equation problem $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ with initial and boundary conditions $u(0,t) = 0$, $u(1,t) = u_0$, $u(x,0) = 0$. Describe a physical problem with this as a mathematical model.

6. In the previous problem, apply the Laplace transform to the differential equation. What is the resulting problem for $U(x,s) = \mathcal{L}\{u(x,t)\}$?

7. In the previous problem, what is the solution $U(x,s)$?

8. In the previous three problems, what is the solution $u(x,t)$? Hint: write the solution for $U(x,s)$ in terms of exponentials and expand in an infinite series.

9. Define the Fourier cosine transform of an even function f.

10. In the previous problem, what is the inverse Fourier cosine transform?

11. Find the Fourier cosine transform of the function $f(x) = e^{-|x|}$.

12. Consider the steady state temperature distribution on a semi–infinite plate determined by $0 < x < \pi$, $y > 0$, with zero temperature at $x = 0$ and at $y = 0$, and with temperature $u = e^{-y}$ along $x = \pi$. Write down the mathematical model for this problem.

13. In the previous problem, apply a Fourier sine or cosine transform in y to produce a transformed function $U(x,\alpha)$. What is the resulting differential equation for U?

14. In the previous two problems, what are the resulting boundary conditions for U?

15. What is the solution $U(x,\alpha)$ of the previous two problems?

16. In the previous four problems, what is the solution for the temperature, $u(x,y)$?

17. Find the Fourier integral representation of the function $f(x) = \left\{ \begin{array}{lll} 0 & \text{if} & x < 0 \\ 1 & \text{if} & 0 < x < 2 \\ 0 & \text{if} & x > 2 \end{array} \right\}$.

18. In the previous problem, what characteristics of the given function guarantee that the integral converges?

19. In the previous two problems, to what value does the integral converge at $x = 1$?

20. In the previous two problems, to what value does the integral converge at $x = 0$?

ANSWER KEY
Zill Differential Equations 9e Chapter 14 Form B

1. $\text{erfc}(x) = 2\int_x^\infty e^{-u^2}du/\sqrt{\pi}$
2. 0
3. $(1 - 1/\sqrt{s+1})/s$
4. $1/(\sqrt{s}(\sqrt{s}+1))$
5. Temperature in a rod of length 1, with zero temperature at $x = 0$, zero temperature initially, and temperature of u_0 at $x = 1$
6. $U_{xx} = sU$, $U(0, s) = 0$, $U(1, s) = u_0/s$
7. $U = \frac{u_0}{s}\frac{\sinh(\sqrt{s}x)}{\sinh\sqrt{s}} = \frac{u_0}{s}\frac{e^{\sqrt{s}(x-1)} - e^{-\sqrt{s}(x+1)}}{1-e^{-2\sqrt{s}}}$
8. $u = u_0\sum_{n=0}^\infty \left[\text{erfc}\left(\frac{2n+1-x}{2\sqrt{t}}\right) - \text{erfc}\left(\frac{2n+1+x}{2\sqrt{t}}\right)\right]$
9. $\mathcal{F}_c\{f(x)\} = \int_0^\infty f(x)\cos(\alpha x)dx = F(\alpha)$
10. $\mathcal{F}_c^{-1}\{F(\alpha)\} = 2\int_0^\infty F(\alpha)\cos(\alpha x)d\alpha/\pi = f(x)$
11. $\mathcal{F}_c\{e^{-|x|}\} = \int_0^\infty e^{-|x|}\cos(\alpha x)dx = 1/(1+\alpha^2)$
12. $u_{xx} + u_{yy} = 0$, $u(0, y) = 0$, $u(x, 0) = 0$, $u(\pi, y) = e^{-y}$
13. $U_{xx} - \alpha^2 U = 0$
14. $U(0, \alpha) = 0$, $U(\pi, \alpha) = \alpha/(1+\alpha^2)$
15. $U = \alpha\sinh(\alpha x)/((1+\alpha^2)\sinh(\alpha\pi))$
16. $u = 2\int_0^\infty[\alpha\sin(\alpha y)\sinh(\alpha x)/((1+\alpha^2)\sinh(\alpha\pi))d\alpha/\pi$
17. $f(x) = \int_0^\infty\{[\sin(2\alpha)\cos(\alpha x) + (1-\cos(2\alpha))\sin(\alpha x)]/\alpha\}d\alpha/\pi$
18. f and f' are piecewise continuous on any finite interval, and f is absolutely integrable.
19. 1
20. 1/2

1. The error function is defined as

   Select the correct answer.

   (a) $\text{erf}(x) = \int_0^x e^{-u^2} du$
   (b) $\text{erf}(x) = \int_x^\infty e^{-u^2} du$
   (c) $\text{erf}(x) = 2\int_0^x e^{-u^2} du/\pi$
   (d) $\text{erf}(x) = 2\int_x^\infty e^{-u^2} du/\sqrt{\pi}$
   (e) $\text{erf}(x) = 2\int_0^x e^{-u^2} du/\sqrt{\pi}$

2. The value of $\lim_{x\to\infty} \text{erf}(x)$ is

   Select the correct answer.

   (a) 0
   (b) 1
   (c) $\pi/2$
   (d) $\sqrt{\pi/2}$
   (e) $\sqrt{\pi}/2$

3. The value of $\mathcal{L}\{\text{erf}(\sqrt{t})\}$ is

   Select the correct answer.

   (a) $1/(s\sqrt{s+1})$
   (b) $1/(s\sqrt{s-1})$
   (c) $1/((s+1)\sqrt{s})$
   (d) $1/((s-1)\sqrt{s})$
   (e) none of the above

4. The value of $\mathcal{L}\{e^t\text{erf}(\sqrt{t})\}$ is

   Select the correct answer.

   (a) $1/(\sqrt{s}(s+1))$
   (b) $1/(\sqrt{s}(s-1))$
   (c) $1/(s(s+1))$
   (d) $1/(s(\sqrt{s}-1))$
   (e) $1/(s(\sqrt{s}+1))$

5. The Laplace transform of a function f is

   Select the correct answer.

   (a) $\mathcal{L}\{f(x)\} = \int_0^\infty e^t f(t)dt$
   (b) $\mathcal{L}\{f(x)\} = \int_0^\infty e^{-t} f(t)dt$
   (c) $\mathcal{L}\{f(x)\} = \int_0^\infty e^{st} f(t)dt$
   (d) $\mathcal{L}\{f(x)\} = \int_0^\infty e^{-st} f(t)dt$
   (e) $\mathcal{L}\{f(x)\} = \int_0^s e^{st} f(t)dt$

6. If $U(x,s) = \mathcal{L}\{u(x,t)\}$, then $\mathcal{L}\{\frac{\partial^2 u}{\partial t^2}\}$ is

   Select the correct answer.

   (a) $U_s(x,s)$
   (b) $sU(x,s)$
   (c) $s^2U(x,s) - u(x,0)$
   (d) $s^2U(x,s) - u(x,0) - su_t(x,0)$
   (e) $s^2U(x,s) - su(x,0) - u_t(x,0)$

7. If $U(x,s) = \mathcal{L}\{u(x,t)\}$, then $\mathcal{L}\{\frac{\partial u}{\partial x}\}$ is

   Select the correct answer.

   (a) $sU(x,s) - u(x,0)$
   (b) $sU(x,s) - u_x(x,0)$
   (c) $sU(x,s) - su(x,0)$
   (d) $U_x(x,s)$
   (e) $U_s(x,s)$

8. Suppose $\mathcal{F}\{f(x)\} = F(\alpha)$, $\mathcal{F}\{g(x)\} = G(\alpha)$. In the convolution theorem, the formula for the Fourier transform is

   Select all that apply.

   (a) $\int_{-\infty}^\infty f(\tau)g(t-\tau)d\tau = \mathcal{F}^{-1}\{F(\alpha)G(\alpha)\}$
   (b) $\int_{-\infty}^\infty f(t-\tau)g(\tau)d\tau = \mathcal{F}^{-1}\{F(\alpha)G(\alpha)\}$
   (c) $\mathcal{F}\{\int_{-\infty}^\infty f(\tau)g(t-\tau)d\tau\} = F(\alpha)G(\alpha)$
   (d) $\mathcal{F}\{\int_{-\infty}^\infty f(t-\tau)g(\tau)d\tau\} = F(\alpha)G(\alpha)$
   (e) none of the above

9. Consider the problem of a vibrating string, tightly-stretched between $x = 0$ and $x = 1$, with a fixed initial position, $f(x)$, and zero initial velocity. The mathematical problem for the deflection, $u(x,t)$, is (with $u(0,t) = 0$, $u(1,t) = 0$)

   Select the correct answer.

   (a) $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, $u(x,0) = f(x)$, $u_t(x,0) = 0$
   (b) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial t} = 0$, $u(x,0) = f(x)$, $u_t(x,0) = 0$
   (c) $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$, $u(x,0) = f(x)$, $u_t(x,0) = 0$
   (d) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = 0$, $u(x,0) = f(x)$, $u_t(x,0) = 0$
   (e) $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$, $u(x,0) = 0$, $u_t(x,0) = f(x)$

10. In the previous problem, apply the Laplace transform. The resulting equation for $U(x,s) = \mathcal{L}\{u(x,t)\}$ is

    Select the correct answer.

    (a) $\frac{\partial^2 U}{\partial x^2} + s^2 U = -sf(x)$
    (b) $\frac{\partial^2 U}{\partial x^2} + s^2 U = f(x)$
    (c) $\frac{\partial^2 U}{\partial x^2} - s^2 U = -f(x)$
    (d) $\frac{\partial^2 U}{\partial x^2} - s^2 U = sf(x)$
    (e) $\frac{\partial^2 U}{\partial x^2} - s^2 U = -sf(x)$

11. In the previous problem, assume that $f(x) = \sin(\pi x)$. The solution for $U(x,s)$ is

    Select the correct answer.

    (a) $U = \sinh(sx) + \frac{s}{s^2+\pi^2}\sin(\pi x)$
    (b) $U = \cosh(sx) + \frac{s}{s^2-\pi^2}\sin(\pi x)$
    (c) $U = \frac{1}{s^2-\pi^2}\sin(\pi x)$
    (d) $U = \frac{s}{s^2+\pi^2}\sin(\pi x)$
    (e) $U = \frac{1}{s^2+\pi^2}\sin(\pi x)$

12. In the three previous problems, the solution for $u(x,t)$ is

    Select the correct answer.

    (a) $u = \sin(\pi x)\sin(\pi t)$
    (b) $u = \sin(\pi x)\cos(\pi t)$
    (c) $u = \sin(\pi x)\cosh(\pi t)$
    (d) $u = \cos(\pi x)\sinh(\pi t)$
    (e) $u = \cos(\pi x)\cos(\pi t)$

13. The Fourier integral representation of a function f is given by

    Select the correct answer.

    (a) $\int_{-\infty}^{\infty} f(x)\cos(\alpha x)dx$
    (b) $\int_{-\infty}^{\infty} f(x)\sin(\alpha x)dx$
    (c) $\int_0^{\infty}[\int_{-\infty}^{\infty} f(t)\cos(\alpha t)dt\cos(\alpha x) + \int_{-\infty}^{\infty} f(t)\sin(\alpha t)dt\sin(\alpha x)]d\alpha$
    (d) $\int_0^{\infty}[\int_{-\infty}^{\infty} f(t)\cos(\alpha t)dt\cos(\alpha x) + \int_{-\infty}^{\infty} f(t)\sin(\alpha t)dt\sin(\alpha x)]d\alpha/\pi$
    (e) $\int_0^{\infty}[\int_{-\infty}^{\infty} f(t)\cos(\alpha t)dt\cos(\alpha x) + \int_{-\infty}^{\infty} f(t)\sin(\alpha t)dt\sin(\alpha x)]d\alpha/(2\pi)$

14. Let $f(x) = \left\{ \begin{array}{ccc} 0 & \text{if} & x < -1 \\ 1 & \text{if} & -1 < x < 1 \\ 0 & \text{if} & x > 1 \end{array} \right\}$. The Fourier integral representation of f is $f(x) = \int_0^{\infty}[A(\alpha)\cos(\alpha x) + B(\alpha)\sin(\alpha x)]d\alpha/\pi$, where

    Select the correct answer.

    (a) $A(\alpha) = \sin\alpha/\alpha$ and $B(\alpha) = \cos\alpha/\alpha$
    (b) $A(\alpha) = 2\cos\alpha/\alpha$ and $B(\alpha) = 2\sin\alpha/\alpha$
    (c) $A(\alpha) = 2\sin\alpha/\alpha$ and $B(\alpha) = 2\cos\alpha/\alpha$
    (d) $A(\alpha) = 0$ and $B(\alpha) = 2\sin\alpha/\alpha$
    (e) $A(\alpha) = 2\sin\alpha/\alpha$ and $B(\alpha) = 0$

15. In the previous problem, the integral representation converges at $x = 1$ to the value

    Select the correct answer.

    (a) 0
    (b) 1
    (c) 1/2
    (d) -1
    (e) none of the above

16. The complex form of the Fourier integral of a function f is

    Select the correct answer.

    (a) $\int_{-\infty}^{\infty} f(t)e^{i\alpha t}dt$
    (b) $\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t)e^{i\alpha t}dt\right)e^{-i\alpha x}d\alpha/(2\pi)$
    (c) $\int_{-\infty}^{\infty} f(t)e^{-i\alpha t}dt$
    (d) $\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t)e^{-i\alpha t}dt\right)e^{i\alpha x}d\alpha/\pi$
    (e) $\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t)e^{\alpha t}dt\right)e^{-i\alpha x}d\alpha/\pi$

17. Consider the problem of finding the temperature in a semi-infinite rod with zero initial temperature and a fixed constant temperature, u_0, at $x = 0$. The mathematical model for this problem is

    Select the correct answer.

    (a) $k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, $u(x,0) = 0$, $u(0,t) = u_0$
    (b) $k\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$, $u(x,0) = 0$, $u(0,t) = u_0$
    (c) $k\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial t} = 0$, $u(x,0) = 0$, $u(0,t) = u_0$
    (d) $k\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = 0$, $u(x,0) = u_0$, $u(0,t) = 0$
    (e) $k\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial t} = 0$, $u(x,0) = u_0$, $u(0,t) = 0$

18. In the previous problem, apply a Fourier sine transform in x. The new problem for the transform $U(\alpha, t)$ is

    Select the correct answer.

    (a) $k\frac{\partial U}{\partial t} + \alpha^2 U = k\alpha u_0$, $U(\alpha,0) = 0$
    (b) $k\frac{\partial U}{\partial t} - \alpha^2 U = k\alpha u_0$, $U(\alpha,0) = 0$
    (c) $\frac{\partial U}{\partial t} + k\alpha^2 U = k\alpha u_0$, $U(\alpha,0) = 0$
    (d) $\frac{\partial U}{\partial t} - k\alpha^2 U = k\alpha u_0$, $U(\alpha,0) = 0$
    (e) $\frac{\partial U}{\partial t} + \alpha^2 U = k\alpha u_0$, $U(\alpha,0) = 0$

19. In the previous problem, the solution is

    Select the correct answer.

    (a) $U = u_0(1 + e^{-k\alpha^2 t})$
    (b) $U = u_0(1 - e^{-k\alpha^2 t})$
    (c) $U = u_0(1 + e^{-k\alpha^2 t})/\alpha$
    (d) $U = u_0(1 - e^{-k\alpha^2 t})/\alpha$
    (e) $U = u_0(1 - e^{-k\alpha^2 t})/\alpha^2$

20. In the three previous problems, the solution for the temperature $u(x,t)$ is

    Select the correct answer.

    (a) $u = u_0 \int_0^\infty [\sin(\alpha x)(1 - e^{-k\alpha^2 t})/\alpha]\, d\alpha$
    (b) $u = u_0 \int_0^\infty [\sin(\alpha x)(1 - e^{-k\alpha^2 t})/\alpha]\, d\alpha/\pi$
    (c) $u = 2u_0 \int_0^\infty [\sin(\alpha x)(1 - e^{k\alpha^2 t})/\alpha]\, d\alpha$
    (d) $u = 2u_0 \int_0^\infty [\sin(\alpha x)(1 - e^{k\alpha^2 t})/\alpha]\, d\alpha/\pi$
    (e) $u = 2u_0 \int_0^\infty [\sin(\alpha x)(1 - e^{-k\alpha^2 t})/\alpha]\, d\alpha/\pi$

ANSWER KEY

Zill Differential Equations 9e Chapter 14 Form C

1. e
2. b
3. a
4. b
5. d
6. e
7. d
8. a, b, c, d
9. c
10. e
11. d
12. b
13. d
14. e
15. c
16. b
17. a
18. c
19. d
20. e

1. The complementary error function is defined as

   Select the correct answer.

   (a) $\operatorname{erfc}(x) = \int_0^x e^{-u^2} du$
   (b) $\operatorname{erfc}(x) = \int_x^\infty e^{-u^2} du$
   (c) $\operatorname{erfc}(x) = 2\int_0^x e^{-u^2} du/\pi$
   (d) $\operatorname{erfc}(x) = 2\int_x^\infty e^{-u^2} du/\sqrt{\pi}$
   (e) $\operatorname{erfc}(x) = 2\int_0^x e^{-u^2} du/\sqrt{\pi}$

2. The value of $\lim_{x\to\infty} \operatorname{erfc}(x)$ is

   Select the correct answer.

   (a) 0
   (b) 1
   (c) $\pi/2$
   (d) $\sqrt{\pi/2}$
   (e) $\sqrt{\pi}/2$

3. The value of $\mathcal{L}\{\operatorname{erfc}(\sqrt{t})\}$ is

   Select the correct answer.

   (a) $(1 + 1/\sqrt{s+1})/s$
   (b) $(1 + 1/\sqrt{s-1})/s$
   (c) $(1 - 1/\sqrt{s+1})/s$
   (d) $(1 - 1/\sqrt{s-1})/s$
   (e) none of the above

4. The value of $\mathcal{L}\{e^t \operatorname{erfc}(\sqrt{t})\}$ is

   Select the correct answer.

   (a) $1/(\sqrt{s}(\sqrt{s}+1))$
   (b) $1/(\sqrt{s}(\sqrt{s}-1))$
   (c) $1/(s(\sqrt{s}+1))$
   (d) $1/(s(\sqrt{s}-1))$
   (e) none of the above

5. Consider a semi–infinite, elastic, vibrating string, with zero initial position and velocity, driven by a vertical force at $x = 0$, so that $u(0,t) = f(t)$. Assume that $lim_{x\to\infty}u(x,t) = 0$. The mathematical model for the deflection, $u(x,t)$, is

   Select the correct answer.

   (a) $a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$, $u(0,t) = f(t)$, $u(x,0) = 0$, $u_t(x,0) = 0$
   (b) $a^2\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = 0$, $u(0,t) = f(t)$, $u(x,0) = f(t)$, $u_t(x,0) = 0$
   (c) $a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, $u(0,t) = f(t)$, $u(x,0) = 0$, $u_t(x,0) = 0$
   (d) $a^2\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial t} = 0$, $u(0,t) = f(t)$, $u(x,0) = 0$, $u_t(x,0) = 0$
   (e) $a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$, $u(0,t) = f(t)$, $u(x,0) = f(t)$, $u_t(x,0) = 0$

6. In the previous problem, apply the Laplace transform. The resulting equation for $U(x,s) = \mathcal{L}\{u(x,t)\}$ is

   Select the correct answer.

   (a) $a^2\frac{\partial^2 U}{\partial x^2} + s^2U = -sf(x)$
   (b) $a^2\frac{\partial^2 U}{\partial x^2} + s^2U = 0$
   (c) $a^2\frac{\partial^2 U}{\partial x^2} - s^2U = -f(x)$
   (d) $a^2\frac{\partial^2 U}{\partial x^2} - s^2U = sf(x)$
   (e) $a^2\frac{\partial^2 U}{\partial x^2} - s^2U = 0$

7. In the previous problem, assume that $f(t) = \sin(\pi t)$. The solution for $U(x,s)$ is

   Select the correct answer.

   (a) $U = \frac{s}{s^2+\pi^2}e^{-sx/a}$
   (b) $U = \frac{s}{s^2-\pi^2}e^{-sx/a}$
   (c) $U = \frac{\pi}{s^2-\pi^2}e^{-sx/a}$
   (d) $U = \frac{\pi}{s^2+\pi^2}e^{-sx/a}$
   (e) $U = \frac{s}{s^2+\pi^2}\sinh(-sx/a)$

8. In the three previous problems, the solution for $u(x,t)$ is

   Select the correct answer.

   (a) $u = \sin(\pi t)\mathcal{U}(t - x/a)$
   (b) $u = \sin(\pi(t - x/a))\mathcal{U}(t - x/a)$
   (c) $u = \sin(\pi x)\cosh(\pi t)$
   (d) $u = \cos(\pi x)\sinh(\pi t)$
   (e) $u = \cos(\pi(t - x/a))\mathcal{U}(t - x/a)$

9. The Fourier cosine integral of a function f defined on $[0, \infty]$ is

   Select the correct answer.

   (a) $2\int_0^\infty [\int_0^\infty f(x)\cos(\alpha x)dx]\cos(\alpha x)d\alpha/\pi$
   (b) $\int_0^\infty [\int_0^\infty f(x)\cos(\alpha x)dx]\cos(\alpha x)d\alpha/\pi$
   (c) $2\int_0^\infty [\int_0^\infty f(x)\cos(\alpha x)dx]\cos(\alpha x)d\alpha$
   (d) $\int_0^\infty [\int_0^\infty f(x)\cos(\alpha x)d\alpha]\cos(\alpha x)dx$
   (e) $2\int_0^\infty [\int_0^\infty f(x)\cos(\alpha x)d\alpha]\cos(\alpha x)dx/\pi$

10. The Fourier cosine integral of $f(x) = e^{-x}$, $x \geq 0$ is

    Select the correct answer.

    (a) $f(x) = 2\int_0^\infty [\alpha\cos(\alpha x)/(1+\alpha)]d\alpha$
    (b) $f(x) = \int_0^\infty [\alpha\cos(\alpha x)/(1+\alpha^2)]d\alpha/\pi$
    (c) $f(x) = \int_0^\infty [\alpha\cos(\alpha x)/(1+\alpha^2)]d\alpha$
    (d) $f(x) = 2\int_0^\infty [\cos(\alpha x)/(1+\alpha^2)]d\alpha/\pi$
    (e) $f(x) = 2\int_0^\infty [\cos(\alpha x)/(1+\alpha)]d\alpha/\pi$

11. In the previous problem, the integral converges at $x = 0$ to the value

    Select the correct answer.

    (a) 0
    (b) 1/2
    (c) 1
    (d) 2
    (e) e

12. In the previous two problems, the integral converges for $x < 0$ to the function

    Select the correct answer.

    (a) e^{-x}
    (b) e^{x}
    (c) x
    (d) 1
    (e) none of the above

13. The solution of the integral equation $\int_0^\infty f(x)\cos(\alpha x)dx = F(\alpha)$ is

    Select the correct answer.

    (a) $f(x) = \int_0^\infty F(\alpha)\cos(\alpha x)d\alpha/\pi$
    (b) $f(x) = 2\int_0^\infty F(\alpha)\cos(\alpha x)d\alpha/\pi$
    (c) $f(x) = 2\int_0^\infty F(\alpha)\cos(\alpha x)d\alpha$
    (d) $f(x) = \int_0^\infty F(\alpha)\cos(\alpha x)d\alpha$
    (e) none of the above

14. In the previous problem, if $F(\alpha) = e^{-\alpha}$, then $f(x)$ is

    Select the correct answer.

    (a) x
    (b) 1
    (c) $1/(\pi(1+x^2))$
    (d) $2/(\pi(1+x^2))$
    (e) 0

15. Let $f(x) = \left\{ \begin{array}{lll} 0 & \text{if} & x < 0 \\ 2 & \text{if} & 0 \le x \le 3 \\ 0 & \text{if} & x > 3 \end{array} \right\}$. The Fourier integral representation of f is

    Select the correct answer.

    (a) $f(x) = \int_0^\infty \{[\sin(\alpha x) + \sin((3-x)\alpha)]/\alpha\}\, d\alpha/\pi$
    (b) $f(x) = 2\int_0^\infty \{[\sin(\alpha x) + \sin((3-x)\alpha)]/\alpha\}\, d\alpha$
    (c) $f(x) = 2\int_0^\infty \{[\sin(\alpha x) + \sin((3-x)\alpha)]/\alpha\}\, d\alpha/\pi$
    (d) $f(x) = \int_0^\infty \{[\sin(\alpha x) + \sin((3-x)\alpha)]/\alpha\}\, d\alpha$
    (e) none of the above

16. In the previous problem, the integral converges at $x = 0$ to the value

    Select the correct answer.

    (a) 0
    (b) 1
    (c) 2
    (d) 3
    (e) 4

17. Consider the temperature, $u(x,t)$, in an infinite rod ($-\infty < x < \infty$), with an initial temperature of $f(x) = e^{-|x|}$. The mathematical model for this is

    Select the correct answer.

    (a) $k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, $u(x,0) = e^{-|x|}$
    (b) $k\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t}$, $u(x,0) = e^{-|x|}$
    (c) $k\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$, $u(x,0) = e^{-|x|}$
    (d) $k\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = 0$, $u(x,0) = e^{-|x|}$
    (e) $k\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial t} = 0$, $u(x,0) = e^{-|x|}$

18. Apply a Fourier transform in x in the previous problem. The resulting equation for $U(\alpha,t) = \mathcal{F}\{u(x,t)\}$ is

    Select the correct answer.

    (a) $k\alpha^2 U = U_t$, $U(\alpha,0) = 2/(1+\alpha^2)$
    (b) $k\alpha U = U_t$, $U(\alpha,0) = 1/(1+\alpha^2)$
    (c) $k\alpha^2 U = U_t$, $U(\alpha,0) = 1/(1+\alpha^2)$
    (d) $-k\alpha^2 U = U_t$, $U(\alpha,0) = 2/(1+\alpha^2)$
    (e) $-k\alpha U = U_t$, $U(\alpha,0) = 2/(1+\alpha^2)$

19. In the previous problem, the solution for $U(\alpha,t)$ is

    Select the correct answer.

    (a) $e^{k\alpha t}/(1+\alpha^2)$
    (b) $e^{k\alpha^2 t}/(1+\alpha^2)$
    (c) $2e^{k\alpha t}/(1+\alpha^2)$
    (d) $2e^{-k\alpha t}/(1+\alpha^2)$
    (e) $2e^{-k\alpha^2 t}/(1+\alpha^2)$

20. In the previous three problems, the solution for $u(x,t)$ is

    Select the correct answer.

    (a) $\int_{-\infty}^{\infty} [e^{-i\alpha x} e^{k\alpha t}/(1+\alpha^2)]\, d\alpha/(2\pi)$
    (b) $\int_{-\infty}^{\infty} [e^{-i\alpha x} e^{-k\alpha^2 t}/(1+\alpha^2)]\, d\alpha/\pi$
    (c) $\int_{-\infty}^{\infty} [e^{-i\alpha x} e^{k\alpha t}/(1+\alpha^2)]\, d\alpha/\pi$
    (d) $\int_{-\infty}^{\infty} [e^{-i\alpha x} e^{-k\alpha t}/(1+\alpha^2)]\, d\alpha/\pi$
    (e) $\int_{-\infty}^{\infty} [e^{-i\alpha x} e^{k\alpha^2 t}/(1+\alpha^2)]\, d\alpha/(2\pi)$

ANSWER KEY
Zill Differential Equations 9e Chapter 14 Form D

1. d
2. a
3. c
4. a
5. a
6. e
7. d
8. b
9. a
10. d
11. c
12. b
13. b
14. d
15. c
16. b
17. a
18. d
19. e
20. b

1. Show that $\text{erf}(\sqrt{t}) = \int_0^t (e^{-\tau}/\sqrt{\tau})\, d\tau/\sqrt{\pi}$.

2. Use the previous problem and the convolution theorem to find $\mathcal{L}\{\text{erf}(\sqrt{t})\}$

3. Using the previous two problems, what is $\mathcal{L}\{\text{erfc}(\sqrt{t})\}$?

4. What is $\mathcal{L}\{e^{-a^2/4t}/\sqrt{\pi t}\}$?

5. What is $\mathcal{L}^{-1}\{e^{-a\sqrt{s}}\}$?

6. The Laplace transform of $f(t)$ is defined as

   Select the correct answer.

   (a) $\int_0^\infty f(t)e^{st}dt$
   (b) $\int_0^\infty f(t)e^{-st}dt$
   (c) $\int_0^t f(t)e^{-st}dt$
   (d) $\int_t^\infty f(t)e^{-st}dt$
   (e) $\int_0^t f(t)e^{st}dt$

7. If $U(x,s) = \mathcal{L}\{u(x,t)\}$, the Laplace transform of $\frac{\partial u}{\partial t}$ is

   Select the correct answer.

   (a) $sU(x,s) - su(x,0)$
   (b) $sU(x,s) - su(x,0) - u_t(x,0)$
   (c) $sU(x,s) - u(x,0)$
   (d) $s^2U(x,s) - su(x,0)$
   (e) $s^2U(x,s) - su(x,0) - u_t(x,0)$

8. If $U(x,s) = \mathcal{L}\{u(x,t)\}$, the Laplace transform of the wave equation $a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$ is

   Select the correct answer.

   (a) $a^2U_{xx} = sU - u(x,0)$
   (b) $a^2U_{xx} = sU - u_t(x,0)$
   (c) $a^2U_{xx} = s^2U - su(x,0)$
   (d) $a^2U_{xx} = s^2U - u(x,0)$
   (e) $a^2U_{xx} = s^2U - su(x,0) - u_t(x,0)$

9. In the previous problem, assume that $a = 1$ and use the boundary and initial conditions $u(0,t) = 0$, $u(1,t) = 0$, $u(x,0) = 0$, $u_t(x,0) = \sin(\pi x)$. This is a mathematical model for

   Select the correct answer.

   (a) Laplace's equation on a semi-infinite region of the $x - -t$ plane
   (b) the heat equation for a rod of length 1 with zero temperatures at the ends and given initial temperature
   (c) the wave equation for a vibrating string, tightly stretched between $x = 0$ and $x = 1$, with zero initial position and a given initial velocity
   (d) the wave equation for a vibrating string, tightly stretched between $x = 0$ and $x = 1$, with zero initial velocity and a given initial position
   (e) none of the above

10. For the previous two problems, the boundary condition(s) on U is (are)

    Select all that apply.

    (a) $U(0,s) = 0$
    (b) $U(1,s) = 0$
    (c) $U(x,0) = 0$
    (d) $U(x,1) = 0$
    (e) $U(0,s) + U(1,s) = 0$

11. In the previous three problems, the solution $U(x,s)$ is

    Select the correct answer.

    (a) $U(x,s) = \sin(\pi x)/(s^2 + \pi^2)$
    (b) $U(x,s) = \sinh(sx) + \sin(\pi x)/(s^2 + \pi^2)$
    (c) $U(x,s) = c_2 \sinh(sx) + \sin(\pi x)/(s^2 + \pi^2)$
    (d) $U(x,s) = \cosh(sx) + \sinh(sx) + \sin(\pi x)$
    (e) $U(x,s) = c_1 \cosh(sx) + \sin(\pi x)$

12. In the previous four problems, the solution $u(x,t)$ is

    Select the correct answer.

    (a) $u(x,t) = \cos(\pi x)\cos(\pi t)$
    (b) $u(x,t) = \cos(\pi x)\sin(\pi t)$
    (c) $u(x,t) = \sin(\pi x)\cos(\pi t)$
    (d) $u(x,t) = \sin(\pi x)\sin(\pi t)/\pi$
    (e) $u(x,t) = \sin(\pi x)\cos(\pi t)/\pi$

13. Write down the Fourier integral representation of a function $f(x)$.

14. Find the Fourier integral representation of the function $f(x) = \left\{ \begin{array}{ccc} 0 & \text{if} & x < 0 \\ 5 & \text{if} & 0 < x < 3 \\ 0 & \text{if} & x > 3 \end{array} \right\}$.

15. Define the function g by $g(x) = f(x)$ if $x > 0$, where f is defined in the previous problem, and, for $x < 0$, let $g(x) = -g(-x)$. Find the Fourier sine integral representation of $g(x)$.

16. Consider the problem $k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, $u(x, 0) = 0$, $u(0, t) = 10$. Describe a physical problem having this as a mathematical model.

17. In the previous problem, apply an appropriate Fourier sine or cosine transform. What is the resulting equation for $U(\alpha, t)$?

18. In the previous two problems, what is the initial condition for $U(\alpha, t)$?

19. In the previous three problems, what is the solution $U(\alpha, t)$?

20. In the four previous problems, what is the solution $u(x, t)$?

ANSWER KEY

Zill Differential Equations 9e Chapter 14 Form E

1. $\text{erf}\sqrt{t} = 2\int_0^{\sqrt{t}} e^{-u^2} du/\sqrt{\pi}$. Let $u = \sqrt{\tau}$, $du = d\tau/(2\sqrt{\tau})$. Then $\text{erf}\sqrt{t} = \int_0^t (e^{-\tau}/\sqrt{\tau}) d\tau/\sqrt{\pi}$
2. $1/(s\sqrt{s+1})$
3. $1/s - 1/(s\sqrt{s+1})$
4. $e^{-a\sqrt{s}}/\sqrt{s}$
5. $ae^{-a^2/4t}/(2\sqrt{\pi t^3})$
6. b
7. c
8. e
9. c
10. a, b
11. a
12. d
13. $f(x) = \int_0^\infty [\int_{-\infty}^\infty f(x)\cos(\alpha x) dx \cos(\alpha x) + \int_{-\infty}^\infty f(x)\sin(\alpha x) dx \sin(\alpha x)] d\alpha/\pi$
14. $f(x) = \int_0^\infty [5\sin(3\alpha)\cos(\alpha x) + 5(1-\cos(3\alpha))\sin(\alpha x)]/\alpha \, d\alpha/\pi$
15. $g(x) = 2\int_0^\infty [5(1-\cos(3\alpha))\sin(\alpha x)]/\alpha \, d\alpha/\pi$
16. Temperature in a semi-infinite rod, with zero temperature initially, and a temperature of 10 at the left end
17. $U(\alpha, t) = \mathcal{F}_s\{u(x,t)\}$, $U_t = -k\alpha^2 U + 10k\alpha$
18. $U(\alpha, 0) = 0$
19. $U(\alpha, t) = 10(1 - e^{-\alpha^2 kt})/\alpha$
20. $u(x,t) = 20\int_0^\infty (1 - e^{-\alpha^2 kt})\sin(\alpha x)/\alpha \, d\alpha/\pi$

1. The error function is defined as

   Select the correct answer.

   (a) $\text{erf}(x) = \int_0^x e^{-u^2} du$
   (b) $\text{erf}(x) = \int_x^\infty e^{-u^2} du$
   (c) $\text{erf}(x) = 2\int_0^x e^{-u^2} du/\pi$
   (d) $\text{erf}(x) = 2\int_x^\infty e^{-u^2} du/\sqrt{\pi}$
   (e) $\text{erf}(x) = 2\int_0^x e^{-u^2} du/\sqrt{\pi}$

2. The value of $\lim_{x\to 0} \text{erf}(x)$ is

   Select the correct answer.

   (a) 0
   (b) 1
   (c) $\pi/2$
   (d) $\sqrt{\pi/2}$
   (e) $\sqrt{\pi}/2$

3. The value of $\mathcal{L}\{\text{erfc}(\sqrt{t})\}$ is

   Select the correct answer.

   (a) $(1 + 1/\sqrt{s+1})/s$
   (b) $(1 + 1/\sqrt{s-1})/s$
   (c) $(1 - 1/\sqrt{s+1})/s$
   (d) $(1 - 1/\sqrt{s-1})/s$
   (e) none of the above

4. The value of $\mathcal{L}\{e^t \text{erfc}(\sqrt{t})\}$ is

   Select the correct answer.

   (a) $1/(\sqrt{s}(\sqrt{s} - 1))$
   (b) $1/(\sqrt{s}(\sqrt{s} + 1))$
   (c) $1/(s(\sqrt{s} + 1))$
   (d) $1/(s(\sqrt{s} - 1))$
   (e) none of the above

5. Show that $\int_a^b e^{-u^2} du = \sqrt{\pi}[\text{erf}(b) - \text{erf}(a)]/2$

6. If $U(x,s) = \mathcal{L}\{u(x,t)\}$, what is $\mathcal{L}\{\frac{\partial u}{\partial t}\}$?

7. Consider the heat equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$. Apply a Laplace transform in t and write down the new equation for $U(x,s) = \mathcal{L}\{u(x,t)\}$.

8. In the previous problem, if the boundary and initial conditions are $u(0,t) = 0$, $u(2,t) = 10$, $u(x,0) = 0$, describe a physical problem with this as a mathematical model.

9. In the previous two problems, what are the new boundary conditions for U?

10. In the previous three problems, what is the solution $U(x,s)$?

11. In the previous four problems, what is the solution $u(x,t)$?

12. Briefly explain how the Fourier integral representation of a function can be derived from its Fourier series.

13. Let $f(x) = e^{-|x|}$. The Fourier integral representation of $f(x)$ is

    Select the correct answer.

    (a) $f(x) = \int_0^\infty [\cos(\alpha x)/(1+\alpha^2)]\, d\alpha$
    (b) $f(x) = 2\int_0^\infty [\cos(\alpha x)/(1+\alpha^2)]\, d\alpha$
    (c) $f(x) = 2\int_0^\infty [\sin(\alpha x)/(1+\alpha^2)]\, d\alpha$
    (d) $f(x) = 2\int_0^\infty [\cos(\alpha x)/(1+\alpha^2)]\, d\alpha/\pi$
    (e) $f(x) = \int_0^\infty [\sin(\alpha x)/(1+\alpha^2)]\, d\alpha/\pi$

14. For the function f of the previous problem, the complex form of the Fourier integral representation of $f(x)$ is

    Select the correct answer.

    (a) $f(x) = 2\int_{-\infty}^\infty [e^{-i\alpha x}/(1+\alpha^2)]d\alpha/\pi$
    (b) $f(x) = 2\int_{-\infty}^\infty [e^{i\alpha x}/(1+\alpha^2)]d\alpha$
    (c) $f(x) = \int_{-\infty}^\infty [e^{i\alpha x}/(1+\alpha^2)]d\alpha/\pi$
    (d) $f(x) = \int_{-\infty}^\infty [e^{-i\alpha x}/(1+\alpha^2)]d\alpha/\pi$
    (e) $f(x) = \int_{-\infty}^\infty [e^{i\alpha x}/(1+\alpha^2)]d\alpha$

15. Let $f(x) = e^{-x}$ for $x > 0$. The Fourier cosine representation of $f(x)$ is

    Select the correct answer.

    (a) $f(x) = 2\int_0^\infty [\sin(\alpha x)/(1+\alpha^2)]d\alpha$
    (b) $f(x) = 2\int_0^\infty [\cos(\alpha x)/(1+\alpha^2)]d\alpha/\pi$
    (c) $f(x) = \int_0^\infty [\sin(\alpha x)/(1+\alpha^2)]d\alpha/\pi$
    (d) $f(x) = \int_0^\infty [\cos(\alpha x)/(1+\alpha^2)]d\alpha$
    (e) $f(x) = 2\int_0^\infty [\cos(\alpha x)/(1+\alpha^2)]d\alpha$

16. Let $f(x) = e^{-x}$ for $x > 0$. The Fourier sine representation of $f(x)$ is

    Select the correct answer.

    (a) $f(x) = \int_0^\infty (\alpha \sin(\alpha x)/(1 + \alpha^2))\, d\alpha$
    (b) $f(x) = \int_0^\infty (\alpha \cos(\alpha x)/(1 + \alpha^2))\, d\alpha$
    (c) $f(x) = 2\int_0^\infty (\alpha \sin(\alpha x)/(1 + \alpha^2))\, d\alpha$
    (d) $f(x) = 2\int_0^\infty (\alpha \cos(\alpha x)/(1 + \alpha^2))\, d\alpha/\pi$
    (e) $f(x) = 2\int_0^\infty (\alpha \sin(\alpha x)/(1 + \alpha^2))\, d\alpha/\pi$

17. Consider the problem $k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, $-\infty < x < \infty$, $u(x, 0) = \left\{ \begin{array}{ccc} 0 & \text{if} & x < -\pi/2 \\ u_0 & \text{if} & -\pi/2 < x < \pi/2 \\ 0 & \text{if} & x > \pi/2 \end{array} \right\}$. Apply a Fourier transform: $U(\alpha, t) = \mathcal{F}\{u(x, t)\}$. What is the transformed equation for U?

18. In the previous problem, what is the new initial condition for $U(\alpha, t)$?

19. In the previous two problems, what is the solution for $U(\alpha, t)$?

20. In the previous three problems, what is the solution for $u(x, t)$?

ANSWER KEY

Zill Differential Equations 9e Chapter 14 Form F

1. e
2. a
3. c
4. b
5. $\int_a^b e^{-u^2} du = \int_0^b e^{-u^2} du - \int_0^a e^{-u^2} du = \sqrt{\pi}[\text{erf}(b) - \text{erf}(a)]/2$
6. $\mathcal{L}\{\frac{\partial u}{\partial t}\} = sU(x,s) - u(x,0)$
7. $U_{xx} = sU - u(x,0)$
8. Temperature in a rod of length 2, initially at a temperature of zero, with the left end held at a temperature of 0 and the right end held at a temperature of 10.
9. $U(0,s) = 0$, $U(2,s) = 10/s$
10. $U(x,s) = \frac{10}{s}\frac{\sinh(\sqrt{s}x)}{\sinh(2\sqrt{s})} = 10\sum_{n=0}^{\infty}(e^{\sqrt{s}(x-2-4n)} - e^{-\sqrt{s}(x+2+4n)})/s$
11. $u(x,t) = 10\sum_{n=0}^{\infty}[\text{erfc}((-x+2+4n)/(2\sqrt{t})) - \text{erfc}((x+2+4n)/(2\sqrt{t}))]$
12. Write a Fourier series for $f(x)$ on the interval $[-p,p]$. Let $p \to \infty$. The (Riemann) sums change into definite integrals, yielding the Fourier integral of $f(x)$.
13. d
14. d
15. b
16. e
17. $-k\alpha^2 U = U_t$
18. $U(\alpha,0) = 2u_0 \sin(\alpha\pi/2)/\alpha$
19. $U(\alpha,t) = 2u_0 \sin(\alpha\pi/2)e^{-k\alpha^2 t}/\alpha$
20. $u(x,t) = u_0 \int_{-\infty}^{\infty}[\sin(\alpha\pi/2)e^{-k\alpha^2 t}e^{-i\alpha x}/\alpha]d\alpha/\pi$

1. What is the relationship between erf(x) and erfc(x)?

2. What is $\mathcal{L}\{\text{erfc}(a/(2\sqrt{t}))\}$?

3. What is $\mathcal{L}^{-1}\{e^{-a\sqrt{s}}/\sqrt{s}\}$?

4. Let $U(x,s) = \mathcal{L}\{u(x,t)\}$. Then $\mathcal{L}\left\{\frac{\partial^2 u}{\partial t^2}\right\} =$

   Select the correct answer.

   (a) $sU - u(x,0)$
   (b) $sU - u_t(x,0)$
   (c) $s^2U - su(x,0) - u_t(x,0)$
   (d) $s^2U - su_t(x,0) - u(x,0)$
   (e) $sU - su(x,0) - u_t(x,0)$

5. Using $U(x,s) = \mathcal{L}\{u(x,t)\}$, the Laplace transform of the wave equation $a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$ is

   Select the correct answer.

   (a) $a^2U_{xx} - s^2U = -su(x,0) + u_t(x,0)$
   (b) $a^2U_{xx} - s^2U = -su(x,0) - u_t(x,0)$
   (c) $a^2U_{xx} - s^2U = su(x,0) + u_t(x,0)$
   (d) $a^2U_{xx} - sU = -u_t(x,0)$
   (e) $a^2U_{xx} - sU = -su(x,0)$

6. In the previous problem, assume that the initial conditions are $u(x,0) = 0$, $u_t(x,0) = \sin(\pi x)$ and the boundary conditions are $u(0,t) = 0$, $u(1,t) = 0$. Then the boundary conditions for U are

   Select all that apply.

   (a) $U(0,s) = 1/s$
   (b) $U(1,s) = 1/s$
   (c) $U(0,s) = 1/s^2$
   (d) $U(0,s) = 0$
   (e) $U(1,s) = 0$

7. In the previous two problems, the solution for $U(x,s)$ is

   Select the correct answer.

   (a) $U = \sin(\pi x)/(s^2 + \pi^2)$
   (b) $U = \cos(\pi x)/(s^2 + \pi^2)$
   (c) $U = \sin(\pi x)/(s^2 + a^2\pi^2)$
   (d) $U = \cos(\pi x)/(s^2 - a^2\pi^2)$
   (e) $U = \sin(\pi x)/(s^2 - a^2\pi^2)$

8. In the previous three problems, the solution for $u(x,t)$ is

   Select the correct answer.

   (a) $\cos(\pi x)\sinh(a\pi t)/(a\pi)$
   (b) $\sin(\pi x)\sinh(a\pi t)/(a\pi)$
   (c) $\sin(\pi x)\sin(\pi t)/\pi$
   (d) $\cos(\pi x)\sin(\pi t)/\pi$
   (e) $\sin(\pi x)\sin(a\pi t)/(a\pi)$

9. Find the Fourier integral representation of $f(x) = e^{-|x|}$.

10. Solve the integral equation $\int_0^\infty f(x)\sin(\alpha x)\,dx = \left\{\begin{array}{ccc} 1 & \text{if} & 0 < \alpha < 2 \\ 0 & \text{if} & \alpha > 2 \end{array}\right\}$.

11. Write down the Fourier cosine transform pair.

12. The Fourier sine representation of $f(x) = e^{-x}$, $x > 0$ is

    Select the correct answer.

    (a) $f(x) = 2\int_0^\infty [\alpha\sin(\alpha x)/(1+\alpha^2)]d\alpha/\pi$
    (b) $f(x) = \int_0^\infty [\alpha\sin(\alpha x)/(1+\alpha^2)]d\alpha/\pi$
    (c) $f(x) = \int_0^\infty [\alpha\sin(\alpha x)/(1+\alpha^2)]d\alpha$
    (d) $f(x) = 2\int_0^\infty [\sin(\alpha x)/(1+\alpha^2)]d\alpha$
    (e) $f(x) = 2\int_0^\infty [\sin(\alpha x)/(1+\alpha^2)]d\alpha/\pi$

13. Consider the problem $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, $0 < x < \pi$, $0 < y$, $u(0,y) = 0$, $u(\pi,y) = e^{-2y}$, $\frac{\partial u}{\partial y}(x,0) = 0$. Describe a physical problem for which this is a mathamatical model.

14. In the previous problem, apply a Fourier cosine transform. The new equation for $U(x,\alpha) = \mathcal{F}_c\{u(x,y)\}$ is

    Select the correct answer.

    (a) $U_x - \alpha^2 U = 0$
    (b) $U_x + \alpha U = 0$
    (c) $U_{xx} - \alpha U = 0$
    (d) $U_{xx} - \alpha^2 U = 0$
    (e) $U_{xx} + \alpha^2 U = 0$

15. In the previous two problems, the boundary conditions for U are

    Select all that apply.

    (a) $U(0, \alpha) = 0$
    (b) $U(0, \alpha) = 2/(4 + \alpha^2)$
    (c) $U(\pi, \alpha) = 2/(4 + \alpha^2)$
    (d) $U(\pi, \alpha) = \alpha/(4 - \alpha^2)$
    (e) $U(\pi, \alpha) = 0$

16. In the previous three problems, the solution $U(x, \alpha)$ is

    Select the correct answer.

    (a) $U = 2\sinh(\alpha x)/((4 - \alpha^2)\sinh(\pi\alpha))$
    (b) $U = 2\sinh(\alpha x)/((4 + \alpha^2)\sinh(\pi\alpha))$
    (c) $U = 2\alpha\sinh(\alpha x)/((4 + \alpha^2)\sinh(\pi\alpha))$
    (d) $U = \alpha\sinh(\alpha x)/((4 + \alpha^2)\sinh(\pi\alpha))$
    (e) $U = \sinh(\alpha x)/((4 + \alpha^2)\sinh(\pi\alpha))$

17. In the previous four problems, the solution $u(x, y)$ is

    Select the correct answer.

    (a) $u = 2\int_0^\infty \sinh(\alpha x)\cos(\alpha y)/((4 - \alpha^2)\sinh(\pi\alpha))\,d\alpha/\pi$
    (b) $u = 2\int_0^\infty \sinh(\alpha x)\cos(\alpha y)/((4 + \alpha^2)\sinh(\pi\alpha))\,d\alpha/\pi$
    (c) $u = 4\int_0^\infty \alpha\sinh(\alpha x)\cos(\alpha y)/((4 + \alpha^2)\sinh(\pi\alpha))\,d\alpha/\pi$
    (d) $u = 4\int_0^\infty \alpha^2\sinh(\alpha x)\cos(\alpha y)/((4 + \alpha^2)\sinh(\pi\alpha))\,d\alpha/\pi$
    (e) $u = 4\int_0^\infty \sinh(\alpha x)\cos(\alpha y)/((4 + \alpha^2)\sinh(\pi\alpha))\,d\alpha/\pi$

18. Let $f(x) = \left\{ \begin{array}{ccc} 0 & \text{if} & x < 0 \\ 1 & \text{if} & 0 < x < 2 \\ 0 & \text{if} & x > 2 \end{array} \right\}$. Find the Fourier integral representation of $f(x)$.

19. In the previous problem, simplify the integral as much as possible using trigonometric identities.

20. Using the formula in the previous two problems, evaluate $\int_0^\infty [\sin\alpha/\alpha]d\alpha$.

ANSWER KEY
Zill Differential Equations 9e Chapter 14 Form G

1. $\operatorname{erf}(x) + \operatorname{erfc}(x) = 1$
2. $e^{-a\sqrt{s}}/s$
3. $e^{-a^2/4t}/\sqrt{\pi t}$
4. c
5. b
6. d, e
7. c
8. e
9. $f(x) = 2\int_0^\infty \cos(\alpha x)/(1+\alpha^2)\, d\alpha/\pi$
10. $f(x) = 2(1 - \cos(2x))/(\pi x)$
11. $A(\alpha) = \int_0^\infty f(x)\cos(\alpha x)\, dx,\ f(x) = 2\int_0^\infty A(\alpha)\cos(\alpha x)\, d\alpha/\pi$
12. a
13. Steady state temperature distribution in a semi-infinite plate with temperature fixed at zero at $x = 0$, temperature fixed at e^{-2y} at $x = \pi$, and insulated along $y = 0$.
14. d
15. a, c
16. b
17. e
18. $f(x) = \int_0^\infty \{[\sin(2\alpha)\cos(\alpha x) + (1 - \cos(2\alpha))\sin(\alpha x)]/\alpha\}\, d\alpha/\pi$
19. $f(x) = 2\int_0^\infty [\sin\alpha \cos(\alpha(x-1))/\alpha]\, d\alpha/\pi$
20. $\pi/2$

1. The complementary error function is defined as

   Select the correct answer.

   (a) $\text{erfc}(x) = \int_0^x e^{-u^2} du$
   (b) $\text{erfc}(x) = \int_x^\infty e^{-u^2} du$
   (c) $\text{erfc}(x) = 2\int_0^x e^{-u^2} du/\pi$
   (d) $\text{erfc}(x) = 2\int_x^\infty e^{-u^2} du/\sqrt{\pi}$
   (e) $\text{erfc}(x) = 2\int_0^x e^{-u^2} du/\sqrt{\pi}$

2. The value of $\lim_{x\to 0} \text{erfc}(x)$ is

   Select the correct answer.

   (a) $\pi/2$
   (b) $\sqrt{\pi/2}$
   (c) $\sqrt{\pi}/2$
   (d) 0
   (e) 1

3. The value of $\mathcal{L}\{\text{erf}(\sqrt{t})\}$ is

   Select the correct answer.

   (a) $1/(s\sqrt{s+1})$
   (b) $1/(s\sqrt{s-1})$
   (c) $(1 - 1/\sqrt{s+1})/s$
   (d) $(1 - 1/\sqrt{s-1})/s$
   (e) none of the above

4. The value of $\mathcal{L}\{e^t \text{erf}(\sqrt{t})\}$ is

   Select the correct answer.

   (a) $1/(\sqrt{s}(s+1))$
   (b) $1/(\sqrt{s}(s-1))$
   (c) $1/(s(\sqrt{s}+1))$
   (d) $1/(s(\sqrt{s}-1))$
   (e) none of the above

5. Show that $\int_{-a}^{a} e^{-u^2} du = \sqrt{\pi}\text{erf}(a)$

6. If $U(x,s) = \mathcal{L}\{u(x,t)\}$, what is $\mathcal{L}\{\frac{\partial^2 u}{\partial x^2}\}$?

7. Consider the wave equation $a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$. Apply a Laplace transform in t and write down the new equation for $U(x,s) = \mathcal{L}\{u(x,t)\}$.

8. In the previous problem, if the boundary and initial conditions are $u(0,t) = 0$, $u(2,t) = 0$, $u(x,0) = \sin(\pi x/2)$, $u_t(x,0) = 0$, describe a physical problem with this as a mathematical model.

9. In the previous two problems, what are the new boundary conditions for U?

10. In the previous three problems, what is the solution for $U(x,s)$?

11. In the previous four problems, what is the solution for $u(x,t)$?

12. Let $f(x) = \left\{ \begin{array}{lcc} x & \text{if} & 0 < x < 2 \\ 0 & \text{if} & x > 2 \end{array} \right\}$. The Fourier cosine representation of $f(x)$ is

    Select the correct answer.

    (a) $f(x) = \int_0^\infty [(2\alpha\cos(2\alpha) + 1 - \sin(2\alpha))\cos(\alpha x)/\alpha^2]\, d\alpha$
    (b) $f(x) = 2\int_0^\infty [(2\alpha\cos(2\alpha) + 1 - \sin(2\alpha))\cos(\alpha x)/\alpha^2]\, d\alpha$
    (c) $f(x) = 2\int_0^\infty [(2\alpha\sin(2\alpha) - 1 + \cos(2\alpha))\cos(\alpha x)/\alpha^2]\, d\alpha/\pi$
    (d) $f(x) = \int_0^\infty [(2\alpha\sin(2\alpha) - 1 + \cos(2\alpha))\cos(\alpha x)/\alpha^2]\, d\alpha/\pi$
    (e) $f(x) = 2\int_0^\infty [(2\alpha\sin(2\alpha) + 1 - \cos(2\alpha))\cos(\alpha x)/\alpha^2]\, d\alpha$

13. In the previous problem, the Fourier sine representation of $f(x)$ is

    Select the correct answer.

    (a) $f(x) = 2\int_0^\infty [(-2\alpha\cos(2\alpha) + \sin(2\alpha))\sin(\alpha x)/\alpha^2] d\alpha/\pi$
    (b) $f(x) = \int_0^\infty [(-2\alpha\cos(2\alpha) + \sin(2\alpha))\sin(\alpha x)/\alpha^2] d\alpha/\pi$
    (c) $f(x) = 2\int_0^\infty [(-2\alpha\cos(2\alpha) + \sin(2\alpha))\sin(\alpha x)/\alpha^2] d\alpha$
    (d) $f(x) = \int_0^\infty [(2\alpha\cos(2\alpha) - \sin(2\alpha))\sin(\alpha x)/\alpha^2] d\alpha$
    (e) $f(x) = 2\int_0^\infty [(2\alpha\cos(2\alpha) - \sin(2\alpha))\sin(\alpha x)/\alpha^2] d\alpha$

14. In the previous two problems, let $f(x)$ be extended as an odd function. The Fourier integral representation of $f(x)$ is

    Select the correct answer.

    (a) $f(x) = 2\int_0^\infty [(-2\alpha\cos(2\alpha) + \sin(2\alpha))\sin(\alpha x)/\alpha^2] d\alpha$
    (b) $f(x) = 2\int_0^\infty [(2\alpha\cos(2\alpha) - \sin(2\alpha))\sin(\alpha x)/\alpha^2] d\alpha$
    (c) $f(x) = 2\int_0^\infty [(2\alpha\cos(2\alpha) - \sin(2\alpha))\sin(\alpha x)/\alpha^2] d\alpha/\pi$
    (d) $f(x) = 2\int_0^\infty [(-2\alpha\cos(2\alpha) + \sin(2\alpha))\sin(\alpha x)/\alpha^2] d\alpha/\pi$
    (e) $f(x) = \int_0^\infty [(-2\alpha\cos(2\alpha) + \sin(2\alpha))\sin(\alpha x)/\alpha^2] d\alpha/\pi$

15. In the previous problem, the value to which the representations of $f(x)$ converges at $x = 2$ is

    Select the correct answer.

    (a) 2
    (b) 1
    (c) 0
    (d) -2
    (e) -1

16. Consider the problem $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$, $x > 0$, $t > 0$, $u(0,t) = 0$, $u(x,0) = xe^{-x}$, $u_t(x,0) = 0$. Apply a Fourier sine transform to form a function $U(\alpha, t)$. What is the Fourier sine transform of $\frac{\partial^2 u}{\partial x^2}$?

17. In the previous problem, what is the new differential equation for U?

18. In the previous two problems, what are the new initial conditions on U?

19. In the previous three problems, what is the solution for $U(\alpha, t)$?

20. In the previous four problems, what is the solution for $u(x, t)$?

ANSWER KEY
Zill Differential Equations 9e Chapter 14 Form H

1. d
2. e
3. a
4. b
5. $\int_{-a}^{a} e^{-u^2} du = 2\int_0^a e^{-u^2} du = \sqrt{\pi}\text{erf}(a)$
6. $\mathcal{L}\{\frac{\partial^2 u}{\partial x^2}\} = U_{xx}(x, \alpha)$
7. $a^2 U_{xx} = s^2 U - su(x, 0) - u_t(x, 0)$
8. Vibrations of a string, tightly stretched between $x = 0$ and $x = 2$, with initial position $\sin(\pi x/2)$ and zero initial velocity
9. $U(0, \alpha) = 0$, $U(2, \alpha) = 0$
10. $U = s\sin(\pi x/2)/(s^2 + (a\pi/2)^2)$
11. $u = \cos(a\pi t/2)\sin(\pi x/2)$
12. c
13. a
14. d
15. b
16. $\mathcal{F}_s\{u_{xx}\} = -\alpha^2 U$
17. $U_{tt} = -\alpha^2 U$
18. $U(\alpha, 0) = 2\alpha/(1 + \alpha^2)^2$, $U_t(\alpha, 0) = 0$
19. $U(\alpha, t) = 2\alpha\cos(\alpha t)/(1 + \alpha^2)^2$
20. $u(x, t) = 2\int_0^\infty [2\alpha\cos(\alpha t)\sin(\alpha x)/(1 + \alpha^2)^2]\, d\alpha/\pi$