1. Classify the heat equation as either hyperbolic, parabolic, or elliptic.

2. Classify the wave equation as either hyperbolic, parabolic, or elliptic.

3. Write down the central difference approximation of $\frac{\partial^2 u}{\partial x^2}$ for a function $u(x,y)$ using a step size of h.

4. Write down the five-point approximation of the Laplacian for a function $u(x,y)$ using a step size of h.

5. Define the term mesh point.

6. When the finite difference method is applied to the problem $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$, $u(0,y)=0$, $u(2,y)=\sin(\pi y/2)$, $u(x,0)=0$, $u(x,2)=\sin(\pi x/2)$, with $n=2, h=1$, what are the values of the unknown function u at the boundary mesh points?

7. In the previous problem, what are the equations to solve for the solution at the interior mesh points?

8. Solve the system of equations in the previous problem for $u(1,1)$.

9. What is the difference between an explicit method and an implicit method?

10. Consider the heat equation for the temperature, $u(x,t)$, in a rod, $c\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}$, $u(0,t)=0$, $u(1,t)=0$, $u(x,0)=f(x)$. Using $u(x,t)=u_{ij}$, the central difference approximation of $\frac{\partial^2 u}{\partial x^2}$, and the forward difference approximation of $\frac{\partial u}{\partial t}$, and using $h=1/n$ and $k=T/m$ for certain integers n and m and an ending time value T, write down the resulting difference equations for the approximation of the problem.

11. In the previous problem, if $f(x)=\sin(\pi x)$, solve for $u(x_i,1)$ if $c=1$, $T=5$, $m=5$, and $n=4$.

12. In the previous problem, continue to solve for $u(x_i,2)$

13. In the previous problem, discuss the issue of stability.

14. For the previous four problems, write down the Crank–Nicholson method for solving the problem.

15. Does the Crank–Nicholson method have the same stability problems as the regular finite difference method for solving the heat equation?

16. For a function $u(x,y)$ and step size h, write down the finite difference approximation for $\frac{\partial^2 u}{\partial y^2}$.

17. Write down the finite difference approximation for the wave equation, $c^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}$ with an x step size of h and a t step size of k.

18. In the previous problem, assume that the boundary and initial conditions are $u(0,t)=0$, $u(1,t)=0$, $u(x,0)=\sin(\pi x)$, $u_t(x,0)=0$. Also assume that $c=1$, $h=0.25$, and $k=0.5$. The equations for $u_{i,1}$ involve $u_{i,-1}$. Explain how you would find those values.

19. In the previous two problems, solve the system of equations for u at the mesh points along the line where $t = 0.5$.

20. In the previous three problems, solve the system of equations for u at the mesh points along the line where $t = 1.0$.

ANSWER KEY
Zill Differential Equations 9e Chapter 15 Form A

1. parabolic
2. hyperbolic
3. $\frac{\partial^2 u}{\partial x^2} \approx (u(x+h,y) - 2u(x,y) + u(x-h,y))/h^2$
4. $\nabla^2 u \approx (u(x+h,y) + u(x,y+h) + u(x-h,y) + u(x,y-h) - 4u(x,y))/h^2$
5. When a region is split into subregions by a set of vertical and horizontal lines, the mesh points are the intersection points of those lines, usually where an approximation is needed for a function.
6. $u = 0$ at $(0,0)$, $(1,0)$, $(2,0)$, $(0,1)$, $(0,2)$, and $(2,2)$, and $u = 1$ at $(2,1)$ and at $(1,2)$
7. $u(2,1) + u(1,2) + u(0,1) + u(1,0) - 4u(1,1) = 0$
8. $u(1,1) = 1/2$
9. In an explicit method, the formula can be solved for the unknown function values in terms of known values. In an implicit method, the formulas each involve more than one unknown function value and must be solved by some other method.
10. $u_{i,j+1} = \lambda u_{i+1,j} + (1-2\lambda)u_{ij} + \lambda u_{i-1,j}$, where $\lambda = ck/h^2$
11. $u_{11} = 16 - 31\sqrt{2}/2 = -5.920$, $u_{21} = 16\sqrt{2} - 31 = -8.373$, $u_{31} = 16 - 31\sqrt{2}/2 = -5.920$
12. $u_{12} = -992 + 1473\sqrt{2}/2 = 49.568$, $u_{22} = 1473 - 992\sqrt{2} = 70.100$, $u_{32} = -992 + 1473\sqrt{2}/2 = 49.568$
13. The method is unstable since $\lambda = 16 > 1/2$.
14. $-u_{i-1,j+1} + \alpha u_{i,j+1} - u_{i+1,j+1} = u_{i+1,j} - \beta u_{i,j} + u_{i-1,j}$, where $\lambda = ck/h^2$, $\alpha = 2(1+1/\lambda)$, $\beta = 2(1-1/\lambda)$
15. No, it is stable.
16. $\frac{\partial^2 u}{\partial y^2} \approx (u(x,y+h) - 2u(x,y) + u(x,y-h))/h^2$
17. $u_{i,j+1} = \lambda^2 u_{i+1,j} + 2(1-\lambda^2)u_{ij} + \lambda^2 u_{i-1,j} - u_{i,j-1}$, where $\lambda = ck/h$
18. Use a central difference approximation for $\frac{\partial u}{\partial t}(x,0) = 0 \approx (u_{i,1} - u_{i,-1})/(2k)$, and solve for $u_{i,-1} = u_{i,1}$.
19. $u_{11} = 2 - 3\sqrt{2}/2 = -0.121$, $u_{21} = 2\sqrt{2} - 3 = -0.172$, $u_{31} = 2 - 3\sqrt{2}/2 = -0.121$
20. $u_{12} = 33\sqrt{2}/2 - 24 = -0.665$, $u_{22} = 33 - 24\sqrt{2} = -0.941$, $u_{32} = 33\sqrt{2}/2 - 24 = -0.665$

1. Classify the wave equation as either hyperbolic, parabolic, or elliptic.

2. Classify Laplace's equation as either hyperbolic, parabolic, or elliptic.

3. Write down the central difference approximation of $\frac{\partial^2 u}{\partial y^2}$ using a step size of h for a function $u(x, y)$.

4. Write down the Dirichlet problem for Laplace's equation on a rectangle.

5. Write down the five-point approximation of Laplace's equation at an interior point using the u_{ij} notation.

6. Write down the system of equations that result when the finite difference method is applied to the problem $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, $u(0, y) = 0$, $u(2, y) = \sin(\pi y/2)$, $u(x, 0) = 0$, $u(x, 2) = \sin(\pi x/2)$, with $n = 4$.

7. Consider the heat equation for the temperature, $u(x, t)$, in a rod, $c\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, $u(0, t) = u_0$, $u(1, t) = u_1$, $u(x, 0) = f(x)$. Using $u(x, t) = u_{ij}$, the central difference approximation of $\frac{\partial^2 u}{\partial x^2}$ with $h = 1/n$, and the forward difference approximation of $\frac{\partial u}{\partial t}$ with $k = T/m$, for certain integers n and m and an ending time value T, write down the resulting equation for the approximation of the problem.

8. In the previous problem, if $u_0 = 0$, $u_1 = 2$, and $f(x) = 2x$, solve for $u(x_i, 1)$ if $c = 1$, $T = 5$, $m = 5$, and $n = 4$.

9. For the values of c, h, and k in the previous problem, will the numerical scheme be stable? Explain.

10. For the previous three problems, write down the Crank–Nicholson method for solving the problem.

11. In the previous problem, solve the Crank–Nicholson equations for the values of u at the first time step.

12. In the previous problem, continue to solve the Crank–Nicholson equations for the values of u at the next time step.

13. Discuss stability for the Crank–Nicholson method of solving the heat equation.

14. Write down the finite difference approximation for $\frac{\partial^2 u}{\partial t^2}$ for a function $u(x, t)$ with a step size of k.

15. Write down the finite difference approximation for the wave equation, $c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$ with an x step size of h and a t step size of k Use the u_{ij} notation.

16. In the previous problem, assume that the boundary and initial conditions are $u(0, t) = 0$, $u(1, t) = 0$, $u(x, 0) = \left\{ \begin{array}{ccc} x & \text{if} & 0 < x \le 1/2 \\ 1 - x & \text{if} & 1/2 < x < 1 \end{array} \right\}$, $u_t(x, 0) = 0$. Also assume that $c = 1$, $h = 0.25$, and $k = 0.5$. The equations for $u_{i,1}$ involve $u_{i,-1}$. Explain how you would find those values.

17. In the previous two problems, solve the system of equations for u at the mesh points along the line where $t = 0.5$.

18. In the previous three problems, solve the system of equations for u at the mesh points along the line where $t = 1.0$.

19. In the previous four problems, calculate λ and discuss whether the calculation is stable?

20. In the previous five problems, if $c = 1$ and $h = 0.25$, what is the least upper bound for k that would guarantee stability of the numerical scheme?

ANSWER KEY

Zill Differential Equations 9e Chapter 15 Form B

1. hyperbolic
2. elliptic
3. $\frac{\partial^2 u}{\partial y^2} \approx (u(x,y+h) - 2u(x,y) + u(x,y-h))/h^2$
4. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, $u(a,y) = f_1(y)$, $u(b,y) = f_2(y)$, $u(x,c) = g_1(x)$, $u(x,d) = g_2(x)$
5. $u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{ij} = 0$
6. $u_{12} + u_{21} - 4u_{11} = 0$, $u_{22} + u_{31} + u_{11} - 4u_{21} = 0$, $u_{32} + u_{21} + \sqrt{2}/2 - 4u_{31} = 0$, $u_{13} + u_{22} + u_{11} - 4u_{12} = 0$, $u_{23} + u_{12} + u_{32} + u_{21} - 4u_{22} = 0$, $u_{33} + u_{22} + 1 + u_{31} - 4u_{32} = 0$, $\sqrt{2}/2 + u_{23} + u_{12} - 4u_{13} = 0$, $1 + u_{13} + u_{33} + u_{22} - 4u_{23} = 0$, $\sqrt{2}/2 + u_{23} + \sqrt{2}/2 + u_{32} - 4u_{33} = 0$
7. $u_{i,j+1} = \lambda(u_{i+1,j} - 2u_{ij} + u_{i-1,j}) + u_{ij}$ where $\lambda = ck/h^2$
8. $u_{11} = 1/2$, $u_{21} = 1$, $u_{31} = 3/2$
9. No, $\lambda = 16 > 1/2$.
10. $-u_{i-1,j+1} + \alpha u_{i,j+1} - u_{i+1,j+1} = u_{i+1,j} - \beta u_{ij} + u_{i-1,j}$, where $\lambda = ck/h^2$, $\alpha = 2(1 + 1/\lambda)$, $\beta = 2(1 - 1/\lambda)$
11. $u_{11} = 1/2$, $u_{21} = 1$, $u_{31} = 3/2$
12. $u_{12} = 1/2$, $u_{22} = 1$, $u_{32} = 3/2$
13. The Crank–Nicholson scheme is always stable.
14. $\frac{\partial^2 u}{\partial t^2} \approx (u(x,t+k) - 2u(x,t) + u(x,t-k))/k^2$
15. $u_{i,j+1} = \lambda^2 u_{i+1,j} + 2(1-\lambda^2)u_{ij} + \lambda^2 u_{i-1,j} - u_{i,j-1}$ where $\lambda = ck/h$
16. Use a central difference approximation for $\frac{\partial u}{\partial t}(x,0) = 0 \approx (u_{i1} - u_{i,-1})/(2k)$, and solve for $u_{i,-1} = u_{i,1}$.
17. $u_{11} = 1/4$, $u_{21} = -1/2$, $u_{31} = 1/4$
18. $u_{12} = -15/4$, $u_{22} = 9/2$, $u_{32} = -15/4$
19. No, $\lambda = 2 > 1$
20. $k = 1/4$

1. The wave equation is

   Select the correct answer.

   (a) hyperbolic
   (b) parabolic
   (c) elliptic
   (d) none of the above

2. Laplace's equation is

   Select the correct answer.

   (a) hyperbolic
   (b) parabolic
   (c) elliptic
   (d) none of the above

3. The wave equation is

   Select the correct answer.

   (a) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$
   (b) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y} = 0$
   (c) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial t} = 0$
   (d) $\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0$
   (e) $\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0$

4. The heat equation is

   Select the correct answer.

   (a) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$
   (b) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y} = 0$
   (c) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial t} = 0$
   (d) $\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0$
   (e) $\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0$

5. The central difference approximation for $\frac{\partial^2 u}{\partial x^2}$ with step size h is

   Select the correct answer.

   (a) $(u(x+h, y) - 2u(x, y) + u(x-h, y))/h$
   (b) $(u(x+h, y) - 2u(x, y) + u(x-h, y))/h^2$
   (c) $(u(x, y+h) - 2u(x, y) + u(x, y-h))/h$
   (d) $(u(x, y+h) - 2u(x, y) + u(x, y-h))/h^2$
   (e) $(u(x+h, y) - u(x-h, y))/(2h)$

6. A Dirichlet problem is a partial differential equation with conditions specifying

   Select the correct answer.

   (a) the values of the unknown function along the boundary
   (b) the values of the derivative of the unknown function along the boundary
   (c) a linear combination of the values of the unknown function along the boundary and the values of the derivative of the unknown function along the boundary
   (d) none of the above

7. The five point approximation of the Laplacian is

   Select the correct answer.

   (a) $[u(x+h, y) + u(x, y+h) + u(x-h, y) + u(x, y-h) - 2u(x, y)]/h$
   (b) $[u(x+h, y) + u(x, y+h) + u(x-h, y) + u(x, y-h) - 4u(x, y)]/h$
   (c) $[u(x+h, y) + u(x, y+h) + u(x-h, y) + u(x, y-h) - 2u(x, y)]/h^2$
   (d) $[u(x+h, y) + u(x, y+h) + u(x-h, y) + u(x, y-h) - 4u(x, y)]/h^2$
   (e) $[u(x+h, y) - u(x, y+h) + u(x-h, y) - u(x, y-h) - 4u(x, y)]/h^2$

8. Consider the problem $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, $u(0, y) = 0$, $u(x, 0) = 0$, $u(1, y) = \sin(\pi y)$, $u(x, 1) = \sin(\pi x)$. A finite difference approximation of the solution is desired, using the approximation of the previous problem. Use a mesh size of $h = 1/3$. The conditions satisfied by the mesh points on the boundary are

   Select all that apply.

   (a) $u = 1/2$ at $(1, 2/3)$ and $(2/3, 1)$
   (b) $u = \sqrt{3}/2$ at $(1, 1/3)$ and $(1/3, 1)$
   (c) $u = 0$ at $(0, 1/3)$ and $(1/3, 0)$
   (d) $u = 0$ at $(0, 2/3)$ and $(2/3, 0)$
   (e) $u = 0$ at $(1/3, 1/3)$ and $(2/3, 2/3)$

9. In the previous problem, using u_{ij} to denote the value of u at the i, j point, the equations for the values of the unknown function at the interior points are

   Select all that apply.

   (a) $-4u_{11} + u_{21} + u_{12} = 0$
   (b) $-4u_{22} + u_{21} + u_{12} = -\sqrt{3}$
   (c) $-4u_{22} + u_{21} + u_{12} = -\sqrt{3}/2$
   (d) $-4u_{12} + u_{11} + u_{22} = -\sqrt{3}/2$
   (e) $-4u_{21} + u_{11} + u_{22} = -\sqrt{3}/2$

10. In the previous three problems, the solution at the interior points is

    Select all that apply.

    (a) $u_{22} = \sqrt{3}/8$
    (b) $u_{22} = \sqrt{3}/4$
    (c) $u_{11} = \sqrt{3}/8$
    (d) $u_{12} = \sqrt{3}/4$
    (e) $u_{21} = \sqrt{3}/4$

11. The forward difference approximation of $\frac{\partial u}{\partial t}$ with step size k is

    Select the correct answer.

    (a) $(u(x+k,t) - u(x,t))/k$
    (b) $(u(x-k,t) - u(x,t))/k^2$
    (c) $(u(x,t+k) - u(x,t))/k$
    (d) $(u(x,t-k) - u(x,t))/k$
    (e) $(u(x,t+k) - u(x,t))/k^2$

12. Consider the problem $c\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, $u(0,t) = 0$, $u(1,t) = 2$, $u(x,0) = 2x^2$. Replace $\frac{\partial^2 u}{\partial x^2}$ with a central difference approximation with $h = 1/3$ and $\frac{\partial u}{\partial t}$ with a forward difference approximation with $k = 1/2$. The resulting equation is

    Select the correct answer.

    (a) $c[u(x+h,t) + 2u(x,t) + u(x-h,t)]/h^2 = (u(x,t+k) - u(x,t))/k$
    (b) $c[u(x+h,t) + 2u(x,t) + u(x-h,t)]/h^2 = (u(x,t+k) + u(x,t))/k$
    (c) $c[u(x+h,t) - 2u(x,t) + u(x-h,t)]/h^2 = (u(x,t+k) - u(x,t))/k$
    (d) $c[u(x+h,t) - 4u(x,t) + u(x-h,t)]/h^2 = (u(x,t+k) + u(x,t))/k$
    (e) $c[u(x+h,t) - 4u(x,t) + u(x-h,t)]/h^2 = (u(x,t+k) - u(x,t))/k$

13. In the previous problem, using the notation $u_{ij} = u(x,t)$, and letting $\lambda = ck/h^2$, the equation becomes

    Select the correct answer.

    (a) $u_{i,j-1} = \lambda u_{i+1,j} + (1+2\lambda)u_{i,j} + \lambda u_{i-1,j}$
    (b) $u_{i,j-1} = \lambda u_{i+1,j} + (1-2\lambda)u_{i,j} + \lambda u_{i-1,j}$
    (c) $u_{i,j+1} = \lambda u_{i+1,j} + (1+2\lambda)u_{i,j} + \lambda u_{i-1,j}$
    (d) $u_{i,j+1} = \lambda u_{i+1,j} + (1-2\lambda)u_{i,j} + \lambda u_{i-1,j}$
    (e) $u_{i,j+1} = \lambda u_{i+1,j} + (1-\lambda)u_{i,j} + \lambda u_{i-1,j}$

14. In the previous two problems, let $c = 1$. The solution for u along the line $t = 0.5$ at the mesh points is

    Select all that apply.

    (a) $u_{11} = 0$
    (b) $u_{11} = 20/9$
    (c) $u_{11} = 30/9$
    (d) $u_{21} = 26/9$
    (e) $u_{21} = 32/9$

15. In the previous problem, is the value of λ such that the scheme is stable?

    Select the correct answer.

    (a) yes
    (b) no
    (c) It is right on the borderline.
    (d) It cannot be determined from the available data.

16. Consider the problem $c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$, $u(0,t) = 0$, $u(1,t) = 0$, $u(x,0) = \left\{ \begin{array}{ccc} x & \text{if} & 0 < x \le 1/2 \\ 1-x & \text{if} & 1/2 < x < 1 \end{array} \right\}$, $u_t(x,0) = g(x)$. Replace $\frac{\partial^2 u}{\partial x^2}$ with a central difference approximation with $h = 1/3$ and $\frac{\partial u}{\partial t}$ with a central difference approximation with $k = 1/2$. The resulting equation is

    Select the correct answer.

    (a) $c^2[u(x+h,t) - 2u(x,t) + u(x-h,t)]/h^2 = (u(x,t+k) - u(x,t))/k^2$
    (b) $c^2[u(x+h,t) - 2u(x,t) + u(x-h,t)]/h^2 = (u(x,t+k) + u(x,t))/k$
    (c) $c^2[u(x+h,t) - 2u(x,t) + u(x-h,t)]/h^2 = (u(x,t+k) - u(x,t))/k$
    (d) $c^2[u(x+h,t) - 2u(x,t) + u(x-h,t)]/h^2 = (u(x,t+k) - u(x,t) + u(x,t-k))/k^2$
    (e) $c^2[u(x+h,t) - 2u(x,t) + u(x-h,t)]/h^2 = (u(x,t+k) - 2u(x,t) + u(x,t-k))/k^2$

17. In the previous problem, using the notation $u_{ij} = u(x,t)$, and letting $\lambda = ck/h$, the equation becomes

    Select the correct answer.

    (a) $u_{i,j-1} = \lambda^2 u_{i+1,j} + 2(1+\lambda^2)u_{ij} + \lambda^2 u_{i-1,j} + u_{i,j-1}$
    (b) $u_{i,j-1} = \lambda u_{i+1,j} + 2(1-\lambda)u_{ij} + \lambda u_{i-1,j} + u_{i,j-1}$
    (c) $u_{i,j+1} = \lambda^2 u_{i+1,j} + 2(1+\lambda^2)u_{ij} + \lambda^2 u_{i-1,j} - u_{i,j-1}$
    (d) $u_{i,j+1} = \lambda^2 u_{i+1,j} + 2(1-\lambda^2)u_{ij} + \lambda^2 u_{i-1,j} - u_{i,j-1}$
    (e) $u_{i,j+1} = \lambda u_{i+1,j} + (1-\lambda)u_{ij} + \lambda u_{i-1,j} - u_{i,j-1}$

18. In the previous two problems, the values $u_{i,1}$ depend on the values $u_{i,-1}$. How do you calculate those values?

    Select the correct answer.

    (a) Use a forward difference approximation in t along the line $t = 0$.
    (b) Use a backward difference approximation in t along the line $t = 0$.
    (c) Use a central difference approximation in t along the line $t = 0$.
    (d) Use a forward difference approximation in x along the line $t = 0$.
    (e) Use a backward difference approximation in x along the line $t = 0$.

19. In the previous three problems, if $g(x) = 0$, then the values of $u_{i,-1}$ are

    Select the correct answer.

    (a) $u_{i,-1} = u_{i,1}$
    (b) $u_{i,-1} = 0$
    (c) $u_{i,-1} = 1$
    (d) $u_{i,-1} = -1$
    (e) none of the above

20. In the four previous problems, let $c = 1$. The calculated values of $u_{i,1}$ are

    Select the correct answer.

    (a) $u_{11} = -13/24, u_{21} = -13/24$
    (b) $u_{11} = -17/24, u_{21} = -17/24$
    (c) $u_{11} = -1/2, u_{21} = -1/2$
    (d) $u_{11} = -1/24, u_{21} = -1/24$
    (e) $u_{11} = -1/4, u_{21} = -1/4$

ANSWER KEY
Zill Differential Equations 9e Chapter 15 Form C

1. a
2. c
3. e
4. d
5. b
6. a
7. d
8. b, c, d
9. a, b, d, e
10. c, d, e
11. c
12. c
13. d
14. b, d
15. b
16. e
17. d
18. c
19. a
20. d

1. The heat equation is

   Select the correct answer.

   (a) hyperbolic
   (b) parabolic
   (c) elliptic
   (d) none of the above

2. Laplace's equation is

   Select the correct answer.

   (a) hyperbolic
   (b) parabolic
   (c) elliptic
   (d) none of the above

3. The wave equation is

   Select the correct answer.

   (a) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$
   (b) $\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0$
   (c) $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y}$
   (d) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial t} = 0$
   (e) $\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} = 0$

4. Laplace's equation is

   Select the correct answer.

   (a) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$
   (b) $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$
   (c) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial t} = 0$
   (d) $\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0$
   (e) $\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0$

5. The central difference approximation for $\frac{\partial u}{\partial x}$ with step size h is

   Select the correct answer.

   (a) $(u(x+h,y)-2u(x,y)+u(x-h,y))/h$
   (b) $(u(x+h,y)-2u(x,y)+u(x-h,y))/h^2$
   (c) $(u(x,y+h)-2u(x,y)+u(x,y-h))/h$
   (d) $(u(x,y+h)-2u(x,y)+u(x,y-h))/h^2$
   (e) $(u(x+h,y)-u(x-h,y))/(2h)$

6. A Dirichlet problem is a partial differential equation with conditions specifying

   Select the correct answer.

   (a) a linear combination of the values of the unknown function along the boundary and the values of the derivative of the unknown function along the boundary
   (b) the values of the unknown function along the boundary
   (c) the values of the derivative of the unknown function along the boundary
   (d) none of the above

7. The five-point approximation of the Laplacian is

   Select the correct answer.

   (a) $[u(x+h,y)+u(x,y+h)+u(x-h,y)+u(x,y+h)-4u(x,y)]$
   (b) $[u(x+h,y)+u(x,y+h)+u(x-h,y)+u(x,y+h)-2u(x,y)]$
   (c) $[u(x+h,y)+u(x,y+h)+u(x-h,y)+u(x,y+h)-4u(x,y)]/h$
   (d) $[u(x+h,y)+u(x,y+h)+u(x-h,y)+u(x,y+h)-4u(x,y)]/h^2$
   (e) $[u(x+h,y)+u(x,y+h)+u(x-h,y)+u(x,y+h)-2u(x,y)]/h^2$

8. Consider the problem $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$, $u(0,y)=0$, $u(x,0)=0$, $u(1,y)=y-y^2$, $u(x,1)=x-x^2$. A finite difference approximation of the solution is desired, using the approximation of the previous problem. Use a mesh size of $h=1/3$. The conditions satisfied by the mesh points on the boundary are

   Select all that apply.

   (a) $u=0$ at $(0,1/3)$ and $(1/3,0)$
   (b) $u=0$ at $(0,2/3)$ and $(2/3,0)$
   (c) $u=0$ at $(1/3,1/3)$ and $(2/3,2/3)$
   (d) $u=2/9$ at $(1,1/3)$ and $(1/3,1)$
   (e) $u=2/3$ at $(1,2/3)$ and $(2/3,1)$

9. In the previous two problems, using u_{ij} to denote the value of u at the i, j point, the equations for the values of the unknown function at the interior points are

   Select all that apply.

   (a) $-4u_{11} + u_{21} + u_{12} = 0$
   (b) $-4u_{22} + u_{21} + u_{12} = -2/9$
   (c) $-4u_{12} + u_{11} + u_{22} = -2/9$
   (d) $-4u_{21} + u_{11} + u_{22} = -2/9$
   (e) $-4u_{22} + u_{21} + u_{12} = -4/9$

10. In the previous three problems, the solution at the interior points is

    Select all that apply.

    (a) $u_{22} = 1/9$
    (b) $u_{22} = 1/6$
    (c) $u_{11} = 1/18$
    (d) $u_{12} = 1/9$
    (e) $u_{21} = 1/9$

11. Consider the problem $c\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, $u(0,t) = 0$, $u(2,t) = 6$, $u(x,0) = 3x^2/2$. Replace $\frac{\partial^2 u}{\partial x^2}$ with a central difference approximation with $h = 1/2$ and $\frac{\partial u}{\partial t}$ with a forward difference approximation with $k = 1/4$. The resulting equation is

    Select the correct answer.

    (a) $c[u(x+h,t) - 2u(x,t) + u(x-h,t)]/h^2 = (u(x,t+k) - u(x,t))/k$
    (b) $c[u(x+h,t) - 4u(x,t) + u(x-h,t)]/h^2 = (u(x,t+k) + u(x,t))/k$
    (c) $c[u(x+h,t) - 4u(x,t) + u(x-h,t)]/h^2 = (u(x,t+k) - u(x,t))/k$
    (d) $c[u(x+h,t) + 2u(x,t) + u(x-h,t)]/h^2 = (u(x,t+k) - u(x,t))/k$
    (e) $c[u(x+h,t) + 2u(x,t) + u(x-h,t)]/h^2 = (u(x,t+k) + u(x,t))/k$

12. In the previous problem, using the notation $u_{ij} = u(x,t)$, and letting $\lambda = ck/h^2$, the equation becomes

    Select the correct answer.

    (a) $u_{i,j+1} = \lambda u_{i+1,j} + (1-\lambda)u_{ij} + \lambda u_{i-1,j}$
    (b) $u_{i,j+1} = \lambda u_{i+1,j} + (1-2\lambda)u_{ij} + \lambda u_{i-1,j}$
    (c) $u_{i,j-1} = \lambda u_{i+1,j} + (1+2\lambda)u_{ij} + \lambda u_{i-1,j}$
    (d) $u_{i,j-1} = \lambda u_{i+1,j} + (1-2\lambda)u_{ij} + \lambda u_{i-1,j}$
    (e) $u_{i,j+1} = \lambda u_{i+1,j} + (1+2\lambda)u_{ij} + \lambda u_{i-1,j}$

13. In the previous two problems, let $c = 1$. The solution for u along the line $t = 0.25$ at the mesh points is

    Select all that apply.

    (a) $u_{31} = 33/8$
    (b) $u_{11} = 9/8$
    (c) $u_{11} = 11/8$
    (d) $u_{21} = 9/4$
    (e) $u_{21} = 11/4$

14. In the previous problem, is the value of λ such that the scheme is stable?

    Select the correct answer.

    (a) yes
    (b) no
    (c) It is right on the borderline.
    (d) It cannot be determined from the available data.

15. Consider the problem $c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$, $u(0,t) = 0$, $u(1,t) = 0$, $u(x,0) = \sin(\pi x)$, $u_t(x,0) = g(x)$. Replace $\frac{\partial^2 u}{\partial x^2}$ with a central difference approximation with $h = 1/4$ and $\frac{\partial^2 u}{\partial t^2}$ with a central difference approximation with $k = 1/3$. The resulting equation is

    Select the correct answer.

    (a) $c^2[u(x+h,t) - 2u(x,t) + u(x-h,t)]/h^2 = (u(x,t+k) - u(x,t) + u(x,t-k))/k^2$
    (b) $c^2[u(x+h,t) - 2u(x,t) + u(x-h,t)]/h^2 = (u(x,t+k) - 2u(x,t) + u(x,t-k))/k^2$
    (c) $c^2[u(x+h,t) - 2u(x,t) + u(x-h,t)]/h^2 = (u(x,t+k) - u(x,t))/k^2$
    (d) $c^2[u(x+h,t) - 2u(x,t) + u(x-h,t)]/h^2 = (u(x,t+k) + u(x,t))/k$
    (e) $c^2[u(x+h,t) - 2u(x,t) + u(x-h,t)]/h^2 = (u(x,t+k) - u(x,t))/k$

16. In the previous problem, using the notation $u_{ij} = u(x,t)$, and letting $\lambda = ck/h$, the equation becomes

    Select the correct answer.

    (a) $u_{i,j+1} = \lambda^2 u_{i+1,j} + (1 + 2\lambda^2) u_{ij} + \lambda^2 u_{i-1,j} - u_{i,j-1}$
    (b) $u_{i,j+1} = \lambda^2 u_{i+1,j} + (1 - 2\lambda^2) u_{ij} + \lambda^2 u_{i-1,j} - u_{i,j-1}$
    (c) $u_{i,j+1} = \lambda u_{i+1,j} + (1 - \lambda) u_{ij} + \lambda u_{i-1,j} - u_{i,j-1}$
    (d) $u_{i,j-1} = \lambda^2 u_{i+1,j} + (1 + 2\lambda^2) u_{ij} + \lambda^2 u_{i-1,j} + u_{i,j-1}$
    (e) $u_{i,j-1} = \lambda u_{i+1,j} + (1 - 2\lambda) u_{ij} + \lambda u_{i-1,j} + u_{i,j-1}$

17. In the previous two problems, the values $u_{i,1}$ depend on the values $u_{i,1}$. How do you calculate those values?

    Select the correct answer.

    (a) Use a central difference approximation in t along the line $t = 0$.
    (b) Use a forward difference approximation in t along the line $t = 0$.
    (c) Use a backward difference approximation in t along the line $t = 0$.
    (d) Use a forward difference approximation in x along the line $t = 0$.
    (e) Use a backward difference approximation in x along the line $t = 0$.

18. In the previous three problems, the values of $u_{i,-1}$ are

    Select the correct answer.

    (a) $u_{i,-1} = u_{i,0} - kg(x_i)$
    (b) $u_{i,-1} = u_{i,0} - 2kg(x_i)$
    (c) $u_{i,-1} = u_{i,1} + 2kg(x_i)$
    (d) $u_{i,-1} = u_{i,1} + kg(x_i)$
    (e) $u_{i,-1} = u_{i,1} - 2kg(x_i)$

19. In the four previous problems, let $c = 1$. The calculated values of $u_{i,1}$ are

    Select all that apply.

    (a) $u_{11} = (16 - 7\sqrt{2} + 6g(1/4))/18$
    (b) $u_{21} = (8\sqrt{2} - 7 + 3g(1/2))/9$
    (c) $u_{21} = (8\sqrt{2} + 7 + 3g(1/2))/9$
    (d) $u_{31} = (8 - 7\sqrt{2}/2 + 3g(3/4))/9$
    (e) $u_{31} = (8 + 7\sqrt{2}/2 + 3g(3/4))/9$

20. In the previous five problems, is the value of λ such that the numerical scheme is stable?

    Select the correct answer.

    (a) yes
    (b) no
    (c) It is in the borderline.
    (d) It cannot be determined from the available data.

ANSWER KEY

Zill Differential Equations 9e Chapter 15 Form D

1. b
2. c
3. b
4. a
5. e
6. b
7. d
8. a, b, d
9. a, c, d, e
10. b, c, d, e
11. a
12. b
13. a, b, d
14. b
15. b
16. b
17. a
18. e
19. a, b, d
20. b

1. The wave equation is

   Select all that apply.

   (a) elliptic
   (b) parabolic
   (c) hyperbolic
   (d) linear
   (e) nonlinear

2. The heat equation is

   Select all that apply.

   (a) elliptic
   (b) parabolic
   (c) hyperbolic
   (d) linear
   (e) nonlinear

3. Write down the system of equations that result when the finite difference method is applied to the problem $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, $u(0, y) = 0$, $u(2, y) = \sin(\pi y/2)$, $u(x, 0) = 0$, $u(x, 2) = \sin(\pi x/2)$, with $n = 3$.

4. In the previous problem, solve the system exactly.

5. Use the Gauss–Seidel method to solve the system of equations from the previous two problems, using guesses of $u_{1,1} = 0.2$, $u_{1,2} = 0.4$, $u_{2,1} = 0.4$, $u_{2,2} = 0.6$. Show one iteration.

6. In the previous problem, show another iteration.

7. In the previous three problems, does it appear that the iteration values are converging on the exact value? What are the errors after two iterations?

8. Consider the heat equation $c\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$. In order to use the finite difference method, the appropriate approximation for $\frac{\partial^2 u}{\partial x^2}$ is a

   Select the correct answer.

   (a) forward difference approximation
   (b) backward difference approximation
   (c) central difference approximation
   (d) none of the above

9. In the previous problem, the appropriate approximation for $\frac{\partial u}{\partial t}$ is a

   Select the correct answer.

   (a) forward difference approximation
   (b) backward difference approximation
   (c) central difference approximation
   (d) none of the above

10. In the previous two problems, after the appropriate approximations are applied, the new equations become (using the u_{ij} notation)

    Select the correct answer.

    (a) $u_{i,j+1} = \lambda u_{i+1,j} + (1-2\lambda)u_{ij} + \lambda u_{i-1,j}$, where $\lambda = ck/h$
    (b) $u_{i,j+1} = \lambda u_{i+1,j} + (1-2\lambda)u_{ij} + \lambda u_{i-1,j}$, where $\lambda = ch/k$
    (c) $u_{i,j+1} = \lambda u_{i+1,j} + (1+2\lambda)u_{ij} + \lambda u_{i-1,j}$, where $\lambda = ch^2/k$
    (d) $u_{i,j+1} = \lambda u_{i+1,j} + (1+2\lambda)u_{ij} + \lambda u_{i-1,j}$, where $\lambda = ck/h^2$
    (e) $u_{i,j+1} = \lambda u_{i+1,j} + (1-2\lambda)u_{ij} + \lambda u_{i-1,j}$, where $\lambda = ck/h^2$

11. In the previous three problems, assume that the boundary and initial conditions are $u(0,t) = 2$, $u(1,t) = 5$, $u(x,0) = 2 + 3x^2$, and that $c = 1$, $h = .25$, $k = .5$. The equations for $u_{i,1}$ are

    Select all that apply.

    (a) $u_{11} = 8u_{2,0} - 15u_{10} + 8u_{00}$
    (b) $u_{11} = 2u_{2,0} - 3u_{10} + 2u_{00}$
    (c) $u_{21} = 8u_{3,0} - 15u_{20} + 8u_{10}$
    (d) $u_{21} = 2u_{3,0} - 3u_{20} + 2u_{10}$
    (e) $u_{31} = 8u_{4,0} - 15u_{30} + 8u_{20}$
    (f) $u_{31} = 2u_{4,0} - 3u_{30} + 2u_{20}$

12. In the previous four problems, the solutions for u_{11}, u_{22}, u_{33} are

    Select the correct answer.

    (a) $u_{11} = 5.1875$
    (b) $u_{11} = 5.75$
    (c) $u_{21} = 5.1875$
    (d) $u_{21} = 5.75$
    (e) $u_{31} = 6.0245$
    (f) $u_{31} = 6.6875$

13. In the previous five problems, the value of k needs to be changed. The value of k that will make the finite difference scheme stable is

    Select the correct answer.

    (a) $k = 1.0$
    (b) $k = 0.5$
    (c) $k = 0.25$
    (d) $k = 0.12$
    (e) $k = 0.03$

14. Write down the Crank–Nicholson scheme for solving the heat equation. Use the $u_{i,j}$ notation.

15. For which values of $\lambda = ck/h$ is the Crank–Nicholson method stable?

16. Write down the wave equation problem for the deflections $u(x,t)$ of a vibrating string, tightly stretched between $x = 0$ and $x = 1$, with initial position $u(x,0) = x - x^2$ and initial velocity zero.

17. In the previous problem, write down the equations derived when central differences are used to replace the partial derivatives. Use step sizes of h in the x direction and k in the y direction, use the u_{ij} notation, and let $\lambda = ck/h$.

18. In the previous two problems, the equation for $u_{i,1}$ involves $u_{i,-1}$. Explain how to determine those values.

19. Using $c = 1$, $h = .25$, $k = .25$, what are the solutions for $u_{i,1}$ in the previous problem.

20. In the previous problem, is the method stable? Explain.

ANSWER KEY
Zill Differential Equations 9e Chapter 15 Form E

1. c, d
2. b, d
3. $u_{21} + u_{12} - 4u_{11} = 0, u_{22} + \sqrt{3}/2 + u_{11} - 4u_{12} = 0, \sqrt{3}/2 + u_{22} + u_{11} - 4u_{21} = 0, \sqrt{3} + u_{12} + u_{21} - 4u_{22} = 0$
4. $u_{11} = \sqrt{3}/8$, $u_{12} = \sqrt{3}/4$, $u_{21} = \sqrt{3}/4$, $u_{22} = 3\sqrt{3}/8$
5. $u_{11} = 0.2$, $u_{12} = 0.415$, $u_{21} = 0.415$, $u_{22} = 0.641$
6. $u_{11} = 0.208$, $u_{12} = 0.429$, $u_{21} = 0.429$, $u_{22} = 0.648$
7. Yes, the errors are 0.009, 0.004, 0.004, 0.002
8. c
9. a
10. e
11. a, c, e
12. a, d, f
13. e
14. $-u_{i-1,j+1} + \alpha u_{i,j+1} - u_{i+1,j+1} = u_{i+1,j} - \beta u_{ij} + u_{i-1,j}$, where $\alpha = 2(1 + 1/\lambda), \beta = 2(1 - 1/\lambda), \lambda = ck/h^2$
15. The Crank–Nicholson method is always stable.
16. $c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$, $u(0, t) = 0$, $u(1, t) = 0$, $u(x, 0) = x - x^2$, $u_t(x, 0) = 0$
17. $u_{i,j+1} = \lambda^2 u_{i+1,j} + 2(1 - \lambda^2)u_{ij} + \lambda^2 u_{i-1,j} - u_{i,j-1}$
18. Use a central difference approximation for $\frac{\partial u}{\partial t}(x, 0) = 0 \approx (u(x, k) - u(x, -k))/(2k)$ to find $u_{i,-1} = u_{i,1}$.
19. $u_{11} = 1/8$, $u_{21} = 3/16$, $u_{31} = 1/8$
20. It is stable. $\lambda = 1$, which is right on the edge of the stability region.

1. The forward difference approximation of $f'(x)$ with step size h is

   Select the correct answer.

   (a) $f'(x) \approx (f(x+h) - f(x-h))/(2h)$
   (b) $f'(x) \approx (f(x) - f(x-h))/h$
   (c) $f'(x) \approx (f(x+h) - f(x))/h$
   (d) $f'(x) \approx (f(x+h) - 2f(x) + f(x-h))/(2h)$
   (e) $f'(x) \approx (f(x+h) - 2f(x) + f(x-h))/h^2$

2. The central difference approximation of $f''(x)$ with step size h is

   Select the correct answer.

   (a) $f''(x) \approx (f(x+h) - f(x-h))/(2h)$
   (b) $f''(x) \approx (f(x) - f(x-h))/h$
   (c) $f''(x) \approx (f(x+h) - f(x))/h$
   (d) $f''(x) \approx (f(x+h) - 2f(x) + f(x-h))/(2h)$
   (e) $f''(x) \approx (f(x+h) - 2f(x) + f(x-h))/h^2$

3. Replace the partial derivatives in Laplace's equation with central differences. The equation becomes (using the u_{ij} notation)

   Select the correct answer.

   (a) $u_{i+1,j} + u_{i,j+1} - u_{i-1,j} - u_{i,j-1} - 4u_{ij} = 0$
   (b) $u_{i+1,j} + u_{i,j+1} - u_{i-1,j} - u_{i,j-1} + 4u_{ij} = 0$
   (c) $u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{ij} = 0$
   (d) $u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} + 4u_{ij} = 0$
   (e) $-u_{i+1,j} + u_{i,j+1} - u_{i-1,j} + u_{i,j-1} - 4u_{ij} = 0$

4. Consider the Dirichlet problem for Laplace's equation with $u(-1, y) = 0$, $u(1, y) = 0$, $u(x, -1) = 0$, $u(x, 1) = 1 - x^2$. In the previous problem, use a step size of $h = 2/3$ in both directions. The equations for the unknown function values are

   Select all that apply.

   (a) $-4u_{11} + u_{12} + u_{21} = 0$
   (b) $-4u_{11} + u_{12} + u_{21} = 8/9$
   (c) $-4u_{12} + u_{11} + u_{22} = -8/9$
   (d) $-4u_{21} + u_{11} + u_{22} = 0$
   (e) $-4u_{21} + u_{11} + u_{22} = 8/9$
   (f) $-4u_{22} + u_{12} + u_{21} = -8/9$

5. The exact solution of the equations of the previous problem are

   Select all that apply.

   (a) $u_{11} = u_{21} = 1/18$
   (b) $u_{11} = u_{21} = 1/9$
   (c) $u_{12} = u_{22} = 1/3$
   (d) $u_{12} = u_{22} = 1/9$
   (e) $u_{12} = u_{22} = 1/18$

6. In the previous two problems, use the Gauss-Seidel method to find an approximate solution of the equations using initial guesses of $u_{11} = .05$, $u_{21} = .05$, $u_{12} = .15$, $u_{22} = .15$. The values for the first iteration are

   Select the correct answer.

   (a) $u_{11} = u_{21} = .05$
   (b) $u_{11} = .125$, $u_{21} = .1$
   (c) $u_{11} = .1$, $u_{21} = .125$
   (d) $u_{12} = .321$, $u_{22} = .322$
   (e) $u_{12} = .272$, $u_{22} = .303$

7. Write down the heat equation problem for a rod of length 2 that is insulated at both ends and has an initial temperature distribution of $4 - x^2$ where x is measured from the left end.

8. In the previous problem, use a central difference approximation with step size h for the second derivative and a forward difference approximation with step size k for the first derivative. Write down the resulting equation, using the u_{ij} notation and letting $\lambda = ck/h^2$.

9. In the previous two problems, explain how to incorporate the boundary conditions.

10. In the previous three problems, let $h = .5$ and $c = 1$. Choose a value for k that will make the finite difference scheme stable.

11. In the previous four problems, using the value of k that you found in the previous problem, write down the specific equations for $u_{i,1}$.

12. In the previous five problems, solve for $u_{i,1}$.

13. The Crank–Nicholson method is

    Select the correct answer.

    (a) an explicit finite difference method
    (b) an implicit finite difference method
    (c) not a finite difference method

14. The Crank–Nicholson method is

    Select the correct answer.

    (a) always stable
    (b) stable if and only if $\lambda < .5$
    (c) stable if and only if $\lambda < 1$
    (d) stable if and only if $\lambda < 2$
    (e) unstable for all values of λ

15. The Crank–Nicholson formula for solving the heat equation is (using the u_{ij} notation)

    Select the correct answer.

    (a) $u_{i-1,j+1} + \alpha u_{i,j+1} + u_{i+1,j+1} = -u_{i+1,j} + \beta u_{ij} - u_{i-1,j}$ where $\alpha = 2(1+1/\lambda)$, $\beta = 2(1-1/\lambda)$, $\lambda = ck/h^2$
    (b) $u_{i-1,j+1} - \alpha u_{i,j+1} + u_{i+1,j+1} = u_{i+1,j} - \beta u_{ij} + u_{i-1,j}$ where $\alpha = 2(1+1/\lambda)$, $\beta = 2(1-1/\lambda)$, $\lambda = ck/h^2$
    (c) $-u_{i-1,j+1} + \alpha u_{i,j+1} - u_{i+1,j+1} = -u_{i+1,j} + \beta u_{ij} - u_{i-1,j}$ where $\alpha = 2(1+1/\lambda)$, $\beta = 2(1-1/\lambda)$, $\lambda = ck/h^2$
    (d) $-u_{i-1,j+1} - \alpha u_{i,j+1} - u_{i+1,j+1} = u_{i+1,j} - \beta u_{ij} + u_{i-1,j}$ where $\alpha = 2(1+1/\lambda)$, $\beta = 2(1-1/\lambda)$, $\lambda = ck/h^2$
    (e) $-u_{i-1,j+1} + \alpha u_{i,j+1} - u_{i+1,j+1} = u_{i+1,j} - \beta u_{ij} + u_{i-1,j}$ where $\alpha = 2(1+1/\lambda)$, $\beta = 2(1-1/\lambda)$, $\lambda = ck/h^2$

16. Consider the wave equation problem $c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$, $u(0,t) = 0$, $u(1,t) = 0$, $u(x,0) = 0$, $u_t(x,0) = g(x)$. Replace the partial derivatives with central differences, with h and k being the step sizes in the x and t directions, respectively. Let $\lambda = ck/h$. Write down the resulting equations, using the u_{ij} notation.

17. In the previous problem, the resulting equations for $u_{i,1}$ involve $u_{i,-1}$. Explain how to find those values.

18. In the previous two problems, what are the resulting equations for $u_{i,1}$?

19. In the previous three problems, use $c = 1$, $h = .25$, $k = .25$ and $g(x) = \sin(\pi x)$. What are the solutions for $u_{i,1}$?

20. Is the method of the previous four problems stable? Explain.

ANSWER KEY
Zill Differential Equations 9e Chapter 15 Form F

1. c
2. e
3. c
4. a, c, d, f
5. b, c
6. a, e
7. $c\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, $u_x(0,t) = 0$, $u_x(2,t) = 0$, $u(x,0) = 4 - x^2$
8. $u_{i,j+1} = \lambda u_{i+1,j} + (1-2\lambda)u_{i,j} + \lambda u_{i-1,j}$
9. Use a forward difference approximation at $x = 0$ to get $\frac{\partial u}{\partial x}(0,t) = 0 \approx (u(h,t) - u(0,t))/h$ or $u_{0,j} = u_{1,j}$. Similarly, use a backward difference approximation at $x = 2$ to derive $u_{n,j} = u_{n-1,j}$
10. $k = 1/8$ (or smaller) makes $\lambda = 1/2$
11. $u_{11} = (u_{20} + u_{10})/2$, $u_{21} = (u_{30} + u_{10})/2$, $u_{31} = (u_{30} + u_{20})/2$
12. $u_{11} = 7/2$, $u_{21} = 11/4$, $u_{31} = 3/2$
13. b
14. a
15. e
16. $u_{i,j+1} = \lambda^2 u_{i+1,j} + 2(1-\lambda^2)u_{ij} + \lambda^2 u_{i-1,j} - u_{i,j-1}$
17. Use a central difference approximation at $t = 0$ for $\frac{\partial u}{\partial t}(x,0) = g(x) \approx (u(x,k) - u(x,-k))/(2k)$, so that $u_{i,-1} = u_{i,1} - 2kg(x_i)$.
18. $u_{i,1} = \lambda^2(u_{i+1,0} + u_{i-1,0})/2 + (1-\lambda^2)u_{i,0} + kg(x_i)$
19. $u_{11} = \sqrt{2}/8$, $u_{21} = 1/4$, $u_{31} = \sqrt{2}/8$
20. It is stable. $\lambda = 1$ is right on the edge of the stability region.

1. The heat equation is

   Select all that apply.

   (a) linear
   (b) nonlinear
   (c) parabolic
   (d) hyperbolic
   (e) elliptic

2. Laplace's equation is

   Select all that apply.

   (a) nonlinear
   (b) linear
   (c) parabolic
   (d) hyperbolic
   (e) elliptic

3. In general, the solution of an elliptic equation depends on

   Select the correct answer.

   (a) only initial conditions
   (b) only boundary conditions
   (c) a combination of initial and boundary conditions
   (d) other conditions

4. Consider the problem of steady state temperature distribution on a rectangular metal plate of size 1 by 2. Assume that the temperature is fixed at 10 all around the boundary except along one of the sides of length 2, where the temperature is 20. Write down the partial differential equation for the temperature, $u(x, y)$, and also the boundary conditions.

5. In the previous problem, use a step size of $1/2$ in both directions and find the five point approximation of the equation.

6. In the previous problem, how many interior points are there? Write down the specific equations for solving for the temperature at those interior points.

7. In the previous three problems, what is the solution for the temperature at the interior points?

8. In the previous four problems, apply the Gauss-Seidel method to solve for the temperature at the interior points. Use starting values of 15 at each of the interior points, and show one iteration.

9. Consider the heat equation problem $c\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, $u(0,t) = 10$, $u(1,t) = 20$, $u(x,0) = 10(1+x^2)$. If a central difference approximation with step size h is used for the x derivative and a forward difference approximation with step size k is used for the t derivative, the new discretized equation becomes (using the u_{ij} notation)

   Select the correct answer.

   (a) $u_{i,j+1} = \lambda u_{i+1,j} - (1-2\lambda)u_{ij} + \lambda u_{i-1,j}$, where $\lambda = ck/h^2$
   (b) $u_{i,j+1} = \lambda u_{i+1,j} - (1-2\lambda)u_{ij} + \lambda u_{i-1,j}$, where $\lambda = ck/h$
   (c) $u_{i,j+1} = \lambda u_{i+1,j} + (1-2\lambda)u_{ij} + \lambda u_{i-1,j}$, where $\lambda = ck/h^2$
   (d) $u_{i,j+1} = \lambda u_{i+1,j} + (1-2\lambda)u_{ij} - \lambda u_{i-1,j}$, where $\lambda = ck/h$
   (e) $u_{i,j+1} = \lambda u_{i+1,j} - (1-2\lambda)u_{ij} - \lambda u_{i-1,j}$, where $\lambda = ck/h^2$

10. In the previous problem, let $c = 1$, $h = 1/3$, $k = 1/10$. For these values, the finite difference scheme is

    Select the correct answer.

    (a) stable
    (b) unstable
    (c) right on the border between the stable and unstable regions
    (d) It cannot be determined from the given data.

11. In the previous two problems, the equations for $u_{i,1}$ are

    Select the correct answer.

    (a) $u_{11} = .9u_{20} + .8u_{10} - .9u_{00}$
    (b) $u_{11} = .9u_{20} - .8u_{10} + .9u_{00}$
    (c) $u_{11} = .9u_{20} + .8u_{10} + .9u_{00}$
    (d) $u_{21} = .9u_{30} + .8u_{20} + .9u_{10}$
    (e) $u_{21} = .9u_{30} - .8u_{20} + .9u_{10}$

12. In the previous three problems, the solutions for $u_{i,1}$ are

    Select all that apply.

    (a) $u_{11} = 118/9$
    (b) $u_{11} = 128/9$
    (c) $u_{11} = 138/9$
    (d) $u_{21} = 138/9$
    (e) $u_{21} = 148/9$

13. In the previous four problems, the least upper bound for k that would ensure stability is

    Select the correct answer.

    (a) $1/9$
    (b) $1/10$
    (c) $1/18$
    (d) $1/15$
    (e) $1/27$

14. In the five previous problems, the equations in the Crank–Nicholson method are (again using the $u_{i,j}$ notation)

    Select the correct answer.

    (a) $-u_{i-1,j+1} + \alpha u_{i,j+1} - u_{i+1,j+1} = u_{i+1,j} - \beta u_{ij} + u_{i-1,j}$, where $\alpha = 2(1 + 1/\lambda)$, $\beta = 2(1 + 1/\lambda)$, $\lambda = ck/h^2$
    (b) $-u_{i-1,j+1} + \alpha u_{i,j+1} - u_{i+1,j+1} = u_{i+1,j} - \beta u_{ij} + u_{i-1,j}$, where $\alpha = 2(1 - 1/\lambda)$, $\beta = 2(1 + 1/\lambda)$, $\lambda = ck/h$
    (c) $-u_{i-1,j+1} + \alpha u_{i,j+1} - u_{i+1,j+1} = u_{i+1,j} - \beta u_{ij} + u_{i-1,j}$, where $\alpha = 2(1 + 1/\lambda)$, $\beta = 2(1 - 1/\lambda)$, $\lambda = ck/h$
    (d) $-u_{i-1,j+1} + \alpha u_{i,j+1} - u_{i+1,j+1} = u_{i+1,j} - \beta u_{ij} + u_{i-1,j}$, where $\alpha = 2(1 + 1/\lambda)$, $\beta = 2(1 - 1/\lambda)$, $\lambda = ck/h^2$
    (e) $-u_{i-1,j+1} + \alpha u_{i,j+1} - u_{i+1,j+1} = u_{i+1,j} - \beta u_{ij} + u_{i-1,j}$, where $\alpha = 2(1 - 1/\lambda)$, $\beta = 2(1 + 1/\lambda)$, $\lambda = ck/h^2$

15. The Crank–Nicholson method is stable for

    Select the correct answer.

    (a) $\lambda < 2$
    (b) $\lambda < 1$
    (c) $\lambda < 1/2$
    (d) $\lambda < 1/4$
    (e) all λ

16. Consider the wave equation problem $a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$, $u(0,t) = 0$, $u(1,t) = 0$, $u(x,0) = x - x^2$, $u_t(x,0) = 0$. If a central difference approximation with step size h is used for the x derivative and a central difference approximation with step size k is used for the t derivative, write down the finite difference approximations, using the u_{ij} notation.

17. In the previous problem, the equation for $u_{i,1}$ involves $u_{i,-1}$. Explain how these values can be obtained.

18. In the previous two problems, what are the equations for $u_{i,1}$?

19. In the previous three problems, if $a = 1$, $h = 1/4$, $k = 1/5$ what are the solutions for $u_{i,1}$?

20. For the values listed in the previous problem, is the numerical scheme stable. Explain.

ANSWER KEY
Zill Differential Equations 9e Chapter 15 Form G

1. a, c
2. b, e
3. b
4. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, $u(0, y) = 10$, $u(2, y) = 10$, $u(x, 0) = 10$, $u(x, 1) = 20$
5. $u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{ij} = 0$
6. 3 interior points, $-4u_{11} + u_{21} + 40 = 0$, $u_{11} - 4u_{21} + u_{31} + 30 = 0$, $u_{21} - 4u_{31} + 40 = 0$
7. $u_{11} = 95/7$, $u_{21} = 100/7$, $u_{31} = 95/7$
8. $u_{11} = 13.75$, $u_{21} = 14.6875$, $u_{31} = 13.671875$
9. c
10. b
11. b, e
12. a, e
13. c
14. d
15. e
16. $u_{i,j+1} = \lambda^2 u_{i+1,j} + 2(1 - \lambda^2)u_{i,j} + \lambda^2 u_{i-1,j} - u_{i,j-1}, \lambda = ak/h$
17. Use a central difference approximation for $\frac{\partial u}{\partial t}(x, 0) = 0 \approx (u(x, k) - u(x, -k))/(2k)$, so that $u_{i,-1} = u_{i,1}$
18. $u_{i,1} = \lambda^2(u_{i+1,0} + u_{i-1,0})/2 + (1 - \lambda^2)u_{i,0}$
19. $u_{11} = 59/400$, $u_{21} = 21/100$, $u_{31} = 59/400$
20. Yes, it is stable: $\lambda = 4/5 < 1$.

1. The wave equation is

   Select all that apply.

   (a) linear
   (b) nonlinear
   (c) parabolic
   (d) hyperbolic
   (e) elliptic

2. Laplace's equation is

   Select all that apply.

   (a) nonlinear
   (b) linear
   (c) parabolic
   (d) hyperbolic
   (e) elliptic

3. In general, the solution of a hyperbolic equation depends on

   Select the correct answer.

   (a) only initial conditions
   (b) only boundary conditions
   (c) a combination of initial and boundary conditions
   (d) other conditions

4. Consider the problem of steady state temperature distribution on a rectangular metal plate of size 2 by 2. Assume that the temperature is fixed at 10 along two parallel sides and 20 along the other two sides. Write down the partial differential equation and the boundary conditions for the temperature $u(x, y)$.

5. In the previous problem, use a step size of 1/2 in both directions and find the five point approximation of the equation.

6. In the previous problem, how many interior points are there? Write down the specific equations for solving for the temperature at those interior points.

7. In the previous three problems, apply the Gauss-Seidel method to solve for the temperature at the interior points. Use starting values of 15 at each of the interior points, and show the results of one iteration.

8. Consider the problem $c\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, $u(0,t) = u_0$, $u(2,t) = u_1$, $u(x,0) = f(x)$. This problem might be a mathematical model for

   Select the correct answer.

   (a) vibrations of a tightly stretched string
   (b) steady-state temperature distribution in a rectangular plate
   (c) heat conduction in a rod
   (d) none of the above

9. In the previous problem, replace $\frac{\partial^2 u}{\partial x^2}$ with a central difference approximation with step size h and $\frac{\partial u}{\partial t}$ with a forward difference approximation with step size k. The resulting finite difference equations are (using the u_{ij} notation)

   Select the correct answer.

   (a) $u_{i,j+1} = \lambda u_{i+1,j} + (1+2\lambda)u_{ij} + \lambda u_{i-1,j}$, where $\lambda = ck/h^2$
   (b) $u_{i,j+1} = \lambda u_{i+1,j} + (1-2\lambda)u_{ij} + \lambda u_{i-1,j}$, where $\lambda = ck/h^2$
   (c) $u_{i,j+1} = \lambda u_{i+1,j} + (1+2\lambda)u_{ij} + \lambda u_{i-1,j}$, where $\lambda = ck/h$
   (d) $u_{i,j+1} = \lambda u_{i+1,j} - (1-2\lambda)u_{ij} + \lambda u_{i-1,j}$, where $\lambda = ck/h$
   (e) $u_{i,j+1} = \lambda u_{i+1,j} - (1-2\lambda)u_{ij} + \lambda u_{i-1,j}$, where $\lambda = ck/h^2$

10. In the previous two problems, let $f(x) = 1 + 2\sin(\pi x/2)$, $u_0 = 1$, $u_1 = 1$, $c = 1$, $h = 1/2$, $k = 1/4$. The equations for $u_{i,1}$ become

    Select the correct answer.

    (a) $u_{i,1} = 2u_{i+1,0} + 3u_{i,0} + 2u_{i-1,0}$
    (b) $u_{i,1} = 2u_{i+1,0} - 3u_{i,0} + 2u_{i-1,0}$
    (c) $u_{i,1} = u_{i+1,0} + 2u_{i,0} + u_{i-1,0}$
    (d) $u_{i,1} = u_{i+1,0} - u_{i,0} + u_{i-1,0}$
    (e) $u_{i,1} = u_{i+1,0} + u_{i,0} + u_{i-1,0}$

11. In the previous problem the solution for $u_{i,1}$ is

    Select all that apply.

    (a) $u_{11} = 5 - \sqrt{2}$
    (b) $u_{11} = 3 - \sqrt{2}$
    (c) $u_{21} = 2\sqrt{2} - 1$
    (d) $u_{21} = 5 + 2\sqrt{2}$
    (e) $u_{31} = 5 - \sqrt{2}$
    (f) $u_{31} = 3 - \sqrt{2}$

12. In the previous four problems, the value of λ implies that the finite difference method is

    Select the correct answer.

    (a) stable
    (b) unstable
    (c) right on the border between the stable and unstable regions
    (d) It cannot be determined from the available data.

13. In the previous five problems, the equations for Crank–Nicholson method are

    Select the correct answer.

    (a) $-u_{i-1,j+1} + \alpha u_{i,j+1} - u_{i+1,j+1} = u_{i+1,j} - \beta u_{i,j} + u_{i-1,j}$, where $\alpha = 2(1 + 1/\lambda)$, $\beta = 2(1 - 1/\lambda)$
    (b) $u_{i-1,j+1} - \alpha u_{i,j+1} + u_{i+1,j+1} = u_{i+1,j} - \beta u_{i,j} + u_{i-1,j}$, where $\alpha = 2(1 + 1/\lambda)$, $\beta = 2(1 - 1/\lambda)$
    (c) $u_{i-1,j+1} + \alpha u_{i,j+1} + u_{i+1,j+1} = u_{i+1,j} + \beta u_{i,j} + u_{i-1,j}$, where $\alpha = 2(1 + 1/\lambda)$, $\beta = 2(1 - 1/\lambda)$
    (d) $-u_{i-1,j+1} + \alpha u_{i,j+1} - u_{i+1,j+1} = u_{i+1,j} - \beta u_{i,j} + u_{i-1,j}$, where $\alpha = 2(1 - 1/\lambda)$, $\beta = 2(1 + 1/\lambda)$
    (e) $u_{i-1,j+1} + \alpha u_{i,j+1} + u_{i+1,j+1} = u_{i+1,j} - \beta u_{i,j} + u_{i-1,j}$, where $\alpha = 2(1 - 1/\lambda)$, $\beta = 2(1 + 1/\lambda)$

14. In the previous problem, the Crank–Nicholson method is

    Select the correct answer.

    (a) stable
    (b) unstable
    (c) right on the border between the stable and unstable regions
    (d) It cannot be determined from the available data.

15. The Crank–Nicholson method for the heat equation is

    Select the correct answer.

    (a) stable if $\lambda < 1/2$
    (b) stable if $\lambda < 1$
    (c) never stable
    (d) always stable
    (e) none of the above

16. Consider the wave equation problem $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$, $u(0,t) = 0$, $u(1,t) = 0$, $u(x,0) = \sin(\pi x)$, $u_t(x,0) = 0$. Replace the partial derivatives with central difference approximations with step size $h = 1/4$ in the $x-$ direction and $k = 1/8$ in the $t-$ direction. Write down the resulting equations using the u_{ij} notation.

17. In the previous problem, when $j = 0$, what values should be used for $u_{i,-1}$?

18. In the previous problem, write down the resulting equations for $u_{i,1}$.

19. In the previous problem, what are the solutions for $u_{i,1}$?

20. In the previous problem, is the method stable? Explain.

ANSWER KEY
Zill Differential Equations 9e Chapter 15 Form H

1. a, d
2. b, e
3. c
4. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, $u(0, y) = 10$, $u(2, y) = 10$, $u(x, 0) = 20$, $u(x, 2) = 20$
5. $u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{i,j} = 0$
6. $-4u_{11} + u_{12} + u_{21} = -30$, $u_{11} - 4u_{21} + u_{31} + u_{22} = -20$, $u_{21} - 4u_{31} + u_{32} = -30$, $u_{11} - 4u_{12} + u_{13} + u_{22} = -10$, $u_{21} + u_{12} - 4u_{22} + u_{32} + u_{23} = 0$, $u_{31} + u_{22} + u_{33} - 4u_{32} = -10$, $u_{12} - 4u_{13} + u_{23} = -30$, $u_{13} - 4u_{23} + u_{33} + u_{22} = -20$, $u_{23} - 4u_{33} + u_{32} = -30$
7. $u_{11} = 15$, $u_{21} = 16.25$, $u_{31} = 15.3125$, $u_{12} = 13.75$, $u_{22} = 15$, $u_{32} = 13.828125$, $u_{13} = 14.6875$, $u_{23} = 16.171875$, $u_{33} = 15$
8. c
9. b
10. d
11. b, c, f
12. b
13. a
14. a
15. d
16. $u_{i,j+1} = \lambda^2 u_{i+1,j} + 2(1 - \lambda^2)u_{i,j} + \lambda^2 u_{i-1,j} - u_{i,j-1}$, where $\lambda = k/h$
17. $u_{i,-1} = u_{i,1}$
18. $u_{i,1} = \lambda^2(u_{i+1,0} + u_{i-1,0})/2 + (1 - \lambda^2)u_{i,0}$
19. $u_{11} = (1 + 3\sqrt{2})/8$, $u_{21} = (6 + \sqrt{2})/8$, $u_{31} = (1 + 3\sqrt{2})/8$
20. yes, $\lambda = 1/2 < 1$