1. Solve the differential equation $y' - 4y = x^2$.
2. Solve the differential equation $(x^2 + y^2)dx + 2xydy = 0$.
3. Use Euler's method to solve the initial value problem $y' = 1 + y$, $y(0) = 1$, with a step size of $h = 0.1$. Find approximations of $y(0.1)$ and $y(0.2)$.
4. A 10 gram sample of a radioactive substance has a half-life of 52 hours. Identify all variables. Assuming the rate of decay is proportional to the amount present, write down the differential equation problem for the amount of the substance as a function of time, and solve it. How much is left after 10 hours?
5. A frozen chicken is taken from a freezer at $32°F$ oven and placed on table at $65°F$. Thirty minutes later it has warmed to $45°F$. Identify all variables in the problem, write down the differential equation problem for the temperature, and solve it for the temperature as a function of time. Assume that Newton's law of cooling applies. What is the temperature one hour after it was taken out of the freezer?
6. Solve the differential equation $y'' + 6y' + 9y = x^2 + 1$.
7. Solve the differential equation $y'' - 6y' + 13y = \cos x$.
8. Solve the differential equation $y'' - 4y' + 3y = x + e^x$.
9. Solve the differential equation $y'' + y = \tan x$.
10. Solve the differential equation $x^2y'' + 3xy + 10y = 0$.
11. A 1-kilogram mass is attached to a spring with spring constant 10-Newtons per meter. The system is immersed in a liquid that imparts a damping force numerically equal to 6 times the instantaneous velocity. The mass is initially released from rest at a point 0.25 meter below the equilibrium point. Identify all variables and find the equation of motion of the mass.
12. Identify the singular points of $(x - 1)^3x(x + 1)^2y'' - 3xy' + 2y = 0$ and tell whether they are regular or irregular.
13. Use power series methods to find the solution of $2xy'' - y' + 2y = 0$ about $x = 0$.
14. Use power series methods to find the solution of $xy'' + 2y' - xy = 0$ about $x = 0$.
15. Use power series methods to find the solution of $xy'' + xy' - y = 0$ about $x = 0$.
16. Use power series methods to find one solution of $x^2y'' + xy' + (x^2 - 9)y = 0$ about $x = 0$.
17. In the previous problem, identify the equation and write down the solution using the special function notation.
18. Find the Laplace transform of te^{2t}.
19. Find the inverse Laplace transform of $5/(s - 4)^4$.
20. Find the inverse Laplace transform of $e^{-2s}/(s(s - 1))$.

21. Find the inverse Laplace transform of $1/(s^2+4s+5)$.

22. Find the inverse Laplace transform of $1/((s-1)(s^2+9))$ using the convolution theorem.

23. Solve the initial value problem $y'-y=\sin(3t)$, $y(0)=0$ using Laplace transform methods. (Hint: the previous problem might be useful.)

24. Solve the initial value problem $y''-y=t\,\mathcal{U}(t-4)$, $y(0)=1$, $y'(0)=0$ using Laplace transform methods.

25. Solve the system $\mathbf{X}'=A\mathbf{X}$ where $A=\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$.

26. Solve the system $\mathbf{X}'=A\mathbf{X}$ where $A=\begin{pmatrix} 1 & 1 \\ -9 & 1 \end{pmatrix}$.

27. Solve the system $\mathbf{X}'=A\mathbf{X}$ where $A=\begin{pmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{pmatrix}$.

28. Consider the initial value problem $y'=3xy, y(0)=1$. Solve for $y(0.1)$ using the improved Euler method with a step size of $h=0.1$.

29. In the previous problem, solve for $y(0.1)$ using the classical fourth order Runge–Kutta method with a step size of $h=0.1$.

30. Let $A=\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$. Classify the critical point $(0,0)$ of the system $X'=AX$ as stable or unstable and saddle point, node, or spiral point.

31. In the previous problem, identify the eigenvectors. Do the solutions enter or leave the origin along the eigenvectors at $t\to\infty$? What happens to all other solutions?

32. Let $A=\begin{pmatrix} -4 & 1 \\ -2 & -1 \end{pmatrix}$. Classify the critical point $(0,0)$ of the system $X'=AX$ as stable or unstable and saddle point, node, or spiral point.

33. In the previous problem, identify the eigenvectors. Do the solutions enter or leave the origin along the eigenvectors at $t\to\infty$?

34. Show that the set of functions $\{\sin x, \sin(2x), \sin(3x), \ldots\}$ is orthogonal on $[0,\pi]$.

35. For the set of functions in the previous problem, find an orthonormal set of functions.

36. Let $f(x)=x$. Write $f(x)$ as a Fourier cosine series on $[0,\pi]$.

37. Consider the heat equation problem $k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}$, $u(0,t)=0$, $u(2,t)=0$, $u(x,0)=f(x)$. Separate variables using $u(x,t)=X(x)T(t)$. What are the corresponding problems for X and T?

38. What are the solutions of the eigenvalue problem and the other problem from the previous problem?

39. In the previous two problems, what is the infinite series solution for $u(x,t)$?

40. Consider the problem of steady-state temperature, $u(r,\theta)$, on a circular plate of radius 5, with given temperature, $f(\theta)$ along the boundary. Write down the differential equation in polar coordinates along with appropriate boundary conditions that will allow you to solve the problem. Separate variables, using $u(r,\theta) = R(r)\Theta(\theta)$. Write down the resulting problems for R and Θ.

41. Find the solutions of the problems of the previous problem.

42. In the previous two problems, what is the infinite series solution for $u(r,\theta)$?

43. Let $f(x) = \left\{ \begin{array}{c} 0 \text{ if } x < 0 \\ 1 \text{ if } 0 < x < 1 \\ 0 \text{ if } x > 1 \end{array} \right\}$. Find the Fourier integral representation of $f(x)$.

44. Consider the heat equation problem $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, $-\infty < x < \infty$, $t > 0$, $u(x,0) = \left\{ \begin{array}{c} u_0 \text{ if } \mid x \mid < 1 \\ 0 \text{ otherwise} \end{array} \right\}$. Apply a Fourier transform in x and write down the resulting problem for $U(\alpha,t) = \mathcal{F}\{u(x,t)\}$.

45. In the previous problem, what is the solution for $U(\alpha,t)$?

46. In the previous two problems, what is the solution for $u(x,t)$?

47. Write down the finite difference approximation for the wave equation, $c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$ with an x step size of h and a t step size of k Use the $u_{i,j}$ notation.

48. In the previous problem, assume that the boundary and initial conditions are $u(0,t) = 0$, $u(1,t) = 0$, $u(x,0) = \left\{ \begin{array}{c} x \text{ if } 0 < x < 1/2 \\ 1 - x \text{ if } 1/2 < x < 1 \end{array} \right\}$, $u_t(x,0) = 0$. Also assume that $c = 1$, $h = 0.25$, and $k = 0.5$. The equations for $u_{i,1}$ involve $u_{i,-1}$. Explain how you would find those values.

49. In the previous two problems, solve the system of equations for u at the mesh points along the line where $t = 0.5$.

50. In the previous three problems, if $c = 1$ and $h = 0.25$, what is the least upper bound for k that would guarantee stability of the numerical scheme?

ANSWER KEY
Zill Differential Equations 9e Final Exam 2 Form A

1. $y = -x^2/4 - x/8 - 1/32 + ce^{4x}$
2. $x^3/3 + xy^2 = c$
3. $y(0.1) \approx 1.2$, $y(0.2) \approx 1.42$
4. $x(t)$ = amount of the substance at time t, $\frac{dx}{dt} = kx$, $x = 10e^{kt}$, $k = \ln(1/2)/52 = -0.0133, x(10) = 8.75g$
5. $T(t)$ = temperature at time t, $\frac{dT}{dt} = k(T-65)$, $T = 65 - 33e^{kt}$, $k = \ln(20/33)/30 = -0.0167$, $T = 65 - 33e^{-0.0167t}$, $T(60) = 52.9°F$
6. $y = c_1e^{-3x} + c_2xe^{-3x} + x^2/9 - 4x/27 + 5/27$
7. $y = c_1e^{3x}\cos(2x) + c_2e^{3x}\sin(2x) + (2\cos x - \sin x)/30$
8. $y = c_1e^{3x} + c_2e^x + x/3 + 4/9 - xe^x/2$
9. $y = c_1\cos x + c_2\sin x - \cos x \ln|\sec x + \tan x|$
10. $y = x^{-1}[c_1\cos(3\ln x) + c_2\sin(3\ln x)]$
11. $x(t)$ = position at time t, $x''+6x'+10x = 0$, $x(0) = 0.25$, $x'(0) = 0$, $x = 0.25e^{-3t}\cos t + 0.75e^{-3t}\sin t$
12. $x = 0, -1$ are regular singular points, $x = 1$ is an irregular singular point
13. $y = c_0[1 + \sum_{n=1}^{\infty}(-2)^n x^n/(n!(-1)\cdot 1\cdots(2n-3)) +$
$c_1x^{3/2}[1 + \sum_{n=1}^{\infty}(-2)^n x^n/(n!5\cdot 7\cdots(2n+3))$
14. $y = c_0\sum_{n=0}^{\infty} x^{2n-1}/(2n)! + c_1\sum_{n=0}^{\infty} x^{2n}/(2n+1)!$
15. $y = c_0x + c_1[-x\ln x - 1 + \sum_{n=2}^{\infty}(-1)^n x^n/(n!(n-1))]$
16. $y = c_0\sum_{n=0}^{\infty}(-1)^n(x/2)^{2n+3}/(n!(n+3)!)$
17. Bessel's equation of order 3, $y = c_0J_3(x) + c_1Y_3(x)$
18. $1/(s-2)^2$
19. $5t^3e^{4t}/6$
20. $\mathcal{U}(t-2)(e^{t-2} - 1)$
21. $e^{-2t}\sin t$
22. $(3e^t - 3\cos(3t) - \sin(3t))/30$
23. $y = (3e^t - 3\cos(3t) - \sin(3t))/10$
24. $y = \cosh t + (-t + 5e^{t-4}/2 + 3e^{4-t}/2)\,\mathcal{U}(t-4)$
25. $X = c_1\begin{pmatrix} 3 \\ -1 \end{pmatrix} e^t + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t}$
26. $X = c_1e^t\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos(3t) - \begin{pmatrix} 0 \\ 3 \end{pmatrix}\sin(3t)\right] + c_2e^t\left[\begin{pmatrix} 0 \\ 3 \end{pmatrix}\cos(3t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin(3t)\right]$

27. $X = c_1 \begin{pmatrix} 4 \\ 5 \\ -2 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} e^{5t} + c_3 \left[\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} te^{5t} + \begin{pmatrix} -5/2 \\ -1/2 \\ 0 \end{pmatrix} e^{5t} \right]$

28. $y_1 = 1.015$

29. $y_1 = 1.015113$

30. unstable saddle point

31. $\lambda_1 = -1$, $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\lambda_2 = 4$, $v_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, the solution approaches the origin along v_1, the solution recedes from the origin along v_2; all other solutions recede from the origin.

32. stable node

33. $\lambda_1 = -2$, $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\lambda_2 = -3$, $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, the solutions approach the origin along both eigenvectors; all other solutions approach the origin tangent to the v_1 line.

34. $\int_0^\pi \sin(nx)\sin(mx)dx = \int_0^\pi [\cos((n-m)x) - \cos((n+m)x)]dx/2 =$ $[\sin((n-m)x)/(n-m) - \sin((n+m)x)/(n+m)]/2|_0^\pi = 0$ if $n \neq m$.

35. $\{\sqrt{2/\pi}\sin x,\ \sqrt{2/\pi}\sin(2x),\ \sqrt{2/\pi}\sin(3x),\ \ldots\}$

36. $f(x) = \pi/2 + \sum_{n=1}^\infty 2(\cos(n\pi) - 1)\cos(nx)/(n^2\pi)$

37. $X'' + \lambda X = 0$, $X(0) = 0$, $X(2) = 0$, $T' + k\lambda T = 0$

38. $\lambda = n^2\pi^2/4$, $X = \sin(n\pi x/2)$, $T = e^{-kn^2\pi^2 t/4}$, $n = 1, 2, 3, \ldots$

39. $u = \sum_{n=1}^\infty c_n \sin(n\pi x/2) e^{-kn^2\pi^2 t/4}$, where $c_n = \int_0^2 f(x)\sin(n\pi x/2)dx$

40. $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$, $u(0,\theta)$ is bounded, $u(5,\theta) = f(\theta)$, $u(r,0) = u(r,2\pi)$. After separation, $r^2R'' + rR' - \lambda R = 0$, $R(0)$ is bounded, $\Theta'' + \lambda\Theta = 0$, $\Theta(0) = \Theta(2\pi)$

41. $\lambda = n^2$, $\Theta = c_n\cos(n\theta) + d_n\sin(n\theta)$, $R = r^n$, $n = 0, 1, 2, \ldots$

42. $u = c_0 + \sum_{n=1}^\infty [c_n\cos(n\theta) + d_n\sin(n\theta)]r^n$, where $c_0 = \int_0^{2\pi} f(\theta)d\theta/(2\pi)$, $c_n = \int_0^{2\pi} f(\theta)\cos(n\theta)d\theta/(5^n\pi)$, $d_n = \int_0^{2\pi} f(\theta)\sin(n\theta)d\theta/(5^n\pi)$

43. $f(x) = \int_0^\infty \{[\sin\alpha\cos(\alpha x) + (1-\cos\alpha)\sin(\alpha x)]/\alpha\}d\alpha/\pi$

44. $U_t + \alpha^2 U = 0$, $U(\alpha, 0) = 2u_0\sin\alpha/\alpha$

45. $U = 2u_0\sin\alpha e^{-\alpha^2 t}/\alpha$

46. $u = u_0\int_{-\infty}^\infty [\sin\alpha e^{-\alpha^2 t}e^{-i\alpha x}/\alpha]d\alpha/\pi$

47. $u_{i,j+1} = \lambda^2(u_{i+1,j} + u_{i-1,j}) + 2(1-\lambda^2)u_{i,j} - u_{i,j-1}$, where $\lambda = ck/h$

48. Use a central difference approximation of $u_t(x,0) = 0 \approx (u(x,k) - u(x,-k))/(2k)$ to get $u_{i,-1} = u_{i,1}$.

49. $u_{1,1} = 1/4$, $u_{2,1} = -1/2$, $u_{3,1} = 1/4$

50. $k = 0.25$

1. The solution of $y' + y = x$ is

   Select the correct answer.

   (a) $y = -x + 1 + ce^{x}$
   (b) $y = -x - 1 + ce^{x}$
   (c) $y = x - 1 + ce^{-x}$
   (d) $y = -x - 1 + ce^{-x}$
   (e) $y = x + 1 + ce^{-x}$

2. The solution of $xy' = (x-1)y^2$

   Select the correct answer.

   (a) $y = 1/(x + \ln x + c)$
   (b) $y = 1/(x - \ln x + c)$
   (c) $y = -c/(x + \ln x)$
   (d) $y = -c/(x - \ln x)$
   (e) $y = -1/(x - \ln x + c)$

3. A frozen chicken at $32°F$ is taken out of the freezer and placed on a table at $70°F$. One hour later the temperature of the chicken is $55°F$. The mathematical model for the temperature $T(t)$ as a function of time t is (assuming Newton's law of warming)

   Select the correct answer.

   (a) $\frac{dT}{dt} = kT$, $T(0) = 32$, $T(1) = 55$
   (b) $\frac{dT}{dt} = k(T - 70)$, $T(0) = 32$, $T(1) = 55$
   (c) $\frac{dT}{dt} = (T - 70)$, $T(0) = 32$, $T(1) = 55$
   (d) $\frac{dT}{dt} = T$, $T(0) = 32$, $T(1) = 55$
   (e) $\frac{dT}{dt} = k(T - 55)$, $T(0) = 32$, $T(1) = 55$

4. In the previous problem, the solution for the temperature is

   Select the correct answer.

   (a) $T(t) = 70 - 38e^{-.930t}$
   (b) $T(t) = 70 - 38e^{.930t}$
   (c) $T(t) = 55 - 32e^{-.930t}$
   (d) $T(t) = 55 - 32e^{.930t}$
   (e) $T(t) = 55e^{-.930t}$

5. The solution of $y'' - 6y' + 8y = 0$ is

   Select the correct answer.

   (a) $y = c_1 e^{-2x} + c_2 e^{-4x}$
   (b) $y = c_1 e^{2x} + c_2 x e^{4x}$
   (c) $y = c_1 e^{-2x} + c_2 x e^{-4x}$
   (d) $y = c_1 e^{2x} + c_2 e^{4x}$
   (e) $y = c_1 e^{2x} + c_2 e^{-4x}$

6. The solution of $y'' - 4y' + 20y = 0$ is

   Select the correct answer.

   (a) $y = c_1 e^{-2x} \cos(4x) + c_2 e^{-2x} \sin(4x)$
   (b) $y = c_1 e^{-2x} \cos(4x) + c_2 e^{2x} \sin(4x)$
   (c) $y = c_1 e^{2x} \cos(4x) + c_2 e^{2x} \sin(4x)$
   (d) $y = c_1 e^{2x} + c_2 e^{4x}$
   (e) $y = c_1 \cos(4x) + c_2 \sin(4x)$

7. The correct form of the particular solution of $y'' + 2y' + y = e^{-x}$ is

   Select the correct answer.

   (a) $y_p = Ae^{-x}$
   (b) $y_p = Axe^{-x}$
   (c) $y_p = Ax^2e^{-x}$
   (d) $y_p = Ax^3e^{-x}$
   (e) none of the above

8. The solution of $y'' + 2y' = x + e^x$ is

   Select the correct answer.

   (a) $y = c_1 + c_2 e^{-2x} + x^2/4 - x/4 + e^x/3$
   (b) $y = c_1 + c_2 e^{-2x} + x^2/4 + x/4 - e^x/3$
   (c) $y = c_1 + c_2 e^{-2x} + x^2/4 + x/4 + e^x/3$
   (d) $y = c_1 + c_2 e^{-2x} - x^2/4 - x/4 - e^x/3$
   (e) $y = c_1 + c_2 e^{-2x} - x^2/4 - x/4 + e^x/3$

9. The solution of $y'' + 3y' - 4y = \cos x$ is

   Select the correct answer.

   (a) $y = c_1e^x + c_2e^{-4x} + (5\sin x + 3\cos x)/34$
   (b) $y = c_1e^x + c_2e^{-4x} + (-5\sin x + 3\cos x)/34$
   (c) $y = c_1e^x + c_2e^{-4x} + (-5\cos x - 3\sin x)/34$
   (d) $y = c_1e^x + c_2e^{-4x} + (5\cos x + 3\sin x)/34$
   (e) $y = c_1e^x + c_2e^{-4x} + (-5\cos x + 3\sin x)/34$

10. The solution of $y'' + y = \tan x$ is

    Select the correct answer.

    (a) $y = c_1\cos x + c_2\sin x + \cos x \ln|\sec x + \tan x|$
    (b) $y = c_1\cos x + c_2\sin x - \cos x \ln|\sec x + \tan x|$
    (c) $y = c_1\cos x + c_2\sin x + \cos x \ln|\sec x|$
    (d) $y = c_1\cos x + c_2\sin x - \cos x \ln|\tan x|$
    (e) $y = c_1\cos x + c_2\sin x - \cos x \ln|\sec x - \tan x|$

11. The solution of $x^2y'' - xy' = 0$ is

    Select the correct answer.

    (a) $y = c_1 + c_2x^{-1}$
    (b) $y = c_1\ln x + c_2x^{-1}$
    (c) $y = c_1 + c_2x^2$
    (d) $y = c_1 + c_2\ln x$
    (e) $y = c_1 + c_2x^{-2}$

12. A 4-pound weight is hung on a spring and stretches it 1 foot. The mass spring system is then put into motion in a medium offering a damping force numerically equal to the velocity. If the mass is pulled down 6 inches from equilibrium and released, the initial value problem describing the position, $x(t)$, of the mass at time t is

    Select the correct answer.

    (a) $x'' - 8x' + 32x = 0,\ x(0) = 6,\ x'(0) = 0$
    (b) $x'' + 8x' + 32x = 0,\ x(0) = 6,\ x'(0) = 0$
    (c) $x'' - 8x' + 32x = 0,\ x(0) = 1/2,\ x'(0) = 0$
    (d) $x'' + 8x' + 32x = 0,\ x(0) = 1/2,\ x'(0) = 0$
    (e) $x'' + 32x = 8,\ x(0) = 1/2,\ x'(0) = 0$

13. In the previous problem, the solution for the position, $x(t)$, is

    Select the correct answer.

    (a) $x = e^{4t}(\cos(4t) + \sin(4t))/2$
    (b) $x = e^{-4t}(\cos(4t) + \sin(4t))/2$
    (c) $x = e^{-4t}(\cos(8t) + \sin(8t))/2$
    (d) $x = e^{4t}(\cos(8t) + \sin(8t))/2$
    (e) $x = 4e^{8t} - 32te^{8t}$

14. Using power series methods, the solution of $2xy'' + y' + 2y = 0$ is

    Select the correct answer.

    (a) $y = c_0 \sum_{n=1}^{\infty}(-2)^n x^n/(n!(1\cdot 3\cdots(2n-1))) + c_1 x^{1/2}\sum_{n=1}^{\infty}(-2)^n x^n/(n!(3\cdot 5\cdots(2n+1)))$
    (b) $y = c_0 \sum_{n=1}^{\infty}(-2)^n x^n/(n!(1\cdot 3\cdots(2n-1))) + c_1 x^{1/2}[1 + \sum_{n=1}^{\infty}(-2)^n x^n/(n!(3\cdot 5\cdots(2n+1)))]$
    (c) $y = c_0[1 + \sum_{n=1}^{\infty}(-2)^n x^n/(n!(1\cdot 3\cdots(2n-1)))] + c_1[1 + \sum_{n=1}^{\infty}(-2)^n x^n/(n!(3\cdot 5\cdots(2n+1)))]$
    (d) $y = c_0[1 + \sum_{n=1}^{\infty}(-2)^n x^n/(n!(1\cdot 3\cdots(2n-1)))] + c_1 x^{1/2}[1 + \sum_{n=1}^{\infty}(-2)^n x^n/(n!(3\cdot 5\cdots(2n+1)))]$
    (e) $y = [1 + \sum_{n=1}^{\infty}(-2)^n x^n/(n!(1\cdot 3\cdots(2n-1)))] + x^{1/2}[1 + \sum_{n=1}^{\infty}(-2)^n x^n/(n!(3\cdot 5\cdots(2n+1)))]$

15. Using power series methods, the solution of $xy'' - xy' + y = 0$ is

    Select the correct answer.

    (a) $y = c_0 x + c_1[x\ln x - 1 + \sum_{n=2}^{\infty} x^n/n!]$
    (b) $y = c_0 x + c_1[x\ln x - 1 + \sum_{n=1}^{\infty} x^n/(n!(n+1))]$
    (c) $y = c_0 x + c_1[x\ln x + \sum_{n=2}^{\infty} x^n/(n!(n-1))]$
    (d) $y = c_0 x + c_1[x\ln x + \sum_{n=1}^{\infty} x^n/(n!(n-1))]$
    (e) $y = c_0 x + c_1[x\ln x - 1 + \sum_{n=2}^{\infty} x^n/(n!(n-1))]$

16. $\mathcal{L}\{t\sin t\} =$

    Select the correct answer.

    (a) $2s/(s^2-1)$
    (b) $(s^2+1)/(s^2-1)^2$
    (c) $(s^2-1)/(s^2+1)$
    (d) $s^2/(s^2+1)$
    (e) $2s/(s^2+1)^2$

17. $\mathcal{L}^{-1}\{s/(s^2+4s+29)\} =$

    Select the correct answer.

    (a) $\cos(5t)e^{-2t}/5$
    (b) $\cos(5t)e^{2t}/5$
    (c) $(\cos(5t) - 2\sin(5t)/5)e^{-2t}$
    (d) $\sin(5t)e^{-2t}/5$
    (e) $\sin(5t)e^{-2t}$

18. Using the convolution theorem, we find that $\mathcal{L}^{-1}\{1/((s+1)(s^2+1))\} =$

    Select the correct answer.

    (a) $(e^{-t} + \sin t - \cos t)/2$
    (b) $(e^{t} + \sin t - \cos t)/2$
    (c) $(e^{-t} + \sin t + \cos t)/2$
    (d) $(e^{t} - \sin t - \cos t)/2$
    (e) $(e^{-t} - \sin t - \cos t)/2$

19. Using Laplace transform methods, the solution of $y' + y = 2\sin t$, $y(0) = 1$ is (Hint: the previous problem might be useful.)

    Select the correct answer.

    (a) $y = 2e^{-t} + \sin t + \cos t$
    (b) $y = e^{t} + e^{-t} - \sin t - \cos t$
    (c) $y = 2e^{-t} - \sin t - \cos t$
    (d) $y = 2e^{-t} + \sin t - \cos t$
    (e) $y = e^{t} + e^{-t} + \sin t - \cos t$

20. Using Laplace transform methods, the solution of $y'' + y = \delta(t - \pi/2)$, $y(0) = 1$, $y'(0) = 0$ is

    Select the correct answer.

    (a) $y = \sin t + \sin(t - \pi/2)\,\mathcal{U}(t - \pi/2)$
    (b) $y = \sin t - \cos(t - \pi/2)\,\mathcal{U}(t - \pi/2)$
    (c) $y = \cos t + \sin(t - \pi/2)\,\mathcal{U}(t - \pi/2)$
    (d) $y = \cos t + \cos(t - \pi/2)\,\mathcal{U}(t - \pi/2)$
    (e) $y = \cos t - \sin(t - \pi/2)\,\mathcal{U}(t - \pi/2)$

21. The solution of $\mathbf{X}' = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix} \mathbf{X}$ is

Select the correct answer.

(a) $\mathbf{X} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t}$

(b) $\mathbf{X} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t}$

(c) $\mathbf{X} = c_1 \begin{pmatrix} -2 \\ -1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t}$

(d) $\mathbf{X} = c_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t}$

(e) $\mathbf{X} = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t}$

22. The eigenvalue-eigenvector pairs for the matrix $A = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \end{pmatrix}$ are

Select all that apply.

(a) $4, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

(b) $2, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

(c) $2, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

(d) $2, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

(e) $2, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

23. The solution of $\mathbf{X}' = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \end{pmatrix} \mathbf{X}$ is

Select the correct answer.

(a) $X = c_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t} + c_3 \left[\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{2t} \right]$

(b) $X = c_1 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{2t} + c_3 \left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{2t} \right]$

(c) $X = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t} + c_3 \left[\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{2t} \right]$

(d) $X = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^{2t} + c_3 \left[\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{2t} \right]$

(e) $X = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{2t} + c_3 \left[\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{2t} \right]$

24. The solution of $\mathbf{X}' = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \mathbf{X}$ is

Select the correct answer.

(a) $\mathbf{X} = c_1 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin t \right] + c_2 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos t \right]$

(b) $\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{\sqrt{3}t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-\sqrt{3}t}$

(c) $\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t}$

(d) $\mathbf{X} = c_1 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\sqrt{3}t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(\sqrt{3}t) \right] +$
$c_2 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin(\sqrt{3}t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(\sqrt{3}t) \right]$

(e) $\mathbf{X} = c_1 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(2t) - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \sin(2t) \right] + c_2 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin(2t) + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos(2t) \right]$

25. A particular solution of $\mathbf{X}' = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}\mathbf{X} + \begin{pmatrix} 2 \\ t \end{pmatrix}$ is

    Select the correct answer.

    (a) $\mathbf{X}_p = \begin{pmatrix} t+2 \\ -t+3 \end{pmatrix}$

    (b) $\mathbf{X}_p = \begin{pmatrix} t+2 \\ -t-3 \end{pmatrix}$

    (c) $\mathbf{X}_p = \begin{pmatrix} -t+2 \\ -t-3 \end{pmatrix}$

    (d) $\mathbf{X}_p = \begin{pmatrix} -t+2 \\ t+3 \end{pmatrix}$

    (e) $\mathbf{X}_p = \begin{pmatrix} -t-2 \\ -t+3 \end{pmatrix}$

26. Using the improved Euler method with a step size of $h = 0.1$, the solution of $y' = 1+y^2$, $y(0) = 0$ at $x = 0.1$ is

    Select the correct answer.

    (a) $y_1 = 0.1015$
    (b) $y_1 = 0.115$
    (c) $y_1 = 0.105$
    (d) $y_1 = 0.10005$
    (e) $y_1 = 0.1005$

27. In the previous problem, the exact solution of the initial value problem is

    Select the correct answer.

    (a) $y = \tan x$
    (b) $y = \sec x$
    (c) $y = (e^{-2x} - 1)/(e^{-2x} + 1)$
    (d) $y = -(e^{-2x} + 1)/(e^{-2x} - 1)$
    (e) $y = -(e^{2x} - 1)/(e^{2x} + 1)$

28. In the previous two problems, the error in the improved Euler method at $x = 0.1$ is

    Select the correct answer.

    (a) 0.00467
    (b) 0.000165
    (c) 0.870
    (d) 0.895
    (e) 0.0897

29. Using the classical Runge–Kutta method of order 4 with a step size of $h = 0.1$, the solution of $y' = 1 + y^2$, $y(0) = 0$ at $x = 0.1$ is

    Select the correct answer.

    (a) 0.099589
    (b) 0.100334589
    (c) 0.10034589
    (d) 0.10334589
    (e) 0.1034589

30. In the previous problem, the error in the classical Runge–Kutta method at $x = 0.1$ is (Hint: see the previous five problems.)

    Select the correct answer.

    (a) 0.00083
    (b) 0.000083
    (c) 0.000000083
    (d) 0.0000083
    (e) 0.00000083

31. Let $A = \begin{pmatrix} -4 & -9 \\ 4 & 8 \end{pmatrix}$, and consider the system $X' = AX$. The critical point $(0, 0)$ of the system is a

    Select the correct answer.

    (a) stable node
    (b) unstable node
    (c) unstable saddle
    (d) stable spiral point
    (e) unstable spiral point

32. Let $A = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}$, and consider the system $X' = AX$. The critical point $(0, 0)$ of the system is a spiral point. The origin is

    Select the correct answer.

    (a) unstable, and the solutions recede from the origin clockwise as $t \to \infty$.
    (b) unstable, and the solutions recede from the origin counter-clockwise as $t \to \infty$.
    (c) stable, and the solutions approach the origin clockwise as $t \to \infty$.
    (d) stable, and the solutions approach the origin counter-clockwise as $t \to \infty$.
    (e) none of the above

33. Consider the non-linear system $x' = 1 - 2xy$, $y' = 2xy - y$. The linearized system about the one critical point, $(1/2, 1)$, is $X' = AX$, where $A =$

    Select the correct answer.

    (a) $\begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}$

    (b) $\begin{pmatrix} 2 & -1 \\ 2 & 0 \end{pmatrix}$

    (c) $\begin{pmatrix} -2 & -1 \\ 2 & 0 \end{pmatrix}$

    (d) $\begin{pmatrix} -2 & 1 \\ -2 & 0 \end{pmatrix}$

    (e) $\begin{pmatrix} -2 & -1 \\ -2 & 0 \end{pmatrix}$

34. In the previous problem, for both the linearized system and the non-linear system, the critical point is a

    Select the correct answer.

    (a) unstable node
    (b) stable node
    (c) saddle point
    (d) unstable spiral point
    (e) stable spiral point

35. The Fourier series of an even function can contain

    Select all that apply.

    (a) sine terms
    (b) cosine terms
    (c) a constant term
    (d) more than one of the above
    (e) none of the above

36. The solutions of a regular Sturm-Liouville problem ($(ry')' + (\lambda p + q)y = 0$, $y(a) = 0$, $y(b) = 0$) have which of the following properties?

    Select the correct answer.

    (a) There exists an infinite number of real eigenvalues.
    (b) The eigenvalues are orthogonal on $[a, b]$.
    (c) For each eigenvalue, there is only one eigenfunction (except for non-zero constant multiples).
    (d) Eigenfunctions corresponding to different eigenvalues are linearly independent.
    (e) The set of eigenfunctions corresponding to the set of eigenvalues is orthogonal with respect to the weight function $r(x)$ on the interval $[a, b]$.

37. The solution of the eigenvalue problem $y'' + \lambda y = 0$, $y(0) = 0$, $y(2) = 0$ is

    Select the correct answer.

    (a) $\lambda = n\pi/2$, $y = \cos(n\pi x/2)$, $n = 1, 2, 3, \ldots$
    (b) $\lambda = (n\pi/2)^2$, $y = \cos(n\pi x/2)$, $n = 1, 2, 3, \ldots$
    (c) $\lambda^2 = n\pi/2$, $y = \sin(n\pi x/2)$, $n = 1, 2, 3, \ldots$
    (d) $\lambda = n\pi/2$, $y = \sin(n\pi x/2)$, $n = 1, 2, 3, \ldots$
    (e) $\lambda = (n\pi/2)^2$, $y = \sin(n\pi x/2)$, $n = 1, 2, 3, \ldots$

38. Consider Laplace's equation on a rectangle, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ with boundary conditions $u_x(0, y) = 0$, $u_x(1, y) = 0$, $u(x, 0) = 0$, $u(x, 2) = f(x)$. When the variables are separated using $u(x, y) = X(x)Y(y)$, the resulting problems for X and Y are

    Select the correct answer.

    (a) $X'' + \lambda X = 0$, $X'(0) = 0$, $X'(1) = 0$, $Y'' - \lambda Y = 0$, $Y(0) = 0$
    (b) $X'' + \lambda X = 0$, $X'(0) = 0$, $X'(1) = 0$, $Y'' + \lambda Y = 0$, $Y(0) = 0$
    (c) $X'' + \lambda X = 0$, $X'(0) = 0$, $X'(1) = 0$, $Y'' - \lambda Y = 0$, $Y(2) = 0$
    (d) $X'' + \lambda X = 0$, $X'(0) = 0$, $Y'' + \lambda Y = 0$, $Y(0) = 0$, $Y(2) = 0$
    (e) $X'' + \lambda X = 0$, $X'(0) = 0$, $Y'' - \lambda Y = 0$, $Y(0) = 0$, $Y(2) = 0$

39. The solutions of the eigenvalue problem and the other problem from the previous problem are

    Select the correct answer.

    (a) $\lambda = n\pi$, $X = \cos(n\pi x)$, $Y = \sinh(n\pi y)$, $n = 1, 2, 3, \ldots$
    (b) $\lambda = n\pi$, $X = \sin(n\pi x)$, $Y = \sinh(n\pi y)$, $n = 1, 2, 3, \ldots$
    (c) $\lambda = n^2\pi^2$, $X = \cos(n\pi x)$, $Y = \sinh(n\pi y)$, $n = 1, 2, 3, \ldots$
    (d) $\lambda = n^2\pi^2$, $X = \sin(n\pi x)$, $Y = \sinh(n\pi y)$, $n = 1, 2, 3, \ldots$
    (e) $\lambda = n^2\pi^2$, $X = \cos(n\pi x)$, $Y = \sinh(n\pi y)$, $n = 0, 1, 2, \ldots$, ($Y = y$ if $n = 0$)

40. In the previous two problems, the solution for $u(x, y)$ is

Select the correct answer.

(a) $u = \sum_{n=1}^{\infty} c_n \cos(n\pi x)\sinh(n\pi y)$, where $c_n = \int_0^2 f(x)\cos(n\pi x)dx/\sinh(2n\pi)$
(b) $u = \sum_{n=1}^{\infty} c_n \sin(n\pi x)\sinh(n\pi y)$, where $c_n = \int_0^2 f(x)\cos(n\pi x)dx/\sinh(2n\pi)$
(c) $u = c_0 y + \sum_{n=1}^{\infty} c_n \cos(n\pi x)\sinh(n\pi y)$, where $c_0 = \int_0^2 f(x)dx/4$ and $c_n = \int_0^2 f(x)\cos(n\pi x)dx/\sinh(2n\pi)$
(d) $u = \sum_{n=1}^{\infty} c_n \cos(n\pi x)\cosh(n\pi y)$, where $c_n = \int_0^2 f(x)\cos(n\pi x)dx/\sinh(2n\pi)$
(e) $u = \sum_{n=1}^{\infty} c_n \sin(n\pi x)\cosh(n\pi y)$, where $c_n = \int_0^2 f(x)\cos(n\pi x)dx/\sinh(2n\pi)$

41. Consider the problem $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$, with boundary conditions $u(r, 0) = 0$, $u(r, \pi) = 0$, $u(1, \theta) = f(\theta)$. Separate variables using $u(r, \theta) = R(r)\Theta(\theta)$. The resulting problems for R and Θ are

Select the correct answer.

(a) $r^2R'' + rR' + \lambda R = 0$, $R(0) = 0$, $\Theta'' + \lambda\Theta = 0$, $\Theta(0) = 0$, $\Theta(\pi) = 0$
(b) $r^2R'' + rR' + \lambda R = 0$, $\Theta'' + \lambda\Theta = 0$, $\Theta(0) = 0$, $\Theta(\pi) = 0$
(c) $r^2R'' + rR' - \lambda R = 0$, $R(0) = 0$, $\Theta'' + \lambda\Theta = 0$, $\Theta(0) = 0$, $\Theta(\pi) = 0$
(d) $r^2R'' + rR' - \lambda R = 0$, $R(0)$ is bounded, $\Theta'' + \lambda\Theta = 0$, $\Theta(0) = 0$, $\Theta(\pi) = 0$
(e) $r^2R'' + rR' - \lambda R = 0$, $\Theta'' + \lambda\Theta = 0$, $\Theta(0) = 0$, $\Theta(\pi) = 0$

42. The solutions for λ, R and Θ from the previous problem are

Select the correct answer.

(a) $\lambda = n^2$, $R = r^n$, $\Theta = \cos(n\theta)$, $n = 1, 2, 3, \ldots$
(b) $\lambda = n^2$, $R = r^n$, $\Theta = \sin(n\theta)$, $n = 1, 2, 3, \ldots$
(c) $\lambda = n^2$, $R = r^n$, $\Theta = \sin(n\theta)$, $n = 0, 1, 2, \ldots$
(d) $\lambda = n$, $R = r^n$, $\Theta = \sin(n\theta)$, $n = 1, 2, 3, \ldots$
(e) $\lambda = n$, $R = r^n$, $\Theta = \cos(n\theta)$, $n = 1, 2, 3, \ldots$

43. In the previous two problems, the infinite series solution for $u(r, \theta)$ is $u = \sum_{n=1}^{\infty} c_n r^n \Theta_n(\theta)$, where Θ_n is found in the previous problem, and

Select the correct answer.

(a) $c_n = 2\int_0^{\pi} f(\theta)\sin(n\theta)d\theta/\pi$
(b) $c_n = 2\int_0^{\pi} f(\theta)\cos(n\theta)d\theta/\pi$
(c) $c_n = \int_0^{\pi} f(\theta)\cos(n\theta)d\theta/\pi$
(d) $c_n = \int_0^{\pi} f(\theta)\sin(n\theta)d\theta/\pi$
(e) $c_n = \int_0^{\pi} f(\theta)\sin(n\theta)d\theta/(2\pi)$

44. Consider the heat problem $k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, $0 < x < \infty$, $t > 0$, $u(x,0) = 0$, $u(0,t) = u_0$. Apply a Fourier sine transform. The resulting problem for $U(\alpha, t) = \mathcal{F}_s\{u(x,t)\}$ is

Select the correct answer.

(a) $U_t = -k\alpha U + k\alpha u_0$, $U(\alpha, 0) = 0$
(b) $U_t = -k\alpha^2 U - k\alpha u_0$, $U(\alpha, 0) = 0$
(c) $U_t = -k\alpha^2 U + k\alpha u_0$, $U(\alpha, 0) = 0$
(d) $U_t = k\alpha^2 U + k\alpha u_0$, $U(\alpha, 0) = 0$
(e) $U_t = k\alpha^2 U - k\alpha u_0$, $U(\alpha, 0) = 0$

45. In the previous problem, the solution for $U(\alpha, t)$ is

Select the correct answer.

(a) $U = u_0(1 + e^{-k\alpha^2 t})/\alpha$
(b) $U = u_0(1 - e^{-k\alpha^2 t})/\alpha$
(c) $U = u_0(1 - e^{k\alpha^2 t})/\alpha$
(d) $U = u_0(1 - e^{k\alpha^2 t})$
(e) $U = u_0(1 + e^{-k\alpha^2 t})$

46. In the previous two problems, the solution for $u(x,t)$ is

Select the correct answer.

(a) $u = 2u_0 \int_0^\infty [(1 - e^{-k\alpha^2 t}) \sin(\alpha x)/\alpha] d\alpha/\pi$
(b) $u = 2u_0 \int_0^\infty [(1 - e^{-k\alpha^2 t}) \sin(\alpha x)/\alpha] d\alpha$
(c) $u = u_0 \int_0^\infty [(1 - e^{-k\alpha^2 t}) \sin(\alpha x)/\alpha] d\alpha$
(d) $u = u_0 \int_0^\infty [(1 - e^{-k\alpha^2 t}) \sin(\alpha x)/\alpha] d\alpha/\pi$
(e) $u = u_0 \int_0^\infty [(1 - e^{-k\alpha^2 t}) \sin(\alpha x)/\alpha] d\alpha/(2\pi)$

47. Consider the problem $c\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, $u(0,t) = 0$, $u(1,t) = 3$, $u(x,0) = 3x^2$. Replace $\frac{\partial^2 u}{\partial x^2}$ with a central difference approximation with $h = 1/3$ and $\frac{\partial u}{\partial t}$ with a forward difference approximation with $k = 1/2$. The resulting equation is

Select the correct answer.

(a) $c[u(x+h,t) + 2u(x,t) + u(x-h,t)]/h^2 = (u(x,t+k) - u(x,t))/k$
(b) $c[u(x+h,t) + 2u(x,t) + u(x-h,t)]/h^2 = (u(x,t+k) + u(x,t))/k$
(c) $c[u(x+h,t) - 2u(x,t) + u(x-h,t)]/h^2 = (u(x,t+k) - u(x,t))/k$
(d) $c[u(x+h,t) - 4u(x,t) + u(x-h,t)]/h^2 = (u(x,t+k) + u(x,t))/k$
(e) $c[u(x+h,t) - 4u(x,t) + u(x-h,t)]/h^2 = (u(x,t+k) - u(x,t))/k$

48. In the previous problem, using the notation $u_{ij} = u(x,t)$, and letting $c = 1$, $\lambda = ck/h^2$, the equation becomes

Select the correct answer.

(a) $u_{i,j-1} = \lambda u_{i+1,j} + (1 + 2\lambda)u_{i,j} + \lambda u_{i-1,j}$
(b) $u_{i,j-1} = \lambda u_{i+1,j} + (1 - 2\lambda)u_{i,j} + \lambda u_{i-1,j}$
(c) $u_{i,j+1} = \lambda u_{i+1,j} + (1 + 2\lambda)u_{i,j} + \lambda u_{i-1,j}$
(d) $u_{i,j+1} = \lambda u_{i+1,j} + (1 - 2\lambda)u_{i,j} + \lambda u_{i-1,j}$
(e) $u_{i,j+1} = \lambda u_{i+1,j} + (1 - \lambda)u_{i,j} + \lambda u_{i-1,j}$

49. In the previous two problems, the solution for u along the line $t = 0.5$ at the mesh points is

Select all that apply.

(a) $u_{11} = 10/3$
(b) $u_{11} = 20/9$
(c) $u_{11} = 20/3$
(d) $u_{21} = 32/3$
(e) $u_{21} = 13/3$

50. Is the value of λ in the previous problem such that the scheme is stable?

Select the correct answer.

(a) yes
(b) no
(c) It is right on the borderline.
(d) It cannot be determined from the available data.

ANSWER KEY

Zill Differential Equations 9e Final Exam 2 Form B

1. c
2. e
3. b
4. a
5. d
6. c
7. c
8. a
9. e
10. b
11. c
12. d
13. b
14. d
15. e
16. e
17. c
18. a
19. d
20. c
21. d
22. a, b
23. c
24. a
25. b
26. e
27. a
28. b
29. b

30. c
31. b
32. d
33. c
34. e
35. b, c, d
36. a, c, d
37. e
38. a
39. e
40. c
41. d
42. b
43. a
44. c
45. b
46. a
47. c
48. d
49. a, e
50. b

1. The solution of $y' - y = x$ is

   Select the correct answer.

   (a) $y = -x - 1 + ce^{-x}$
   (b) $y = x - 1 + ce^{-x}$
   (c) $y = x - 1 + ce^{x}$
   (d) $y = -x + 1 + ce^{x}$
   (e) $y = -x - 1 + ce^{x}$

2. The solution of $y \sec^2 x dx + \tan x dy = 0$ is

   Select the correct answer.

   (a) $y \tan x$
   (b) $y \cot x$
   (c) $y \tan x = c$
   (d) $y \cot x = c$
   (e) $y \cot x - c$

3. Use Euler's method to solve the initial value problem $y' = 1 + y^2, y(0) = 0$, with a step size of $h = 0.1$. Find approximations of $y(0.1)$ and $y(0.2)$.

4. The half life of a C-14 is 5600 years. A charcoal sample is discovered with 10% of its original C-14 remaining. How old is the sample?

5. Consider the logistic equation for a population of a city that is at 5,000, and that has a limiting size of 10,000. The intrinsic growth rate of the city is 2%. Write down the mathematical model for the population, $P(t)$, as a function of t.

6. In the previous problem, what is the population as a function of t?

7. The solution of $y'' - 6y' + 9y = 0$ is

   Select the correct answer.

   (a) $y = c_1 e^{3x} + c_2 x e^{3x}$
   (b) $y = c_1 e^{-6x} + c_2 e^{-3x}$
   (c) $y = c_1 e^{(7+\sqrt{13})x/2} + c_2 e^{(7-\sqrt{13})x/2}$
   (d) $y = c_1 e^{7x/2} \cos(\sqrt{13}x/2) + c_2 e^{7x/2} \sin(\sqrt{13}x/2)$
   (e) $y = c_1 e^{-3x} + c_2 x e^{-3x}$

8. The solution of $y'' - 4y' + 13y = 0$ is

   Select the correct answer.

   (a) $y = c_1e^{-2x}\cos(3x) + c_2e^{2x}\sin(3x)$
   (b) $y = c_1e^{-2x}\cos(3x) + c_2e^{-2x}\sin(3x)$
   (c) $y = c_1e^{2x}\cos(3x) + c_2e^{2x}\sin(3x)$
   (d) $y = c_1e^{2x} + c_2e^{2x}$
   (e) $y = c_1\cos(3x) + c_2\sin(3x)$

9. Solve the differential equation $y'' - 4y' + 13y = \cos(2x)$.

10. Solve the differential equation $y'' + 4y' + 3y = x + e^{-x}$.

11. Solve the differential equation $y'' + y = \tan x$.

12. A 1-kilogram mass is hung on a spring with a spring constant of $4N/m$. The mass spring system is then put into motion in a medium offering a damping force numerically equal to four times the velocity. If the mass is pulled down 10 centimeters from equilibrium and released, and a forcing function equal to $2e^{-3t}$ is applied to the system, the initial value problem describing the position, $x(t)$, of the mass at time t is

    Select the correct answer.

    (a) $x'' - 4x' + 4x = 2e^{-3t}$, $x(0) = .1$, $x'(0) = 0$
    (b) $x'' + 4x' + 4x = 2e^{-3t}$, $x(0) = .1$, $x'(0) = 0$
    (c) $x'' - 4x' + 4x = 2e^{-3t}$, $x(0) = 10$, $x'(0) = 0$
    (d) $x'' + 4x' + 4x = 2e^{-3t}$, $x(0) = 10$, $x'(0) = 0$
    (e) $x'' + 4x = 4 + 2e^{-3t}$, $x(0) = 10$, $x'(0) = 0$

13. In the previous problem, the solution for the position, $x(t)$, is

    Select the correct answer.

    (a) $x = 2.2e^{-2t} - 1.9te^{-2t} + 2e^{-3t}$
    (b) $x = 2.2e^{-2t} + 1.9te^{-2t} + 2e^{-3t}$
    (c) $x = 2.2e^{-2t} - 1.9te^{-2t} - 2e^{-3t}$
    (d) $x = 1.9e^{-2t} + 2.2te^{-2t} + 2e^{-3t}$
    (e) $x = -1.9e^{-2t} + 2.2te^{-2t} + 2e^{-3t}$

14. Use power series methods to find the solution of $xy'' - xy' + y = 0$ about $x = 0$.

15. Use power series methods to find the solution of $xy'' + 2y' - xy = 0$ about $x = 0$.

16. Use power series methods to find the solution of $x^2y'' + xy' + (x^2 - 1/9)y = 0$ about $x = 0$.

17. In the previous problem, identify the equation and write down the solution using the special function notation.

18. $\mathcal{L}\{t^2 \cos t\} =$

    Select the correct answer.

    (a) $(2s^3 - 2s)/(s^2+1)^3$
    (b) $(6s - 2s^3)/(s^2+1)^2$
    (c) $(2s^3 - 6s)/(s^2+1)^2$
    (d) $(2s^3 - 6s)/(s^2+1)^3$
    (e) $(6s - 2s^3)/(s^2+1)^3$

19. $\mathcal{L}^{-1}\{(s+1)/(s^2+4s+29)\} =$

    Select the correct answer.

    (a) $\cos(5t)e^{-2t} - 2\sin(5t)e^{-2t}/5$
    (b) $\cos(5t)e^{-2t} + \sin(5t)e^{-2t}$
    (c) $\cos(5t)e^{-2t} - \sin(5t)e^{-2t}$
    (d) $\cos(5t)e^{-2t} - \sin(5t)e^{-2t}/5$
    (e) $\cos(5t)e^{-2t} + \sin(5t)e^{-2t}/5$

20. Using the convolution theorem, we find that $\mathcal{L}^{-1}\{1/((s+1)(s^2+4))\} =$

    Select the correct answer.

    (a) $(e^{-t} - \sin(2t) - \cos(2t))/10$
    (b) $(2e^{-t} - \sin(2t) - \cos(2t))/10$
    (c) $(e^{-t} - \sin(2t) - 2\cos(2t))/10$
    (d) $(2e^{-t} + \sin(2t) + 2\cos(2t))/10$
    (e) $(2e^{-t} + \sin(2t) - 2\cos(2t))/10$

21. Using Laplace transform methods, the solution of $y' + y = 2\sin(2t)$, $y(0) = 0$ is (Hint: the previous problem might be useful.)

    Select the correct answer.

    (a) $y = 2(2e^{-t} + \sin(2t) + 2\cos(2t))/5$
    (b) $y = 2(2e^{-t} + \sin(2t) - 2\cos(2t))/5$
    (c) $y = 2(e^{-t} - \sin(2t) - \cos(2t))/5$
    (d) $y = 2(2e^{-t} - \sin(2t) - \cos(2t))/5$
    (e) $y = 2(e^{-t} + \sin(2t) - 2\cos(2t))/5$

22. Solve the initial value problem $y'' - y = t\,\mathcal{U}(t-3)$, $y(0) = 1$, $y'(0) = 0$ using Laplace transform methods.

23. Solve the system $\mathbf{X}' = A\mathbf{X}$ where $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$.

24. Find a particular solution of the system $\mathbf{X}' = A\mathbf{X} + F(t)$ where $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ and $F(t) = \begin{pmatrix} e^t \\ 0 \end{pmatrix}$.

25. Solve the system $\mathbf{X}' = A\mathbf{X}$ where $A = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$.

26. Solve the system $\mathbf{X}' = A\mathbf{X}$ where $A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

27. Using the improved Euler method with a step size of $h = 0.1$, the solution of $y' = 1+y^2$, $y(0) = 1$ at $x = 0.1$ is

    Select the correct answer.

    (a) $y_1 = 1.211$
    (b) $y_1 = 1.2$
    (c) $y_1 = 1.221$
    (d) $y_1 = 1.222$
    (e) $y_1 = 1.121$

28. Using the classical Runge–Kutta method of order 4 with a step size of $h = 0.1$, the solution of $y' = 1 + y^2$, $y(0) = 1$ at $x = 0.1$ is

    Select the correct answer.

    (a) 1.228
    (b) 1.231
    (c) 1.023
    (d) 1.218
    (e) 1.223

29. What is the expected order of the local truncation error in the improved Euler method and in the classical Runge–Kutta method?

30. What is the expected order of the global truncation error in the improved Euler method and in the classical Runge–Kutta method?

31. Consider a simple non-linear pendulum with displacement angle θ. The differential equation model for this system is

    Select the correct answer.

    (a) $\frac{d^2\theta}{dt^2} + g\sin\theta/l = 0$
    (b) $\frac{d^2\theta}{dt^2} = g\sin\theta/l$
    (c) $\frac{d^2\theta}{dt^2} + g\sin\theta/l$
    (d) $\frac{d\theta}{dt} = g\sin\theta/l$
    (e) $\frac{d\theta}{dt} + g\sin\theta/l = 0$

32. With the substitution $x = \theta$, $y = \theta'$, the equation in the previous problem can be rewritten as

    Select the correct answer.

    (a) $x' = y$, $y' = x$
    (b) $x' = y$, $y' = g\sin x/l$
    (c) $x' = y$, $y' = -g\sin x/l$
    (d) $x' = y$, $y' = -x$
    (e) $x' = y$, $y' = -gx/l$

33. In the previous problem, the critical point at the origin is a

    Select the correct answer.

    (a) stable spiral point
    (b) unstable spiral point
    (c) stable node
    (d) unstable node
    (e) stable center point
    (f) saddle point

34. In the previous two problems, the critical point at $(\pi, 0)$ is a

    Select the correct answer.

    (a) stable spiral point
    (b) unstable spiral point
    (c) stable node
    (d) unstable node
    (e) stable center point
    (f) saddle point

35. Consider the parametric Bessel problem $x^2y'' + xy' + (\alpha^2x^2 - 4)y = 0$, $y(0) = 0$, $y(1) = 0$. Write the differential equation in self-adjoint form and identify the functions r, p, q, and the parameter λ in the standard Sturm-Liouville differential equation.

36. In the previous problem, what is the solution of the differential equation?

37. In the previous two problems, what are the eigenvalues and eigenfunctions?

38. For a function $f(x)$ defined on $[0, 1]$, write the Fourier-Bessel series using the solutions of the previous three problems.

39. Consider the wave equation problem $a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$, $u(0,t) = 0$, $u(1,t) = 0$, $u(x,0) = 0$, $u_t(x,0) = f(x)$. Separate variables using $u(x,t) = X(x)T(t)$. The resulting problems for X and T are

    Select the correct answer.

    (a) $X'' + \lambda X = 0,\ X(0) = 0,\ X(1) = 0,\ T'' - \lambda T = 0,\ T(0) = 0$
    (b) $X'' + \lambda X = 0,\ X(0) = 0,\ X(1) = 0,\ T'' + \lambda T = 0,\ T(0) = 0$
    (c) $X'' + \lambda X = 0,\ X(0) = 0,\ X(1) = 0,\ T'' - a^2\lambda T = 0,\ T(0) = 0$
    (d) $X'' + \lambda X = 0,\ X(0) = 0,\ X(1) = 0,\ T'' + a^2\lambda T = 0,\ T(0) = 0$
    (e) $X'' + \lambda X = 0,\ X(0) = 0,\ X(1) = 0,\ T'' + \lambda T/a^2 = 0,\ T(0) = 0$

40. The solutions of the eigenvalue problem and the other problem of the previous problem are

    Select the correct answer.

    (a) $\lambda = n\pi,\ X = \sin(n\pi x),\ T = \sin(an\pi t),\ n = 1, 2, 3, \ldots$
    (b) $\lambda = n\pi,\ X = \cos(n\pi x),\ T = \sin(an\pi t),\ n = 1, 2, 3, \ldots$
    (c) $\lambda = n^2\pi^2,\ X = \sin(n\pi x),\ T = \sin(an\pi t),\ n = 1, 2, 3, \ldots$
    (d) $\lambda = n^2\pi^2,\ X = \cos(n\pi x),\ T = \cos(an\pi t),\ n = 1, 2, 3, \ldots$
    (e) $\lambda = n^2\pi^2,\ X = \sin(n\pi x),\ T = \cos(an\pi t),\ n = 1, 2, 3, \ldots$

41. In the previous two problems, the solution for $u(x,t)$ is $u = \sum_{n=1}^{\infty} c_n X_n(x)T_n(t)$, where X_n and T_n are the solutions from the previous problem, and

    Select the correct answer.

    (a) $c_n = 2\int_0^1 f(x)\cos(n\pi x)dx/(an\pi)$
    (b) $c_n = 2\int_0^1 f(x)\sin(n\pi x)dx/(an\pi)$
    (c) $c_n = 2\int_0^1 f(x)\sin(n\pi x)dx/(n\pi)$
    (d) $c_n = 2\int_0^1 f(x)\sin(n\pi x)dx$
    (e) $c_n = 2\int_0^1 f(x)\cos(n\pi x)dx$

42. Consider the displacement $u(r,t)$ of a circular membrane of radius 1 which is clamped along the boundary. Its initial displacement is $f(r)$, and its initial velocity is zero. Write down the initial/boundary value problem for this membrane.

43. Separate variables in the previous problem using $u(r,t) = R(r)T(t)$. Find the resulting problems for R and T.

44. In the previous problem, find the solutions for R and T.

45. In the previous three problems, what is the infinite series solution for $u(r,t)$?

46. Write down Laplace's equation for $u(r,\phi,\theta)$ in spherical coordinates.

47. In the previous problem, write down the resulting equations if a separation of variables method is used with $u(r,\theta)$ being independent of ϕ.

48. Consider the wave equation problem $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$, $u(0,t) = 0$, $u(1,t) = 0$, $u(x,0) = 0$, $u_t(x,0) = \sin(\pi x)$. Apply a Laplace transform in t. Write down the resulting problem for $U(x,s) = \mathcal{L}\{u(x,t)\}$.

49. In the previous problem, what is the solution for $U(x,s)$?

50. In the previous two problems, what is the solution for $u(x,t)$?

1. e
2. c
3. $y_1 = 0.1,\ y_2 = 0.201$
4. $t = \ln(0.1)/(-0.000124) = 18570$ years
5. $\frac{dP}{dt} = 0.000002P(10000 - P),\ P(0) = 5000$
6. $P = 10000/(1 + e^{-0.02t})$
7. a
8. c
9. $y = c_1 e^{2x}\cos(3x) + c_2 e^{2x}\sin(3x) + (9\cos(2x) - 8\sin(2x))/145$
10. $y = c_1 e^{-x} + c_2 e^{-3x} + x/3 - 4/9 + xe^{-x}/2$
11. $y = c_1 \cos x + c_2 \sin x - \cos x \ln|\sec x + \tan x|$
12. b
13. e
14. $y = c_1 x + c_2(x \ln x - 1 + \sum_{n=2}^{\infty} x^n/(n!(n-1))$
15. $y = c_0 \sum_{n=0}^{\infty} x^{2n-1}/(2n)! + c_1 \sum_{n=0}^{\infty} x^{2n}/(2n+1)!$
16. $y = c_0 \sum_{n=0}^{\infty}(-1)^n (x/2)^{2n+1/3}/(n!(\Gamma(n+4/3)) + c_1 \sum_{n=0}^{\infty}(-1)^n (x/2)^{2n-1/3}/(n!(\Gamma(n+2/3))$
17. Bessel's equation of order 1/3, $y = c_1 J_{-1/3}(x) + c_2 J_{1/3}(x)$
18. d
19. d
20. e
21. b
22. $y = (-t + 2e^{t-3} + e^{3-t})\,\mathcal{U}(t-3) + \cosh t$
23. $\mathbf{X} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$
24. $\mathbf{X}_p = \begin{pmatrix} te^t \\ 0 \end{pmatrix}$
25. $\mathbf{X} = c_1 e^{2t}\left[\begin{pmatrix} 2 \\ -1 \end{pmatrix}\cos t - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin t\right] + c_2 e^{2t}\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos t + \begin{pmatrix} 2 \\ -1 \end{pmatrix}\sin t\right]$

26. $\mathbf{X} = c_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{3t} + c_3 \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} te^{3t} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{3t} \right]$

27. d

28. e

29. $O(h^3)$, $O(h^5)$

30. $O(h^2)$, $O(h^4)$

31. a

32. c

33. e

34. f

35. $(xy')' + (\alpha^2 x - 4/x)y = 0$, $r(x) = x$, $p(x) = x$, $q(x) = -4/x$, $\lambda = \alpha^2$

36. $y = c_1 J_2(\alpha x) + c_2 Y_2(\alpha x)$

37. $\lambda = (\alpha_n)^2$, where $J_2(\alpha_n) = 0$, $y = J_2(\alpha_n x)$

38. $f(x) = \sum_{n=1}^{\infty} c_n J_2(\alpha_n x)$, where $c_n = \int_0^1 x J_2(\alpha_n x) f(x) dx / \int_0^1 x {J_2}^2(\alpha_n x) dx$

39. d

40. c

41. b

42. $a^2(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}) = \frac{\partial^2 u}{\partial t^2}$, $u(1,t) = 0$, $u(r,0) = f(r)$, $u_t(r,0) = 0$

43. $rR'' + R' + \lambda r R = 0$, $R(0)$ is bounded, $R(1) = 0$, $T'' + a^2 \lambda T = 0$, $T'(0) = 0$

44. $\lambda = \alpha_n^2$, $R = J_0(\alpha_n r)$, $T = \cos(a\alpha_n t)$, $n = 1, 2, 3, \ldots$, where $J_0(\alpha_n) = 0$

45. $u = \sum_{n=1}^{\infty} c_n J_0(\alpha_n r) \cos(a\alpha_n t)$, where $c_n = \int_0^1 r J_0(\alpha_n r) f(r) dr / \int_0^1 r {J_0}^2(\alpha_n r) dr$

46. $\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} = 0$

47. $r^2 R'' + 2rR' - \lambda R = 0$, $\sin\theta\Theta'' + \cos\theta\Theta' + \lambda\sin\theta\Theta = 0$

48. $U_{xx} - s^2 U = -\sin(\pi x)$, $U(0,s) = 0$, $U(1,s) = 0$

49. $U = \sin(\pi x)/(s^2 + \pi^2)$

50. $u = \sin(\pi x)\sin(\pi t)/\pi$