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Take home Final

Exercise 5.2

3] Nonhomogeneous Fredholm equation is given by,

$$u(x) = \cos 2x + 2 \int_0^{\pi/2} k(x,t)\, u(t)\, dt$$

with kernel

$$k(x,t) = \begin{cases} \sin x \cos t, & 0 \leqslant x \leqslant t \\ \sin t \cos x, & t \leqslant x \leqslant \pi/2 \end{cases}$$

a)

For the kernel to be symmetric,

$$k(x,t) = k(t,x)$$

Here,

$$k(x,t) = \begin{cases} \sin x \cos t \\ \sin t \cos x \end{cases}$$

$$k(t,x) = \begin{cases} \sin t \cos x \\ \cos x \sin t \end{cases}$$

As, $k(x,t) = k(t,x)$, the given kernel is symmetric.

Now,

$$\int_0^{\pi/2}\int_0^{\pi/2} k^2(x,t)\,dx\,dt$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} \sin^2 x \cos^2 t \cdot dx\,dt$$

$$= \int_0^{\pi/2} \sin^2 x\,dx \int_0^{\pi/2} \cos^2 t \cdot dt$$

$$= \frac{\pi^2}{16} = B^2 < \infty$$

As,

$$\left[\int_a^b\int_a^b k^2(x,t)\,dx\,dt\right] < \infty,$$ the kernel is square integrable.

b) Homogeneous equation is given as,

$$u(x) = \lambda \int_0^{\pi/2} k(x,t)\,u(t)\,dt.$$

$u(x)$ can be written as,

$$u(x) = \lambda \int_0^{x} \sin x \cdot \cos t \cdot u(t) \cdot dt$$

$$+ \lambda \int_x^{\pi/2} \cos x \,\sin t \cdot u(t)\,dt.$$

②

$$= \lambda \sin x \int_0^x \cos(t) \cdot u(t)\, dt + \lambda \cos x \int_x^{\pi/2} \sin(t) \cdot u(t)\, dt$$

Differentiating above eqⁿ -

$$u'(x) = \lambda \cos x \int_0^x \cos(t) \cdot u(t) \cdot dt + \lambda \sin x \cos x \cdot u(x)$$

$$- \lambda \sin x \int_x^{\pi/2} \sin(t) \cdot u(t)\, dt \quad + \lambda \cos x \cdot \sin x\, u(x)$$

$$= \lambda \cos x \int_0^x \cos(t) \cdot u(t)\, dt - \lambda \sin x \int_0^{\pi/2} \sin(t) \cdot u(t)\, dt$$

Differentiating again -

$$u''(x) = -\lambda \sin x \int_0^x \cos(t) \cdot u(t)\, dt - \lambda \cos x \cdot \cos x \cdot u(x)$$

$$- \lambda \cos x \int_x^{\pi/2} \sin(t) \cdot u(t) \cdot dt - \lambda \sin x \cdot \sin x \cdot u(x)$$

$$= -\left[\lambda \int_0^x \sin x \cdot \cos t \cdot u(t)\, dt + \lambda \int_x^{\pi/2} \cos x \cdot \sin t \cdot u(t)\, dt\right]$$

$$- \lambda\, (u(x))$$

$$= -u(x) - \lambda\, u(x)$$

$$\therefore\ \underline{u''(x) + (1+\lambda)\, u(x) = 0}$$

$u(0) = 0$ & $u\left(\frac{\pi}{2}\right) = 0$ are the boundary conditions for above differential equations.

c) Now, we have to solve nonhomogeneous equation (a) using information in (b)

eqⁿ in (b) can be written in its eigen values & eigen ~~vect~~ functions as,

$$u(x) = c_1 \cos\sqrt{(\lambda_k+1)}\; x + c_2 \sin\sqrt{\lambda_k+1}\; x$$

$$u(0) = c_1 + 0 = 0 \quad \therefore\; c_1 = 0$$

$$u\left(\frac{\pi}{2}\right) = c_2 \sin\left(\frac{\pi}{2}\sqrt{\lambda_k+1}\right) = 0$$

Thus,
$$\frac{\pi}{2}\sqrt{\lambda_k+1} = k\pi$$

$$\therefore\; \sqrt{\lambda_k+1} = 2k$$

$$\therefore\; \lambda_k = 4k^2-1$$

Thus, solution to equation in (a) is written as,

$$u(x) = \cos 2x + \lambda \sum_{k=1}^{\infty} \frac{2a_k}{\sqrt{\pi}} \frac{\sin\sqrt{\lambda_k+1}\; x}{\lambda_k - \lambda}$$

where,

$$a_k = \int_0^{\pi/2} f(x) \cdot \phi_k(x) \cdot dx$$

$$\phi_k(x) = \frac{\|u_k\|^2}{\|u_k\|} = \frac{\int_0^{\pi/2} \sin^2 k\pi x \cdot dx}{\int_0^{\pi/2} \sin k\pi x \cdot dx}$$

$$= \frac{2}{\sqrt{\pi}} \sin k\pi x$$

$$\therefore a_k = \int_0^{\pi/2} \cos 2x \cdot \frac{2}{\sqrt{\pi}} \sin k\pi x \cdot dx$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\pi/2} \cos 2x \cdot \sin k\pi x \cdot dx$$

$$= \begin{cases} -\frac{2}{\sqrt{\pi}} \cdot \frac{k}{k^2-1} & , \quad k \text{ is even} \\ 0 & , \quad k \text{ is odd.} \end{cases}$$

d) Here, $\lambda = 2 \neq \lambda_k = 4k^2 - 1$, $k = 1, 2, 3 \ldots$

Thus, according to theorem 1, eq^n in (a) has a unique solution.

Exercise 5.3

7] Using collocation method -

$$u(x) = x + \int_{-1}^{1} xt\, u(t)\, dt.$$

$f(x) = x$, $K(x,t) = xt$

(i) When three linearly independant functions
$\phi_1(x) = 1$, $\phi_2(x) = \sin x$, $\phi_3(x) = \cos x$

are assumed.

$$u(x) = S_3(x) = C_1 + C_2 \sin x + C_3 \cos x$$

$$= \sum_{k=1}^{3} C_k \phi_k(x)$$

$$= x + \int_{-1}^{1} xt\left[C_1 + C_2 \sin t + C_3 \cos t\right] dt$$

$$+ \epsilon(x, c_1, c_2, c_3)$$

where. $\epsilon \rightarrow$ Error

$$\therefore u(x) = x + x\int_{-1}^{1}\left(C_1 t + C_2 t \sin t + C_3 t \cos t\right) dt + \epsilon$$

$$= x + x\left[-2C_2(\cos 1 - \sin 1)\right] + \epsilon$$

$$\therefore c_1 + c_2 \sin x + c_3 \cos x = x(1 + 0.6023 c_2) + \epsilon(x, c_1, c_2, c_3)$$

Lets assume that error vanishes at $x_1 = 1$, $x_2 = 0$ & $x_3 = 1$

Thus, above eq^n becomes,

$$c_1 + c_2 \left[\sin(1) - 0.6023\right] + c_3 \cos(1) = 1 \quad \text{--- (1)}$$

$$c_1 + c_2 \sin(0) + c_3 \cancel{\sin} \cos(0) = 0 \quad \text{--- (2)}$$

$$c_1 + c_2 \left(0.6023 - \sin(1)\right) + c_3 \cos(1) = -1 \quad \text{--- (3)}$$

Solving (1), (2) & (3) simultaniously to get c_1, c_2 & c_3

$$\begin{bmatrix} 1 & \sin(1) - 0.6023 & \cos(1) \\ 1 & \sin 0 & \cos(0) \\ 1 & 0.6023 - \sin(1) & \cos(1) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\therefore c_1 = 0, \quad c_2 = 4.1811, \quad c_3 = 0$$

$$\therefore \underline{S_3(x) = 4.1811 \sin x}$$

(ii) $\phi_1(x) = 1$ $\qquad$ $\phi_5(x) = \cos 2x$

$\phi_2(x) = \sin x$ $\qquad$ $\phi_6(x) = \sin 3x$

$\phi_3(x) = \cos x$ $\qquad$ $\phi_7(x) = \cos 3x$

$\phi_4(x) = \sin 2x$ $\qquad$ $\phi_8(x) = \sin 4x$

Thus,

$$S_8(x) = C_1 + C_2 \sin x + C_3 \cos x + C_4 \sin 2x + C_5 \cos 2x + C_6 \sin 3x + C_7 \cos 3x + C_8 \sin 4x$$

$$S_8(x) = \sum_{k=1}^{\infty} C_k \phi_k(x)$$

$$= x + \int_{-1}^{1} x \cdot t \cdot S(t) \cdot dt + \text{Error}$$

Now,

$$\int_{-1}^{1} t \cdot S(t) \cdot dt = \int_{-1}^{1} \begin{bmatrix} C_1 t + C_2 t \sin t + C_3 t \cos t + C_4 t \sin 2t \\ + C_5 t \cos 2t + C_6 t \sin 3t + C_7 t \cos 3t \\ C_8 t \sin 4t \end{bmatrix} dt$$

$$= -2C_2(\cos 1 - \sin 1) + \frac{1}{2} C_4(-2\cos 2 + \sin 2) + \frac{2}{9} C_6(-3\cos 3 + \sin 3) + \frac{1}{8} C_8(-4\cos 4 + \sin 4)$$

$$= 0.6023\, C_2 + 0.8708\, C_4 + 0.6914\, C_6 + 0.2322\, C_8$$

Lets assume that, error vanishes at

$x_1 = 1$

$x_2 = 0.75$

$x_3 = 0.50$

$x_4 = 0.25$

$x_5 = 0$

$x_6 = -0.25$

$x_7 = -0.50$

$x_8 = -1$

Substituting these values in ~~above~~ below eq^n, we get 8 simultanious equations

$$C_1 + C_2 \sin x + C_3 \cos x + C_4 \sin 2x + C_5 \cos 2x + C_6 \sin 3x + C_7 \cos 3x + C_8 \sin 4x$$
$$= x + x(0.6023\, C_2 + 0.8708\, C_4 + 0.6914\, C_6 + 0.2322\, C_8) + \text{Error}.$$

Solving simultanious eq^ns with the help of MATLAB,

$C_1 = 0$

$C_2 = 4.9284$

$C_3 = 0$

$C_4 = -1.3345$

$C_5 = 0$

$C_6 = +0.2906$

$C_7 = 0$

$C_8 = -0.0329$

Thus, solution is given as -

$$S_8(x) = 4.9284 \sin x - 1.3345 \sin 2x$$
$$+ 0.2906 \sin 3x - 0.0329 \sin 4x.$$

MATLAB code to solve set of homogeneous eq^{n}s is attached.

b) Table (1) gives comparison of approx. solutions & exact solution $u(x) = 3x$

Figure (1) is the plot of the above comparison.

Exercise 5.3

9) Use the Galerkin method to find the solution for the equation.

$$u(x) = x + \int_{-1}^{1} x t \, u(t) \, dt$$

(i) With three independant functions,

$\phi_1(x) = 1$ $\phi_2(x) = \sin x$ $\phi_3(x) = \cos x$

Thus,

$$S_3(x) = c_1 + c_2 \sin x + c_3 \cos x$$

Error is approximated as,

$$\epsilon(x, c_1, c_2, c_3) = c_1 + c_2 \sin x + c_3 \cos x - x - \int_{-1}^{1} x t \,(c_1 + c_2 \sin t + c_3 \cos t) \, dt$$

To find out c_1, c_2 & c_3 Galerkin method needs this error to be orthogonal to three independant functions, $\psi_1(x)$, $\psi_2(x)$ and $\psi_3(x)$ which we choose as 1, $\sin x$ and $\cos x$ resp.

Thus, we get three linear equations in c_1, c_2 & c_3 as follows,

$$\int_{-1}^{1} 1\left[c_1 + c_2\sin x + c_3\cos x - \int_{-1}^{1} xt\,(c_1 + c_2\sin \cancel{t} + c_3\cos \cancel{t})\,dt\right] dx$$

$$= \int_{-1}^{1} 1\,(x)\cdot dx \quad \text{----} \quad [f(x) = x]$$

$$\therefore \quad 2c_1 + 2c_3\sin(1) = 0 \quad \text{———} \ (1)$$

$$\int_{-1}^{1} \sin x\left[c_1 + c_2\sin x + c_3\cos x - \int_{-1}^{1} xt\,(c_1 + c_2\sin t + c_3\cos t)\,dt\right] dx$$

$$= \int_{-1}^{1} \sin x\,(x)\cdot dx$$

$$\therefore \quad 0.182563\,c_2 = -2\cos(1) + 2\sin(1) \quad (2)$$

$$\int_{-1}^{1} \cos x\left[c_1 + c_2\sin x + c_3\cos x - \int_{-1}^{1} xt\,(c_1 + c_2\sin t + c_3\cos t)\,dt\right] dx$$

$$= \int_{-1}^{1} \cos x\,(x)\,dx$$

$$\therefore \quad c_3 + 2c_1\sin(1) + c_3\cos(1)\sin(1) = 0 \quad (3)$$

Solving (1), (2) & (3),

$c_1 = 0$, $c_2 = 3.2997$, $c_3 = 0$

$$\therefore \quad \underline{S_3(x) = 3.2997\sin x}$$

9

(ii) With eight independant functions,

$\phi_1(x) = 1$ $\qquad$ $\phi_5(x) = \cos 2x$

$\phi_2(x) = \sin x$ $\qquad$ $\phi_6(x) = \sin 3x$

$\phi_3(x) = \cos x$ $\qquad$ $\phi_7(x) = \cos 3x$

$\phi_4(x) = \sin 2x$ $\qquad$ $\phi_8(x) = \sin 4x$

Thus,

$$S_8(x) = c_1 + c_2 \sin x + c_3 \cos x + c_4 \sin 2x + c_5 \cos 2x + c_6 \sin 3x + c_7 \cancel{\cos 3} \cos 3x + c_8 \sin 4x$$

Error is approximated by,

$$\epsilon(x, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8) = S_8(x) - x - \int_{-1}^{1} x \cdot t \, S_8(t) \cdot dt$$

For the error to be orthogonal we need eight independant functions which we choose as below,

$\psi_1(x) = 1$ $\qquad$ $\psi_5(x) = \cos 2x$

$\psi_2(x) = \sin x$ $\qquad$ $\psi_6(x) = \sin 3x$

$\psi_3(x) = \cos x$ $\qquad$ $\psi_7(x) = \cos 3x$

$\psi_4(x) = \sin 2x$ $\qquad$ $\psi_8(x) = \sin 4x$

This would lead to eight linear equations which would give us $c_1, c_2 \ldots\ldots c_8$.

$$2c_1 + 2c_3 \sin(1) + c_5 \sin(2) + \frac{2}{3} c_7 \sin(3) = 0 \quad \text{— (1)}$$

$$0.182563\, c_2 + 0.269916\, c_4 + 0.22739\, c_6 + 0.0989621\, c_8 = -2\cos(1) + 2\sin(1) \quad \text{— (2)}$$

$$c_3 + 2c_1 \sin(1) + c_5 \sin(1) + c_3 \cos(1)\sin(1) + 2c_7 \cos^3(1)\sin(1) + \frac{1}{3} c_5 \sin(3) = 0 \quad \text{— (3)}$$

$$0.269951\, c_2 + 0.430912\, c_4 + 0.431188\, c_6 + 0.2990\, c_8 = \frac{1}{2}\left(-2\cos(2) + \sin(2)\right) \quad \text{— (4)}$$

$$c_5 + c_3 \sin(1) + c_7 \sin(1) + c_1 \sin(2) + \frac{1}{3} c_3 \sin(3) + \frac{1}{4} c_5 \sin(4) + \frac{1}{5} c_7 \sin(5) = 0 \quad \text{— (5)}$$

$$0.227446\, c_2 + 0.43122\, c_4 + 0.5656\, c_6 + 0.587083\, c_8 = \frac{2}{9}\left(-3\cos(3) + \sin(3)\right) \quad \text{— (6)}$$

$$c_7 + c_5 \sin(1) + 2c_3 \cos^3(1)\sin(1) + \frac{2}{3} c_1 \sin(3) + \frac{1}{5} c_5 \sin(5) + \frac{1}{6} c_7 \sin(6) = 0 \quad \text{— (7)}$$

$$0.09895\, c_2 + 0.2988\, c_4 + 0.5870\, c_6 + 0.8224\, c_8$$

$$= \frac{1}{8}\left(-4\cos(4) + \sin(4)\right) \qquad (8)$$

These equations are solved simultaneously to get $c_1, c_2, c_3 \ldots c_8$

$\therefore$ $c_1 = 0$ $\qquad$ $c_5 = 0$

$c_2 = 4.3924$ $\qquad$ $c_6 = -0.0098$

$c_3 = 0$ $\qquad$ $c_7 = 0$

$c_4 = -0.7423$ $\qquad$ $c_8 = 0.0305$

Thus, solution is given as,

$$S_8(x) = 4.3924 \sin x - 0.7423 \sin 2x - 0.0098 \sin 3x + 0.0305 \sin 4x$$

b) Table (2) gives comparison of approx. solutions obtained in (i) & (ii) with exact solution, $u(x) = 3x$

figure (2) is the plot of above comparison.

c) Use least squares criterion on 7(a,i) and 9(a,i)

According to least square Method, integral of the square of the error should be minimum.

$$\int_a^b e^2(x, c_1, c_2 \cdots c_N)\, dx = \text{minimum}$$

Now, according to 7(a,i) & 9(a,i)

$$\int_{-1}^{1}\left\{[c_1 + c_2 \sin x + c_3 \cos x] - x - \int_{-1}^{1} xt\,(c_1 + c_2 \sin t + c_3 \cos t)\, dt\right\}^2 dx$$

$= \underline{0.012392}$

where, $c_1 = 0$, $c_2 = 3.2997$, $c_3 = 0$

d) Use least square criterion on 7(a,ii) and 9(a,ii)

$$\int_a^b e^2(x, c_1, c_2, \cdots c_N)$$

$$= \int_{-1}^{1}\Big\{[c_2 \sin x + c_4 \sin 2x + c_6 \sin 3x + c_8 \sin 4x] - x - x\,[0.6023\, c_2 + 0.8708\, c_4 + 0.0614\, c_6 + 0.2322\, c_8]\Big\}^2 dx$$

$= \underline{7.11412 \times 10^{-7}}$

Exercise 5.4

3] Fredholm integral equation with first kind is given as,

$$f(x) = \int_a^b k(x,t)\, u(t)\, dt$$

with degenerate kernel,

$$k(x,t) = \sum_{k=1}^{n} a_k(x)\, b_k(t)$$

f(x) cab be written as,

$$f(x) = \int_a^b \sum_{k=1}^{n} a_k(x)\, b_k(t)\, u(t)\, dt$$

$$= \sum_{k=1}^{n} a_k(x) \int_a^b b_k(t)\, u(t)\, dt$$

Now,

$$C_k = \int_a^b b_k(t)\, u(t)\, dt$$

$$\therefore \quad f(x) = \sum_{k=1}^{n} a_k(x)\, C_k$$

where C_k is a constant.

Thus, $f(x)$ is restricted to a linear combination of functions $a_k(x)$

b) $f(x) = \int_a^b k(x,t)\, u(t)\, dt$

kernel $k(x,t)$ & $f(x)$ are continuous. Here, $u(t)$ need not be necessarily continuous. Piecewise continuous function $u(t)$ can give continuous $f(x)$ as an output.

c) $f(x)$ & the solution $u(x)$ have each a fourier series expansion in terms of $\{\phi_n(x)\}$

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

$$u(x) = \sum c_n \lambda_n \phi_n(x)$$

For the solution to exist, $u(x)$ series must converge. With the assumption that λ_n increases with n, c_n should reduce which puts restrain on $f(x)$.

Exercise 5.4

4] Fredholm integral equation of the first kind is given by,

$$f(x) = \int_a^b k(x,t)\, u(t)\, dt$$

$k(x,t)$ is continuous, real & symmetric.
$k(x,t)$ has only finite number of eigenfunctions.

ie. $k(x,t) = \sum_{i=1}^{n} C_i(t)\phi_i(x)$

$$\therefore f(x) = \int_a^b \sum_{i=1}^{n} \phi_i(x) \cdot C_i(t) \cdot u(t) \cdot dt$$

$$= \sum_{i=1}^{n} \phi_i(x) \int_a^b C_i(t)\, u(t)\, dt$$

let, $b_i = \int_a^b C_i(t) \cdot u(t)\, dt$

$$\therefore f(x) = \sum_{i=1}^{n} b_i\, \phi_i(x)$$

Thus, $f(x)$ is restricted to linear combinations of eigenfunctions $\phi_i(x)$

Exercise 5.4

5] $$f(x) = \int_0^{2\pi} \sin(x+t)\, u(t)\, dt$$

a)

$$k(x,t) = \sin(x+t)$$
$$= \sin x \cos t + \cos x \sin t$$
$$= \sum_{k=1}^{n} a_k(x)\, b_k(t)$$

$a_1(x) = \sin x$ $\qquad$ $b_1(t) = \cos t$

$a_2(x) = \cos x$ $\qquad$ $b_2(t) = \sin t$

$$a_{11} = \int_0^{2\pi} b_1(t)\, a_1(t)\, dt = 0$$

$$a_{12} = \int_0^{2\pi} b_1(t)\, a_2(t)\, dt = \pi$$

$$a_{21} = \int_0^{2\pi} b_2(t)\, a_1(t)\, dt = \pi$$

$$a_{22} = \int_0^{2\pi} b_2(t)\, a_2(t)\, dt = 0$$

$$\therefore A = \begin{bmatrix} 0 & \pi \\ \pi & 0 \end{bmatrix}$$

For the fredholm eq^n of the first kind,

$$(I - \lambda A)\, C = 0$$

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 0 & \pi \\ \pi & 0 \end{bmatrix} \right\} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\pi\lambda \\ -\pi\lambda & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This will have a solution if,

$$\begin{vmatrix} 1 & -\lambda\pi \\ -\lambda\pi & 1 \end{vmatrix} = 0 \qquad \therefore\ 1 - \lambda^2 \pi^2 = 0$$

$\therefore\ \lambda_1 = \frac{1}{\pi}$ & $\lambda_2 = \frac{-1}{\pi}$ are the two eigen values.

To find eigenfunctions, put $f(x) = 0$

$$u(x) = \lambda \sum_{k=1}^{n} c_k a_k(x)$$

For $\lambda_1 = \frac{1}{\pi}$,

$$\begin{bmatrix} +1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$c_1 - c_2 = 0$

$-c_1 + c_2 = 0$

Thus, $c_1 = c_2$

$$\therefore \phi_1(x) = \frac{1}{\pi}\sum_{k=1}^{2} c_k a_k(x)$$

$$= \frac{1}{\pi}\left[c_1 \sin x + c_1 \cos x\right]$$

$$= \sin x + \cos x \quad \cdots \left[\text{Normalising } c_1 = \pi\right]$$

For $\lambda_2 = \frac{-1}{\pi}$

$$\phi_2 = \frac{-1}{\pi}\sum_{k=1}^{2} c_k a_k(x)$$

$$= \frac{-1}{\pi}\left[-c_1 \sin x + c_1 \cos x\right]$$

$$= \sin x - \cos x \quad \cdots \left[\text{Normalising } c_1 = \pi\right]$$

$\phi_1(x)$ & $\phi_2(x)$ are eigen functions.

ii) $$f(x) = \int_0^{2\pi} \sin(x+t)\, u(t)\, dt$$

$$= \int_0^{2\pi} (\sin x \cos t + \cos x \sin t)\, u(t)\, dt$$

$$= \sin x \int_0^{2\pi} \cos t \cdot u(t)\, dt + \cancel{\sin}\,\cos x \int_0^{2\pi} \sin t \cdot u(t)\, dt$$

$$= b_1 \sin x + b_2 \cancel{\sin}\,\cos x$$

$$= \sum_{k=1}^{2} b_k \phi_k(x)$$

Thus $f(x)$ is restricted to linear combinations of two eigenfunctions found in part (ai)

b) Show that solution to above $f(x)$ is not unique.

Solution to above integral equation is given as,

$$u(x) = \lambda \sum_{k=1}^{2} C_k a_k(x)$$

$$= \sum C_K \lambda_k \phi_K(x)$$

Let $g(x)$ be the function which is orthogonal to the eigen function $\phi_1(x)$ & $\phi_2(x)$

$$\therefore \int_a^b g(x)\phi_1(x)\,dx = 0$$

$$\int_a^b g(x)\phi_2(x)\,dx = 0$$

Thus, if we add $g(x)$ to $u(x)$

$$u(x) + g(x) = \sum_{k=1}^{2} C_k \lambda_n \phi_k(x) + \sum g(x)\phi_i(x)$$

$$= \sum_{k=1}^{2} C_k \lambda_n \phi_k(x)$$

$\therefore$ ~~sol~~ $u(x) + g(x)$ is still a solution.

thus solution $u(x)$ is not unique.

c) According to theorem 7, the fredholm integral equation of the first kind with closed symmetric kernel has a unique solution if and only if the following series,

$\sum_{n=1}^{\infty} |\lambda_n a_n|^2$ converges where $\{\lambda_n\}$ are

the eigen values & a_n are fourier coefficients of the given function $f(x)$

Homework

Exercise 5.1

1] Solve following fredholm equations in $u(x)$

a) $u(x) = \sin x + \lambda \int_0^{\pi/2} \sin x \cos t \cdot u(t)\, dt$

Let $C = \int_0^{\pi/2} \cos(t)\, u(t)\, dt$

$\therefore u(x) = \sin x + \lambda c \sin x$

$\therefore C = \int_0^{\pi/2} \cos(t)\,(\sin t + \lambda c \sin t)\, dt$

$= \frac{1}{2} c \lambda$

$\therefore c\left[1 - \frac{\lambda}{2}\right] = 1$

$\therefore c_1 = \frac{2}{2-\lambda}$ ... [$\lambda \neq 2$ for the sol^n to exist]

$\therefore u(x) = \sin x + \frac{2\lambda}{2-\lambda} \sin x$

$= \frac{2}{2-\lambda} \sin x$

b) $u(x) = x + \lambda \int_{-\pi}^{\pi} \left(\cos x \sin t + x \cos t + t^2 \sin x\right) u(t)\,dt.$

$f(x) = x$

$a_1(x) = \cos x \qquad b_1(t) = \sin t$

$a_2(x) = x \qquad b_2(t) = \cos t$

$a_3(x) = \sin x \qquad b_3(t) = t^2$

$$a_{11} = \int_{-\pi}^{\pi} b_1(t)\, a_1(t)\, dt = 0$$

$$a_{12} = \int_{-\pi}^{\pi} b_1(t)\, a_2(t)\,dt = 2\pi$$

$$a_{13} = \int_{-\pi}^{\pi} b_1(t)\, a_3(t)\,dt = \pi$$

$$a_{21} = \int_{-\pi}^{\pi} b_2(t)\, a_1(t)\,dt = \pi$$

$$a_{22} = \int_{-\pi}^{\pi} b_2(t)\, a_2(t)\, dt = 0$$

$$a_{23} = \int_{-\pi}^{\pi} b_2(t)\, a_3(t)\, dt = 0$$

$$a_{31} = \int_{-\pi}^{\pi} b_3(t)\, a_1(t)\, dt = -4\pi$$

$$a_{32} = \int_{-\pi}^{\pi} b_3(t)\, a_3(t)\, dt = 0$$

(17)

$$a_{33} = \int_{-\pi}^{\pi} b_3(t) \cdot a_3(t)\, dt = 0$$

$$\therefore A = \begin{bmatrix} 0 & 2\pi & \pi \\ \pi & 0 & 0 \\ 4\pi & 0 & 0 \end{bmatrix}$$

$$f_1 = \int_{-\pi}^{\pi} b_1(t) f(t)\, dt = \int_{-\pi}^{\pi} \sin t \cdot t \cdot dt = 2\pi$$

$$f_2 = \int_{-\pi}^{\pi} b_2(t) f(t)\, dt = \int_{-\pi}^{\pi} \cos t \cdot t \cdot dt = 0$$

$$f_3 = \int_{-\pi}^{\pi} b_3(t) f(t)\, dt = \int_{-\pi}^{\pi} t^2 \cdot t \cdot dt = 0$$

$$\therefore F = \begin{bmatrix} 2\pi \\ 0 \\ 0 \end{bmatrix}$$

Now, $(I - \lambda A) C = F$

$$\therefore \begin{bmatrix} 1 & -2\pi\lambda & -\pi\lambda \\ -\pi\lambda & 1 & 0 \\ 4\pi\lambda & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2\pi \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore c_1 = \frac{2\pi}{2\lambda^2\pi^2+1} \qquad c_3 = \frac{-8\lambda\pi}{2\lambda^2\pi^2+1}$$

$$c_2 = \frac{2\lambda\pi^2}{2\lambda^2\pi^2+1}$$

$$u(x) = f(x) + \lambda C_1 a_1(x) + \lambda C_2 a_2(x) + C_3 \lambda a_3(x)$$

$$= x + \frac{2\lambda\pi}{2\lambda^2\pi^2+1}\left(\lambda\pi x - 4\lambda\pi \sin x + \cos x\right)$$

c)

$$u(x) = 2x - \pi + 4\int_0^{\pi/2} \sin^2 x \cdot u(t)\, dt$$

$$f(x) = 2x - \pi$$

$$a_1(x) = \sin^2 x \qquad b_1(t) = 1$$

$$C = \int_0^{\pi/2} u(t)\, dt$$

$$\therefore u(x) = (2x - \pi) + 4C \sin^2 x$$

$$\therefore C = \int_0^{\pi/2} \left[(2t-\pi) + 4C \sin^2 t\right] dt$$

$$\therefore C = C\pi - \frac{\pi^2}{4}$$

$$\therefore \frac{\pi^2}{4} = C_1(\pi - 1)$$

$$\therefore C_1 = \frac{\pi^2}{4(\pi-1)}$$

$$\therefore u(x) = 2x - \pi + \frac{\pi^2}{\pi-1} \cdot \sin^2 x$$

d) $u(x) = \cos x + \lambda \int_0^{\pi} \sin(x-t)\, u(t)\, dt$

$= \cos x + \lambda \int_0^{\pi} (\sin x \cos t - \cos x \sin t)\, u(t)\, dt$

$f(x) = \cos x$

$a_1(x) = \sin x$ $\quad\quad$ $b_1(t) = \cos t$

$a_2(x) = \cos x$ $\quad\quad$ $b_2(t) = -\sin t$

$$a_{11} = \int_0^{\pi} b_1(t)\, a_1(t)\, dt = 0$$

$$a_{12} = \int_0^{\pi} b_1(t)\, a_2(t)\, dt = \frac{\pi}{2}$$

$$a_{21} = \int_0^{\pi} b_2(t)\, a_1(t)\, dt = -\frac{\pi}{2}$$

$$a_{22} = \int_0^{\pi} b_2(t)\, a_2(t)\, dt = 0$$

$$\therefore A = \begin{bmatrix} 0 & \pi/2 \\ -\pi/2 & 0 \end{bmatrix}$$

$$f_1 = \int_0^{\pi} b_1(t)\, f(t)\, dt = \frac{\pi}{2}$$

$$f_2 = \int_0^{\pi} b_2(t)\, f(t)\, dt = 0$$

$$[I - \lambda A]\, C = [F]$$

$$\therefore \begin{bmatrix} 1 & -\lambda\pi/2 \\ \lambda\pi/2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \pi/2 \\ 0 \end{bmatrix}$$

$$\therefore c_1 = \frac{2\pi}{4+\lambda^2\pi^2}, \qquad c_2 = \frac{-\lambda\pi^2}{4+\lambda^2\pi^2}$$

$$\therefore u(x) = \cos x + \lambda\left(\frac{2\pi}{4+\lambda^2\pi^2}\right)\sin x - \lambda\left(\frac{2\pi^2}{4+\lambda^2\pi^2}\right)\cos x$$

$$= \frac{4}{4+\pi^2\lambda^2}\cos x + \frac{2\pi\lambda}{4+\pi^2\lambda^2}\sin x$$

e) $u(x) = e^x + \lambda \int_0^1 (5x^3-3)t^2 \cdot u(t)\, dt$

$f(x) = e^x$

$a_1(x) = 5x^3 - 3 \qquad\qquad b_1(t) = t^2$

$$C = \int_0^1 b_1(t)\cdot u(t)\, dt$$

$$= \int_0^1 t^2 \cdot u(t)\, dt.$$

$u(x) = e^x + \lambda(5x^3-3)\, c$

$$\therefore C = \int_0^1 t^2\left(\cancel{e^x}\, e^t + \lambda(5t^3-3)c\right) dt$$

$$= e - 2$$

(19)

$\therefore \quad u(x) = e^{x} + x(e-2)(5x^2-3)$

f) $u(x) = 2x - 6 - 2\int_0^1 u(t)\,dt$

$f(x) = 2x - 6 \quad , \quad \lambda = 2$

$c = \int_0^1 u(t)\,dt$

$\therefore \quad u(x) = (2x-6) - 2c$

$\therefore \quad c = \int_0^1 [(2t-6) - 2c]\,dt$

$= -5 - 2c$

$\therefore \quad c = \dfrac{-5}{3}$

$\therefore \quad u(x) = 2x - 6 + 2\left(\dfrac{5}{3}\right)$

$= 2x + \dfrac{8}{3}$

g) $u(x) = 201x^2 - 80x + 52 + \int_0^1 (4xt^2 - 3x^2t - t^3)\, u(t)\,dt.$

$f(x) = 201x^2 - 80x + 52$

$a_1(x) = 4x \qquad b_1(t) = t^2$

$a_2(x) = 3x^2 \qquad b_2(t) = -t$

$a_3(x) = 1 \qquad b_3(t) = -t^3$

$$a_{11} = \int_0^1 (t^2)(4t)\,dt = 1$$

$$a_{12} = \int_0^1 (t^2)(3t^2)\,dt = \frac{3}{5}$$

$$a_{13} = \int_0^1 (t^2)(1)\,dt = \frac{1}{3}$$

$$a_{21} = \int_0^1 (-t)(4t)\,dt = \frac{-4}{3}$$

$$a_{22} = \int_0^1 (-t)(3t^2)\,dt = \frac{-3}{4}$$

$$a_{23} = \int_0^1 (-t)(1)\,dt = \frac{-1}{2}$$

$$a_{31} = \int_0^1 (-t^3)(4t)\,dt = \frac{-4}{5}$$

$$a_{32} = \int_0^1 (-t^3)(3t^2)\,dt = \frac{-1}{2}$$

$$a_{33} = \int_0^1 (-t^3)(1)\,dt = \frac{-1}{4}$$

$$f_1 = \int_0^1 (t^2)(201t^2 - 80t + 52)\,dt = \frac{563}{15}$$

$$f_2 = \int_0^1 (-t)(201t^2 - 80t + 52)\,dt = \frac{-595}{12}$$

$$f_3 = \int_0^1 (-t^3)(201t^2 - 80t + 52)\,dt = \frac{-61}{2}$$

Now,

$$[I - \lambda A]\, C = [F] \quad --- [\lambda = 1]$$

$$\begin{bmatrix} 0 & -\frac{3}{5} & -\frac{1}{3} \\ \frac{4}{3} & 1+\frac{3}{4} & \frac{1}{2} \\ \frac{4}{5} & \frac{1}{2} & 1+\frac{1}{4} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \frac{563}{15} \\ -\frac{595}{12} \\ -\frac{61}{2} \end{bmatrix}$$

$c_1 = 35$, $c_2 = -47$, $c_3 = -28$

$$\therefore u(x) = 201x^2 - 80x + 52 + (35 \times 4x) + (-47 \times 3x^2) + (-28 \times 1)$$

$$= 60x^2 + 60x + 24$$

2] Solve following eigenfunctions by finding eigenvalues & eigenfunctions.

a) $u(x) = \lambda \int_0^{2\pi} \sin x \sin t \cdot u(t)\, dt$

$$C = \int_0^{2\pi} \sin(t)\, u(t)\, dt$$

$$\therefore u(x) = \lambda C \sin x$$

$$\therefore C = \int_0^{2\pi} \sin(t) \cdot \lambda C \sin(t)\, dt$$

$$= \lambda \pi C$$

$$\therefore (1 - \lambda\pi) C = 0$$

For non trivial solution,

$1 - \lambda\pi = 0 \quad \therefore \underline{\lambda_1 = \frac{1}{\pi}}$

$$\therefore u_1(x) = \frac{1}{\pi} \cdot \sin x$$

$$\therefore u(x) = \frac{1}{\pi} C \sin x$$

$= A \sin x \quad \ldots$ [A is any constant]

21

b) $u(x) = \lambda \int_0^{\pi/2} \sin x \cos t \; u(t)\, dt$

$c = \int_0^{\pi/2} \cos t \cdot u(t)\, dt$

$u(x) = \lambda c \sin x$

$c = \int_0^{\pi/2} \cos t \cdot \lambda c \sin t \; dt$

$= \frac{\lambda c}{2}$

$\therefore \left(1 - \frac{\lambda}{2}\right) c_1 = 0$

For non trivial solutions,

$1 - \frac{\lambda}{2} = 0 \qquad \therefore \underline{\lambda_1 = 2}$

$\therefore u_1(x) = 2 \sin x$

& $u(x) = A \sin x$ --- (A is any constant)

c) $u(x) = \lambda \int_0^{\pi} \cos(x+t)\, u(t)\, dt$

$= \lambda \int_0^{\pi} (\cos x \cos t - \sin x \sin t) \cancel{dt} \cdot u(t)\, dt$

$a_1(x) = \cos x \qquad b_1(t) = \cos t$

$a_2(x) = -\sin x \qquad b_2(t) = \sin t$

$$a_{11} = \int_0^{\pi} \cos t \cdot \cos t \cdot dt = \frac{\pi}{2}$$

$$a_{12} = \int_0^{\pi} \cos t \cdot (-\sin t)\, dt = 0$$

$$a_{21} = \int_0^{\pi} \sin t \cdot \cos t \cdot dt = 0$$

$$a_{22} = \int_0^{\pi} \sin t \cdot (-\sin t)\, dt = \frac{-\pi}{2}$$

$$(I - \lambda A)\, c = 0$$

$$\therefore \begin{bmatrix} 1 - \pi\lambda/2 & 0 \\ 0 & 1 + \pi\lambda/2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For non trivial solution,

$$\begin{vmatrix} 1 - \pi\lambda/2 & 0 \\ 0 & 1 + \pi\lambda/2 \end{vmatrix} = 0$$

$$\left(1 - \frac{\lambda\pi}{2}\right)\left(1 + \frac{\lambda\pi}{2}\right) = 0$$

$$\therefore \lambda_1 = \frac{2}{\pi}, \quad \lambda_2 = \frac{-2}{\pi}$$

For $\lambda_1 = 2/\pi$, $u_1(x) = A \cos x$

for $\lambda_2 = -2/\pi$, $u_2(x) = B \sin x$

where, A & B are arbitrary constants.

d) $u(x) = 2\lambda \int_0^1 x(t-2x)\, u(t)\, dt$

$a_1(x) = x \qquad\qquad b_1(t) = 2t$

$a_2(x) = -2x^2 \qquad\qquad b_2(t) = 2$

$$a_{11} = \int_0^1 (2t)(t)\,dt = \frac{2}{3}$$

$$a_{12} = \int_0^1 (2t)(-2t^2)\,dt = -\frac{2}{2}$$

$$a_{21} = \int_0^1 (2)(t)\,dt = 1$$

$$a_{22} = \int_0^1 (2)(-2t^2)\,dt = -\frac{4}{3}$$

$$(I - \lambda A)c = 0$$

$$\therefore \begin{bmatrix} 1 - \frac{2\lambda}{3} & \lambda \\ -\lambda & 1 + \frac{4\lambda}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For a non-trivial solution,

$$\left(1-\frac{2\lambda}{3}\right)\left(1+\frac{4\lambda}{3}\right)-(\lambda)(-\lambda)=0$$

$\therefore \lambda_1 = -3 = +\lambda_2$

$\therefore u_1(x) = u_2(x) = x - 2x^2$

e) $u(x) = \lambda \int_0^1 (5x^2-3)\, t^2 \cdot u(t)\, dt$

$a_1(x) = 5x^2 - 3$ $\qquad$ $b_1(t) = t^2$

$c = \int_0^1 t^2 \cdot u(t)\, dt$

$u(x) = \lambda c (5x^2 - 3)$

$\therefore c = \int_0^1 [\lambda c t^2 (5t^2 - 3)]\, dt$

$= \lambda c (0)$

$\therefore$ For non trivial solution,

$\lambda_1 = 0$

$\therefore u(x) = 0$

3] Discuss the existance of the solution to the eq^n.

$$u(x) = f(x) + \lambda \int_0^{2\pi} \sin(x+t)\, u(t)\, dt.$$

a) As solved in exercise 5.4.5 & example 4 in the book, the eigen values of the kernel are,

$$\lambda_1 = \frac{1}{\pi} \quad \& \quad \lambda_2 = \frac{-1}{\pi}$$

Eigenfunctions are given by,

$$\phi_1(x) = \frac{1}{\sqrt{2\pi}}(\sin x + \cos x)$$

$$\phi_2(x) = \frac{1}{\sqrt{2\pi}}(\sin x - \cos x) \quad \text{resp.}$$

i) $\lambda = \frac{1}{\sqrt{\pi}} \neq \pm\frac{1}{\pi}$ is not an eigenvalue

Thus, $u(x)$ has a unique solution with $f(x) = x^2$

ii) $\lambda = \frac{1}{\pi}$ is a eigenvalue which corrosponds to eigen function $\phi_1(x) = \frac{1}{2\sqrt{\pi}}(\sin x + \cos x)$

Thus ~~so~~ the eq^n has infinite solutions provided that $f(x)$ is orthogonal with eigenfunction corrosponding to $\lambda_1 = \frac{1}{\pi}$

$$\int_0^{2\pi} (\sin x + \cos x) \cdot \sin 3x \, dx = 0$$

$\therefore$ $f(x)$ is orthogonal with $\phi_1(x)$

b) (i) $\lambda = \frac{1}{\sqrt{\pi}}$, $f(x) = x^2$

$$[I - \lambda A] = \begin{bmatrix} 1 & -\pi\lambda \\ -\pi\lambda & 1 \end{bmatrix}$$

$$f_1 = \int_0^{2\pi} f(t) \cdot b_1(t) \, dt$$

$$= \int_0^{2\pi} t^2 \cdot \cos t \cdot dt = 4\pi$$

$$f_2 = \int_0^{2\pi} t^2 \cdot \sin t \cdot dt = -4\pi$$

$\therefore$ $[I - \lambda A] \, c = [F]$

$$\Longrightarrow \begin{bmatrix} 1 & -\lambda\pi \\ -\lambda\pi & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4\pi \\ -4\pi \end{bmatrix}$$

24

$$\therefore\ c_1 = \frac{4\sqrt{\pi}}{1-\pi}(1-\sqrt{\pi})$$

$$c_2 = \frac{4\sqrt{\pi}\cdot\sqrt{\pi}}{1-\pi}(1-\sqrt{\pi})$$

$$\therefore\ u(x) = x^2 + \frac{4\sqrt{\pi}}{1-\pi}\left((1-\sqrt{\pi})\sin x + \sqrt{\pi}(1-\sqrt{\pi})\cos x\right.$$

(ii) $f(x) = \sin 3x$

$$f_1 = \int_0^{2\pi} \sin 3t \cdot \cos t \cdot dt = 0$$

$$f_2 = \int_0^{2\pi} \sin 3t \cdot \sin t \cdot dt = 0$$

$$\therefore\ [I - \lambda A]\, c = [0]$$

$$\therefore\ \begin{bmatrix} 1 & -\lambda\pi \\ -\lambda\pi & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore\ c_1 = c_2 = \frac{c}{\pi}$$

$$\therefore\ u(x) = \sin 3x + \frac{c}{\pi}\left[\sin x + \cos x\right],$$ where

c is any constant.