Orbits

Kinematics

Throughout this notebook we take the origin of our polar coordinates at the sun and assume that all motion is in a single plane (the plane of the ecliptic) and measure the angle in this plane with respect to some point in the fixed stars.

The motion of a planet is described by the functions r[t] and θ[t] which give the time dependence of the radial distance and angle. It is convenent to define a set of orthonormal unit vectors

```
Clear[θ, er, eθ]
er⃗[θ_] = {Cos[θ], Sin[θ]};
eθ⃗[θ_] = {-Sin[θ], Cos[θ]};
```

We use these vectors to decompose vector equations into radial and azimuthal components.

Newton's Law of Gravity

Newton's law of gravitational attraction is simple. Two masses attract each other with a force proportional to the product of their masses and inversely proportional to the square of the distance bewteen them. Writing $\mathbb{M}$ for the mass of the sun and $\mathbb{m}$ for the mass of the planet we have the simple expression for the gravitational force on the planet due to sun

$\vec{F} = G\,\mathbb{M}\,\mathbb{m} / r^3\ \vec{er}$

where in SI units the constant of porportionality is

$$G = \frac{6.67266\times10^{-11}\ \text{Meter}^2\ \text{Newton}}{\text{Kilogram}^2}$$

Single Newtonian Planet Equations and Numerics

The equations governing the motion of a single planet around a sun are fairly simple. The position at time t is given by

```
Clear[p, t, r, θ]
p[t_] = r[t] er⃗[θ[t]];
```

with these definitions it is easy to compute the radial and azimuthal components of the vector Newtonian force balance

$\vec{F} = m\,\vec{a}$

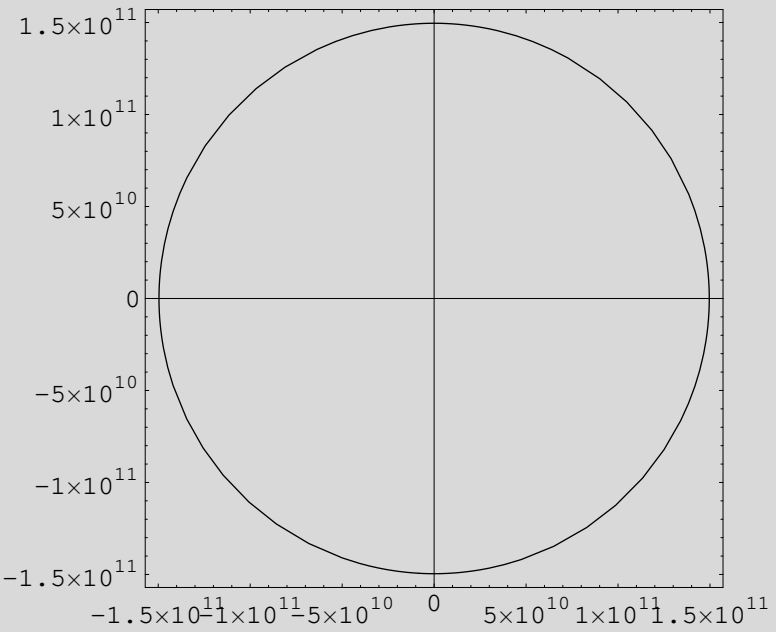
```
Clear[ a, t, M, G]
a[t_] = p''[t];
Eq_θ = Simplify[ a[t].êθ[θ[t]]] == 0
Eq_r = Simplify[ a[t].êr[θ[t]]] == -M G / r[t]^2
```

$$2\,r'(t)\,\theta'(t) + r(t)\,\theta''(t) == 0$$

$$r''(t) - r(t)\,\theta'(t)^2 == -\frac{G\,\mathbb{M}}{r(t)^2}$$

These equations can be readily solved numerically for specific parameter values. Here is a test using the parameter values for the earths orbit.

```
Clear[EarthRad, Earthω, G, M, m, r, θ, t]
EarthRad = 1.496 × 10^11;
Earthω = 2 π / (365 × 24 × 60^2);
G = 6.67266 × 10^-11;
M = 1.9918 × 10^30;
m = 5.9737 × 10^24 ;

TempSol = NDSolve[ {
     Eq_θ, Eq_r,
     r[0] == EarthRad, r'[0] == 0,
     θ[0] == 0, θ'[0] == Earthω
    },
    {r[t], θ[t]},
    {t, 0, 365 × 24 × 60^2}
   ][[1]];

Clear[G, M, m, r, θ, t]

{rSol[t_], θSol[t_] } = {r[t], θ[t]} /. TempSol;
ParametricPlot[
  rSol[t] er[θSol[t]],
  {t, 0, 365 × 24 × 60^2},
  Frame → True,
  AspectRatio → Automatic];
```



Of course it is not very interesting since the Earths orbit is almost completely circular.

Single Newtonian Planet
Analytic Solution

Multiplying the azimuthal equation $\texttt{Eq}_\theta$ by r and integrating gives the conserved quantity

$\texttt{r}^2\,\theta' = \text{Constant}$

which implies Kepler's equal area law. Substituting

$\theta' \to \texttt{p} / \mathbb{m}\,\texttt{r}^{-2}$

(the constant is written as p/m because then p is the angular maomentum) in $\texttt{Eq}_\texttt{r}$ reduces the original system of ODEs to the single ODE

```
Eq_r2 = Eq_r /. θ'[t] → (p / m) r[t]^-2
```

$$r''(t) - \frac{p^2}{m^2\, r(t)^3} == -\frac{G\mathbb{M}}{r(t)^2}$$

The last trick is to solve for the radius as a function of angle R[θ] rather than a function r of time t. In other words we assume there is a function R satisfying

$\texttt{R}[\theta[\texttt{t}]] = \texttt{r}[\texttt{t}]$

and susbtitute

r	→	R
$\partial_\texttt{t}\,\texttt{r}$	→	$\partial_\theta \texttt{R}\,\partial_\texttt{t}\,\theta = \partial_\theta \texttt{R}\,(\texttt{p}/\mathbb{m})\,\texttt{R}^{-2} = -(\texttt{p}/\mathbb{m})\,\partial_\theta \texttt{R}^{-1}$
$\partial_{\texttt{t},\texttt{t}}\,\texttt{r}$	→	$-(\texttt{p}/\mathbb{m})^2\,\texttt{R}^{-2}\,\partial_{\theta,\theta}\,\texttt{R}^{-1}$

to get the following

```
Simplify[Eq_r2 /. {
   r[t] → 1 / W[θ],
   r'[t] → -(p / m) W'[θ],
   r''[t] → -(p / m)^2 W[θ]^2 W''[θ]
  }]
```

$$W(\theta)^2\left(G\mathbb{M} - \frac{p^2\,(W(\theta) + W''(\theta))}{m^2}\right) == 0$$

which reduces to the following constant coefficient linear ODE for $\texttt{W} = \texttt{R}^{-1}$ as a function of θ.

$\texttt{W''}[\theta] + \texttt{W}[\theta] == (\mathbb{m}/\texttt{p})^2\,\mathbb{M}\,\texttt{G}$

which has the general solution

$\texttt{W} = 1/\texttt{R} = (\epsilon\,\text{Cos}[\theta - \theta_0] + 1)/\texttt{a}$

which is the polar for of an ellipse with mean radius

$\texttt{a} = (\texttt{p}/\mathbb{m})^2\,1/(\mathbb{M}\,\texttt{G})$

and eccentricity ϵ, and orientation θ_0. To compute the location of the planet on the ellipse at any time t all one has to do is to integrate the conservation equation

$\partial_\texttt{t}\,\theta = (\texttt{p}/\texttt{m})\,1/\texttt{R}^2 = (\texttt{p}/\texttt{m})\,\texttt{a}^2\,(1 + \epsilon\,\text{Cos}[\theta - \theta_0])^2$

In particular the orbit is periodic with period $(\texttt{p}/\mathbb{m})\,\texttt{a}^2\,\pi\,(2 + \epsilon^2)$.

Two Newtonian Planet Numerical Solution

With two planest mass $\mathbb{m}_1$ and $\mathbb{m}_2$ the equations of motion (again negelecting motions of the massive central sun) are

```
p[t]
```

```
Clear[ p1, p2, a1, a2, t, M, m1, m2, θ1, θ2, G, r1, r2]
p1[t_] = r1[t] er[θ1[t]];
p2[t_] = r2[t] er[θ2[t]];
r12 = p1[t] - p2[t];
F12 = Simplify[ m1 m2 G r12 / (r12.r12)^3/2 ]

a1[t_] = p1''[t];
a2[t_] = p2''[t];

Eq1θ = Simplify[ a1[t].eθ[θ1[t]] == 0 - 1 / m1 F12.eθ[θ1[t]] ]
Eq1r = Simplify[ a1[t].er[θ1[t]] == -M G / r1[t]^2 - 1 / m1 F12.er[θ1[t]] ]
Eq2θ = Simplify[ a2[t].eθ[θ2[t]] == 0 + 1 / m2 F12.eθ[θ2[t]] ]
Eq2r = Simplify[ a2[t].er[θ2[t]] == -M G / r2[t]^2 + 1 / m2 F12.er[θ2[t]] ]
```

$$\left\{\frac{G\,m1\,m2\,(\cos(\theta1(t))\,r1(t)-\cos(\theta2(t))\,r2(t))}{(r1(t)^2-2\cos(\theta1(t)-\theta2(t))\,r2(t)\,r1(t)+r2(t)^2)^{3/2}},\ \frac{G\,m1\,m2\,(r1(t)\sin(\theta1(t))-r2(t)\sin(\theta2(t)))}{(r1(t)^2-2\cos(\theta1(t)-\theta2(t))\,r2(t)\,r1(t)+r2(t)^2)^{3/2}}\right\}$$

$$\frac{G\,m2\,r2(t)\sin(\theta1(t)-\theta2(t))}{(r1(t)^2-2\cos(\theta1(t)-\theta2(t))\,r2(t)\,r1(t)+r2(t)^2)^{3/2}}+2\,r1'(t)\,\theta1'(t)+r1(t)\,\theta1''(t)==0$$

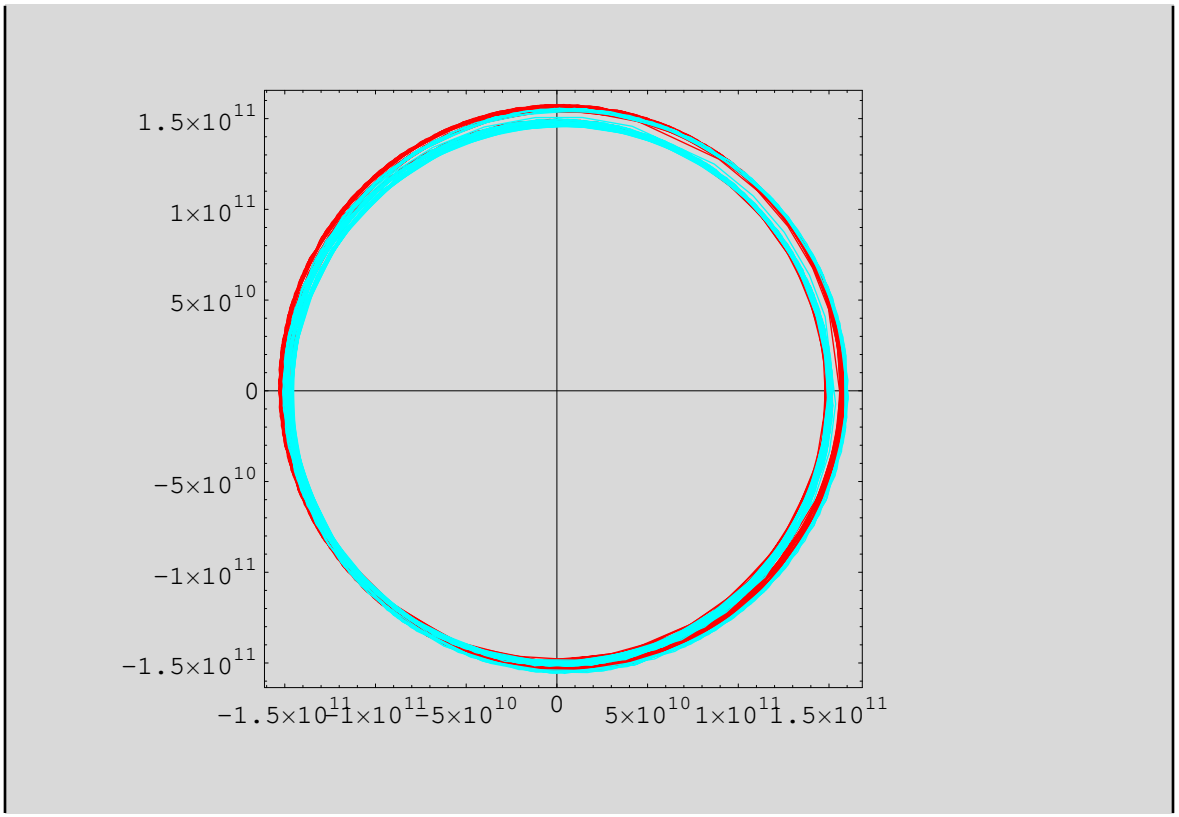
$$\frac{G\left(m2\,r1(t)^3-m2\cos(\theta1(t)-\theta2(t))\,r2(t)\,r1(t)^2+M\,(r1(t)^2-2\cos(\theta1(t)-\theta2(t))\,r2(t)\,r1(t)+r2(t)^2)^{3/2}\right)}{r1(t)^2\,(r1(t)^2-2\cos(\theta1(t)-\theta2(t))\,r2(t)\,r1(t)+r2(t)^2)^{3/2}}$$
$$+\,r1''(t)==r1(t)\,\theta1'(t)^2$$

$$2\,r2'(t)\,\theta2'(t)+r2(t)\,\theta2''(t)==\frac{G\,m1\,r1(t)\sin(\theta1(t)-\theta2(t))}{(r1(t)^2-2\cos(\theta1(t)-\theta2(t))\,r2(t)\,r1(t)+r2(t)^2)^{3/2}}$$

$$\frac{G\left(m1\,r2(t)^3-m1\cos(\theta1(t)-\theta2(t))\,r1(t)\,r2(t)^2+M\,(r1(t)^2-2\cos(\theta1(t)-\theta2(t))\,r2(t)\,r1(t)+r2(t)^2)^{3/2}\right)}{r2(t)^2\,(r1(t)^2-2\cos(\theta1(t)-\theta2(t))\,r2(t)\,r1(t)+r2(t)^2)^{3/2}}$$
$$+\,r2''(t)==r2(t)\,\theta2'(t)^2$$

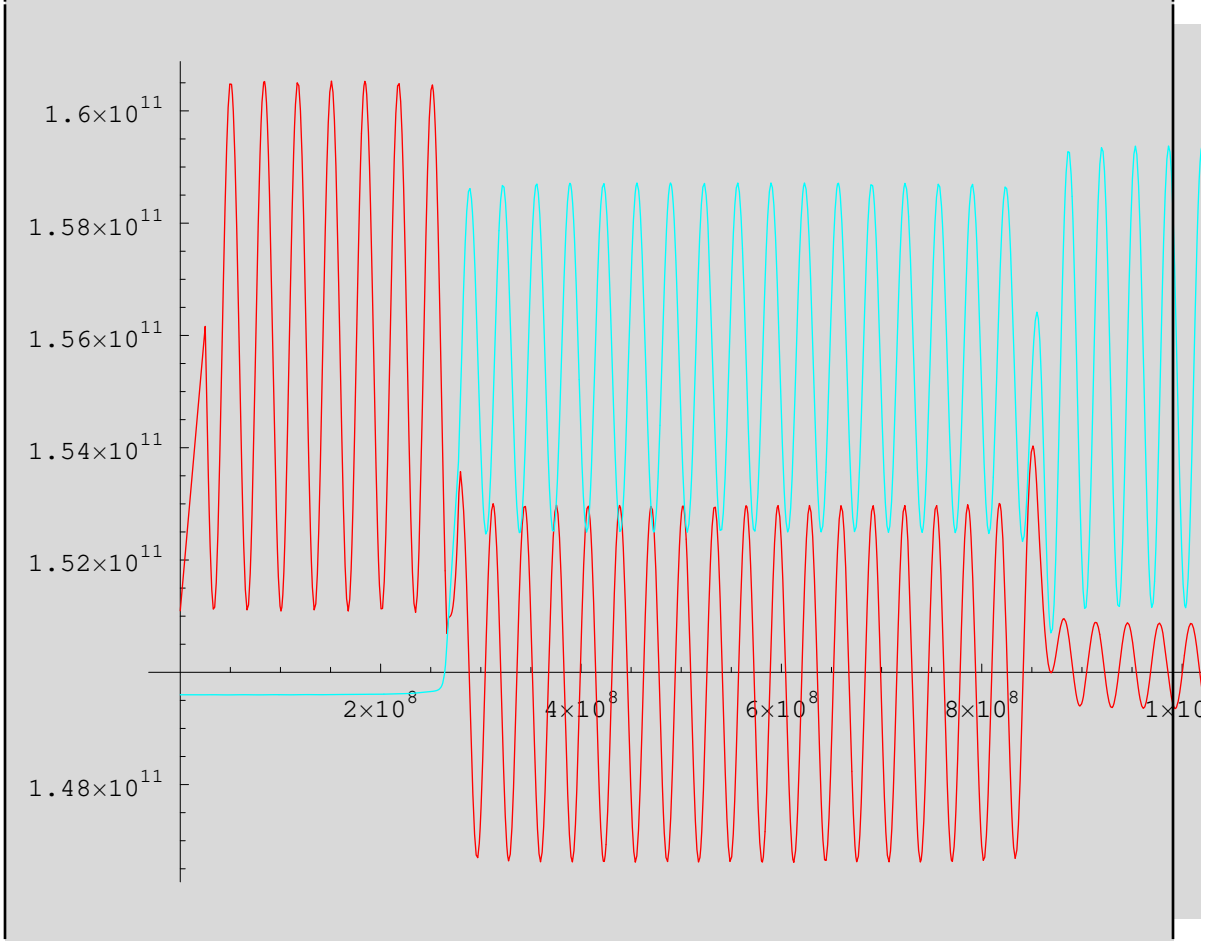
Again these equations are easily simulated numerically. However, to accurately accumulate the tiny effects of the additional terms on the overall motion is very challenging for a numerical solver! As an example here is a simulation of two earth like planets in very similar orbits. Something interesting happens!

```
Clear[EarthRad, Earthω, G, M, m1, m2, r, θ, t]
EarthRad = 1.496 × 10^11;
Earthω = 2 π / (365 × 24 × 60^2);
G = 6.67266 × 10^-11;
M = 1.9918 × 10^30;
m1 = 5.9737 × 10^24 ;
m2 = m1;

TempSol = NDSolve[ {
     Eq1θ, Eq1r, Eq2θ, Eq2r,
     r1[0] == EarthRad, r1'[0] == 0,
     r2[0] == 1.01 EarthRad, r2'[0] == 0,
     θ1[0] == 0, θ1'[0] == Earthω,
     θ2[0] == π, θ2'[0] == Earthω
    },
    {r1[t], θ1[t], r2[t], θ2[t]},
    {t, 0, 40 × 365 × 24 × 60^2},
    MaxSteps → 10^4
   ][[1]];

Clear[G, M, m1, m2, r1, r2, θ1, θ2, t]

{r1Sol[t_], θ1Sol[t_], r2Sol[t_], θ2Sol[t_] } =
  {r1[t], θ1[t], r2[t], θ2[t]} /. TempSol;
ParametricPlot[
  {r1Sol[t] er[θ1Sol[t]], r2Sol[t] er[θ2Sol[t]]},
  {t, 0, 40 × 365 × 24 × 60^2},
  Frame → True,
  PlotStyle → {Hue[0], Hue[0.5]},
  AspectRatio → Automatic];
```



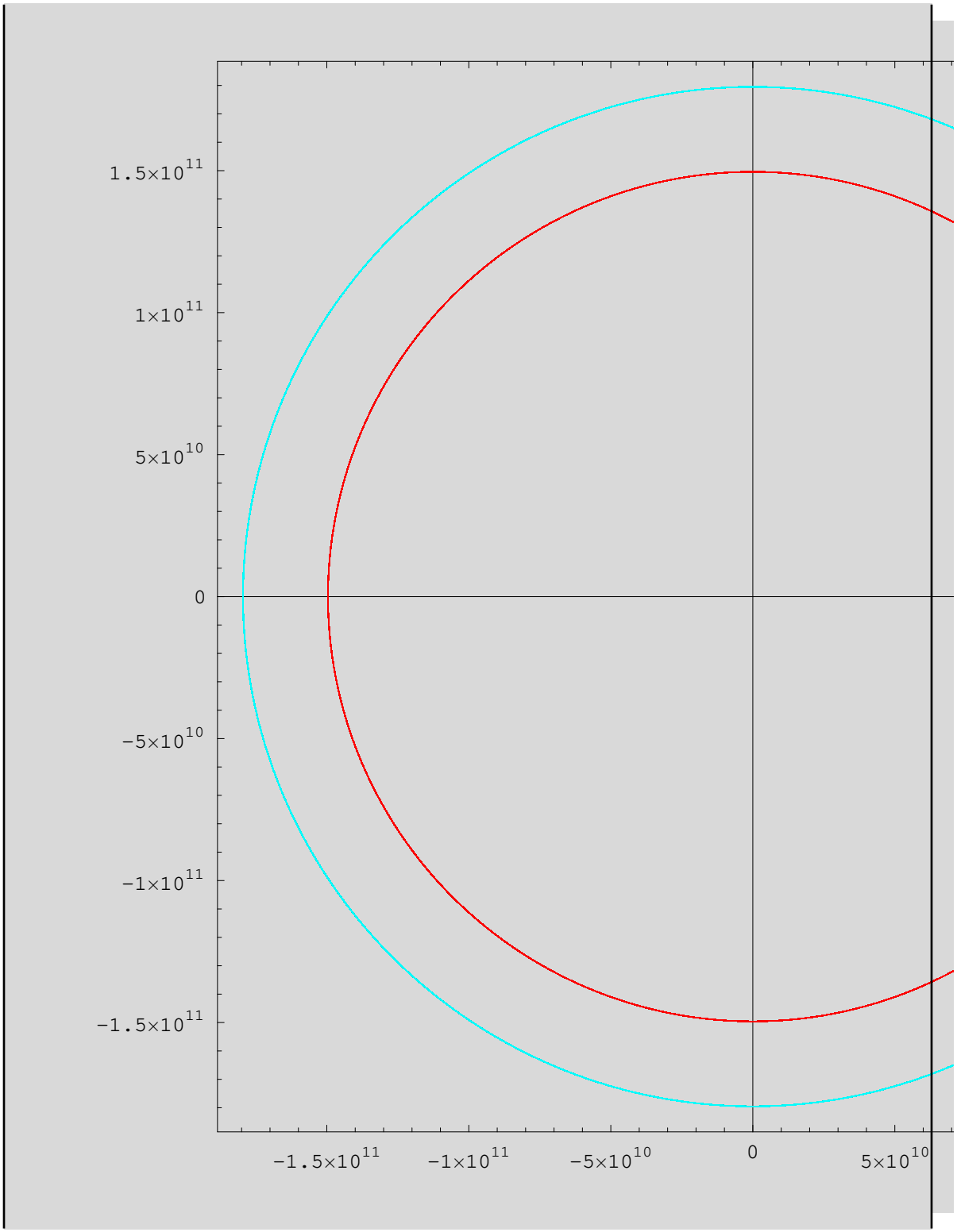
It is easier to see on a plot of the radius vectors alone!

```
Plot[{r2Sol[t], r1Sol[t]},
  {t, 0, 40 × 365 × 24 × 60^2},
  PlotRange → All,
  PlotStyle → {Hue[0], Hue[0.5]}
 ];
```



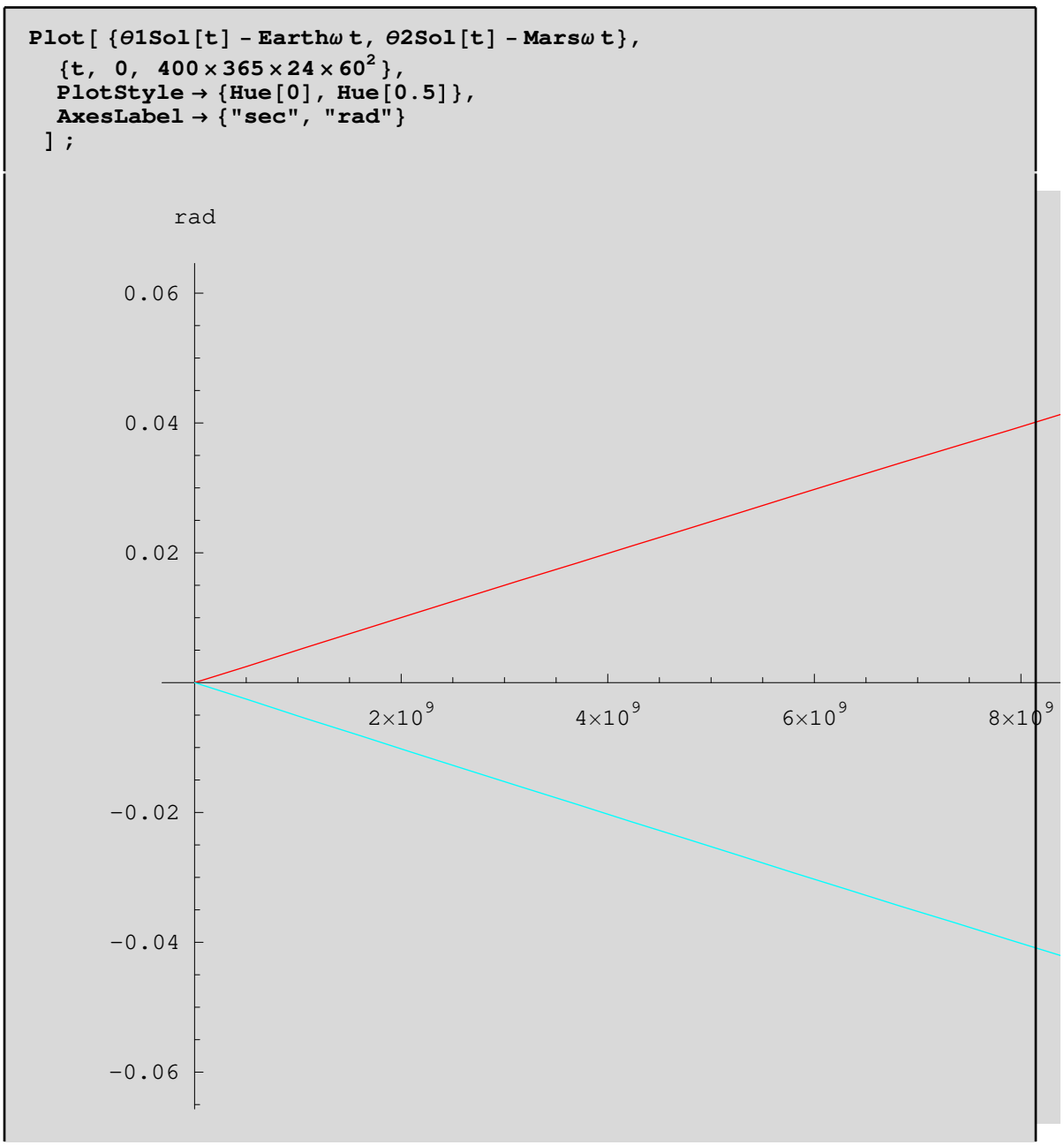
Lets try a slighly different scenario with a Mars and Earth like pair and a subtler effect.

```
Clear[EarthRad, Earthω, MarsRad, Marsω, G, M, m1, m2, r, θ, t]
EarthRad = 1.496 × 10^11;
MarsRad = 1.2 EarthRad;
G = 6.67266 × 10^-11;
M = 1.9918 × 10^30;

Earthω = 2 π / (365 × 24 × 60^2);
Marsω = Marsω /.
   Solve[ Eq2_r /.
      {m1 → 0, r2''[t] → 0, r2[t] → MarsRad, θ2'[t] → Marsω}, Marsω][[1]];

m1 = 5.9737 × 10^24 ;
m2 = 0.7 × m1;

TempSol = NDSolve[ {
     Eq1_θ, Eq1_r, Eq2_θ, Eq2_r,
     r1[0] == EarthRad, r1'[0] == 0,
     r2[0] == MarsRad, r2'[0] == 0,
     θ1[0] == 0, θ1'[0] == Earthω,
     θ2[0] == 0, θ2'[0] == Marsω
    },
    {r1[t], θ1[t], r2[t], θ2[t]},
    {t, 0, 400 × 365 × 24 × 60^2},
    MaxSteps → 10^5
   ][[1]];

Clear[G, M, m1, m2, r1, r2, θ1, θ2, t]

{r1Sol[t_], θ1Sol[t_], r2Sol[t_], θ2Sol[t_] } =
  {r1[t], θ1[t], r2[t], θ2[t]} /. TempSol;
ParametricPlot[
  {r1Sol[t] er[θ1Sol[t]], r2Sol[t] er[θ2Sol[t]]},
  {t, 0, 400 × 365 × 24 × 60^2},
  Frame → True,
  PlotPoints → 10^5,
  PlotStyle → {Hue[0], Hue[0.5]},
  AspectRatio → Automatic];
```



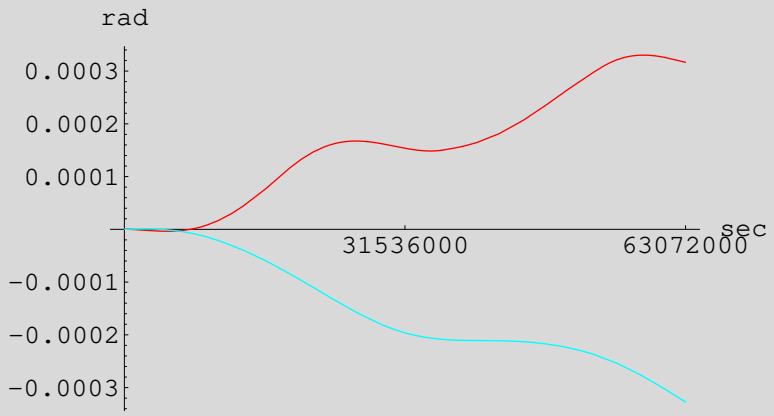
Looks as though nothing much is happening! but look at the angles over the course of 400 years!

```
Plot[ {θ1Sol[t] - Earthω t, θ2Sol[t] - Marsω t},
  {t, 0, 400 × 365 × 24 × 60^2},
  PlotStyle → {Hue[0], Hue[0.5]},
  AxesLabel → {"sec", "rad"}
 ] ;
```



Our goal is to find an analytical formula to explain this! In essence the red guy (Earth) is steadily being dragged ahead while the blue guy (Mars) is steadily being dragged back by the interaction! A close up shows some more interesting features of the interaction!

```
Plot[ {θ1Sol[t] - Earthω t, θ2Sol[t] - Marsω t},
  {t, 0, 2 × 365 × 24 × 60^2},
  PlotStyle → {Hue[0], Hue[0.5]},
  AxesLabel → {"sec", "rad"},
  Ticks → { Table[ i × 365 × 24 × 60^2, {i, 0, 2}], Automatic}
 ] ;
```



Two Newtonian Planet Analysis

We have made the semi−explicitassumption that the planets are small compared to the sun. In other words

$m_1 / \mathbb{M} << 1$ and $m_2 / \mathbb{M} << 1$.

The easiest way to make this explicit is to write

$m_1 = \delta \mathbb{M}$ and $m_2 = \delta \gamma \mathbb{M}$

where $\delta = m_1 / \mathbb{M}$ and $\gamma = m_2 / m_1$ and then look at an expansion of the equations in terms of the small parameter δ.

```
{δEq1_θ, δEq1_r, δEq2_θ, δEq2_r} =
 {Eq1_θ, Eq1_r, Eq2_θ, Eq2_r} /. {m1 → δ M, m2 → δ γ M};
TableForm[{δEq1_θ, δEq1_r, δEq2_θ, δEq2_r}]
```

$$\frac{G\mathbb{M}\gamma\delta\, r2(t)\sin(\theta1(t)-\theta2(t))}{\left(r1(t)^2-2\cos(\theta1(t)-\theta2(t))\,r2(t)\,r1(t)+r2(t)^2\right)^{3/2}} + 2\,r1'(t)\,\theta1'(t) + r1(t)\,\theta1''(t) == 0$$

$$\frac{G\left(\mathbb{M}\gamma\delta\, r1(t)^3 - \mathbb{M}\gamma\delta\cos(\theta1(t)-\theta2(t))\,r2(t)\,r1(t)^2 + \mathbb{M}\left(r1(t)^2-2\cos(\theta1(t)-\theta2(t))\,r2(t)\,r1(t)+r2(t)^2\right)^{3/2}\right)}{r1(t)^2\left(r1(t)^2-2\cos(\theta1(t)-\theta2(t))\,r2(t)\,r1(t)+r2(t)^2\right)^{3/2}} + r1''(t) == r1(t)\,\theta1'(t)^2$$

$$2\,r2'(t)\,\theta2'(t) + r2(t)\,\theta2''(t) == \frac{G\mathbb{M}\delta\, r1(t)\sin(\theta1(t)-\theta2(t))}{\left(r1(t)^2-2\cos(\theta1(t)-\theta2(t))\,r2(t)\,r1(t)+r2(t)^2\right)^{3/2}}$$

$$\frac{G\left(\mathbb{M}\delta\, r2(t)^3 - \mathbb{M}\delta\cos(\theta1(t)-\theta2(t))\,r1(t)\,r2(t)^2 + \mathbb{M}\left(r1(t)^2-2\cos(\theta1(t)-\theta2(t))\,r2(t)\,r1(t)+r2(t)^2\right)^{3/2}\right)}{r2(t)^2\left(r1(t)^2-2\cos(\theta1(t)-\theta2(t))\,r2(t)\,r1(t)+r2(t)^2\right)^{3/2}} + r2''(t) == r2(t)\,\theta2'(t)^2$$

If $\delta = 0$ we recover the original equations which we know the solutions to. The plan is to develop a sequence of corrections terms in the form

$$\begin{aligned}
\text{r1}[t] &= \text{r1}_0[t] + \delta\,\text{r1}_1[\text{t}] + \delta^2\,\text{r1}_2[t] + \ldots \\
\theta 1[t] &= \theta 1_0[t] + \delta\,\theta 1_1[\text{t}] + \delta^2\,\theta 1_2[t] + \ldots \\
\text{r2}[t] &= \text{r2}_0[t] + \delta\,\text{r2}_1[\text{t}] + \delta^2\,\text{r2}_2[t] + \ldots \\
\theta 2[t] &= \theta 2_0[t] + \delta\,\theta 2_1[\text{t}] + \delta^2\,\theta 2_2[t] + \ldots
\end{aligned}$$

In practice we sincereley hope that we do not need to compute too many terms in this series! For the moment lets just do the first corrections.

```
FullForm[r1'[t]]
```

```
Derivative[1][r1][t]
```

```
{Newδ Eq1θ[δ_], Newδ Eq1r[δ_], Newδ Eq2θ[δ_], Newδ Eq2r[δ_]} =
 {δ Eq1θ, δ Eq1r, δ Eq2θ, δ Eq2r} /. {
   r1[t] → r1₀[t] + δ r1₁[t] + δ² r1₂[t],
   Derivative[n_][r1][t] → Derivative[n][r1₀][t] +
     δ Derivative[n][r1₁][t] + δ² Derivative[n][r1₂][t],
   θ1[t] → θ1₀[t] + δ θ1₁[t] + δ² θ1₂[t],
   Derivative[n_][θ1][t] → Derivative[n][θ1₀][t] +
     δ Derivative[n][θ1₁][t] + δ² Derivative[n][θ1₂][t],
   r2[t] → r2₀[t] + δ r2₁[t] + δ² r2₂[t],
   Derivative[n_][r2][t] → Derivative[n][r2₀][t] +
     δ Derivative[n][r2₁][t] + δ² Derivative[n][r2₂][t],
   θ2[t] → θ2₀[t] + δ θ2₁[t] + δ² θ2₂[t],
   Derivative[n_][θ2][t] → Derivative[n][θ2₀][t] +
     δ Derivative[n][θ2₁][t] + δ² Derivative[n][θ2₂][t]
  }
```

$$\Big\{(G\,\mathbb{M}\,\gamma\,\delta\,\sin(\theta 1_2(t)\,\delta^2-\theta 2_2(t)\,\delta^2+\theta 1_1(t)\,\delta-\theta 2_1(t)\,\delta+\theta 1_0(t)-\theta 2_0(t))\,(\mathrm{r}2_2(t)\,\delta^2+\mathrm{r}2_1(t)\,\delta+\mathrm{r}2_0(t)))\Big/$$
$$\Big((\mathrm{r}1_2(t)\,\delta^2+\mathrm{r}1_1(t)\,\delta+\mathrm{r}1_0(t))^2-$$
$$2\cos(\theta 1_2(t)\,\delta^2-\theta 2_2(t)\,\delta^2+\theta 1_1(t)\,\delta-\theta 2_1(t)\,\delta+\theta 1_0(t)-\theta 2_0(t))\,(\mathrm{r}2_2(t)\,\delta^2+\mathrm{r}2_1(t)\,\delta+$$
$$\mathrm{r}2_0(t))\,(\mathrm{r}1_2(t)\,\delta^2+\mathrm{r}1_1(t)\,\delta+\mathrm{r}1_0(t))+(\mathrm{r}2_2(t)\,\delta^2+\mathrm{r}2_1(t)\,\delta+\mathrm{r}2_0(t))^2\Big)\text{^}(3/2)+$$
$$2\,(\mathrm{r}1_2'(t)\,\delta^2+\mathrm{r}1_1'(t)\,\delta+\mathrm{r}1_0'(t))\,(\theta 1_2'(t)\,\delta^2+\theta 1_1'(t)\,\delta+\theta 1_0'(t))+$$
$$(\mathrm{r}1_2(t)\,\delta^2+\mathrm{r}1_1(t)\,\delta+\mathrm{r}1_0(t))\,(\theta 1_2''(t)\,\delta^2+\theta 1_1''(t)\,\delta+\theta 1_0''(t))==$$
$$0,\ \mathrm{r}1_2''(t)\,\delta^2+\mathrm{r}1_1''(t)\,\delta+$$
$$\Big(G\Big(\mathbb{M}\,\gamma\,\delta\,(\mathrm{r}1_2(t)\,\delta^2+\mathrm{r}1_1(t)\,\delta+\mathrm{r}1_0(t))^3-\mathbb{M}\,\gamma\,\delta\cos(\theta 1_2(t)\,\delta^2-\theta 2_2(t)\,\delta^2+\theta 1_1(t)\,\delta-\theta 2_1(t)\,\delta+$$
$$\theta 1_0(t)-\theta 2_0(t))\,(\mathrm{r}2_2(t)\,\delta^2+\mathrm{r}2_1(t)\,\delta+\mathrm{r}2_0(t))\,(\mathrm{r}1_2(t)\,\delta^2+\mathrm{r}1_1(t)\,\delta+\mathrm{r}1_0(t))^2+$$
$$\mathbb{M}\Big((\mathrm{r}1_2(t)\,\delta^2+\mathrm{r}1_1(t)\,\delta+\mathrm{r}1_0(t))^2-2\cos(\theta 1_2(t)\,\delta^2-\theta 2_2(t)\,\delta^2+\theta 1_1(t)$$
$$\delta-\theta 2_1(t)\,\delta+\theta 1_0(t)-\theta 2_0(t))\,(\mathrm{r}2_2(t)\,\delta^2+\mathrm{r}2_1(t)\,\delta+\mathrm{r}2_0(t))$$
$$(\mathrm{r}1_2(t)\,\delta^2+\mathrm{r}1_1(t)\,\delta+\mathrm{r}1_0(t))+(\mathrm{r}2_2(t)\,\delta^2+\mathrm{r}2_1(t)\,\delta+\mathrm{r}2_0(t))^2\Big)\text{^}(3/2)\Big)\Big)\Big/$$
$$\Big((\mathrm{r}1_2(t)\,\delta^2+\mathrm{r}1_1(t)\,\delta+\mathrm{r}1_0(t))^2\,\Big((\mathrm{r}1_2(t)\,\delta^2+\mathrm{r}1_1(t)\,\delta+\mathrm{r}1_0(t))^2-2\cos(\theta 1_2(t)\,\delta^2-$$
$$\theta 2_2(t)\,\delta^2+\theta 1_1(t)\,\delta-\theta 2_1(t)\,\delta+\theta 1_0(t)-\theta 2_0(t))\,(\mathrm{r}2_2(t)\,\delta^2+\mathrm{r}2_1(t)\,\delta+\mathrm{r}2_0(t))$$
$$(\mathrm{r}1_2(t)\,\delta^2+\mathrm{r}1_1(t)\,\delta+\mathrm{r}1_0(t))+(\mathrm{r}2_2(t)\,\delta^2+\mathrm{r}2_1(t)\,\delta+\mathrm{r}2_0(t))^2\Big)\text{^}(3/2)\Big)+$$
$$\mathrm{r}1_0''(t)==(\mathrm{r}1_2(t)\,\delta^2+\mathrm{r}1_1(t)\,\delta+\mathrm{r}1_0(t))\,(\theta 1_2'(t)\,\delta^2+\theta 1_1'(t)\,\delta+\theta 1_0'(t))^2,$$
$$2\,(\mathrm{r}2_2'(t)\,\delta^2+\mathrm{r}2_1'(t)\,\delta+\mathrm{r}2_0'(t))\,(\theta 2_2'(t)\,\delta^2+\theta 2_1'(t)\,\delta+\theta 2_0'(t))+$$
$$(\mathrm{r}2_2(t)\,\delta^2+\mathrm{r}2_1(t)\,\delta+\mathrm{r}2_0(t))$$
$$(\theta 2_2''(t)\,\delta^2+\theta 2_1''(t)\,\delta+\theta 2_0''(t))==$$
$$(G\,\mathbb{M}\,\delta\,\sin(\theta 1_2(t)\,\delta^2-\theta 2_2(t)\,\delta^2+\theta 1_1(t)\,\delta-\theta 2_1(t)\,\delta+\theta 1_0(t)-\theta 2_0(t))$$
$$(\mathrm{r}1_2(t)\,\delta^2+\mathrm{r}1_1(t)\,\delta+\mathrm{r}1_0(t)))\Big/$$
$$\Big((\mathrm{r}1_2(t)\,\delta^2+\mathrm{r}1_1(t)\,\delta+\mathrm{r}1_0(t))^2-2\cos(\theta 1_2(t)\,\delta^2-\theta 2_2(t)\,\delta^2+\theta 1_1(t)\,\delta-\theta 2_1(t)\,\delta+\theta 1_0(t)-\theta 2_0(t))$$
$$(\mathrm{r}2_2(t)\,\delta^2+\mathrm{r}2_1(t)\,\delta+\mathrm{r}2_0(t))\,(\mathrm{r}1_2(t)\,\delta^2+\mathrm{r}1_1(t)\,\delta+\mathrm{r}1_0(t))+$$
$$(\mathrm{r}2_2(t)\,\delta^2+\mathrm{r}2_1(t)\,\delta+\mathrm{r}2_0(t))^2\Big)\text{^}(3/2),\ \mathrm{r}2_2''(t)\,\delta^2+\mathrm{r}2_1''(t)\,\delta+$$
$$\Big(G\Big(\mathbb{M}\,\delta\,(\mathrm{r}2_2(t)\,\delta^2+\mathrm{r}2_1(t)\,\delta+\mathrm{r}2_0(t))^3-\mathbb{M}\,\delta\cos(\theta 1_2(t)\,\delta^2-\theta 2_2(t)\,\delta^2+\theta 1_1(t)\,\delta-\theta 2_1(t)\,\delta+$$
$$\theta 1_0(t)-\theta 2_0(t))\,(\mathrm{r}1_2(t)\,\delta^2+\mathrm{r}1_1(t)\,\delta+\mathrm{r}1_0(t))\,(\mathrm{r}2_2(t)\,\delta^2+\mathrm{r}2_1(t)\,\delta+\mathrm{r}2_0(t))^2+$$
$$\mathbb{M}\Big((\mathrm{r}1_2(t)\,\delta^2+\mathrm{r}1_1(t)\,\delta+\mathrm{r}1_0(t))^2-2\cos(\theta 1_2(t)\,\delta^2-\theta 2_2(t)\,\delta^2+\theta 1_1(t)$$
$$\delta-\theta 2_1(t)\,\delta+\theta 1_0(t)-\theta 2_0(t))\,(\mathrm{r}2_2(t)\,\delta^2+\mathrm{r}2_1(t)\,\delta+\mathrm{r}2_0(t))$$
$$(\mathrm{r}1_2(t)\,\delta^2+\mathrm{r}1_1(t)\,\delta+\mathrm{r}1_0(t))+(\mathrm{r}2_2(t)\,\delta^2+\mathrm{r}2_1(t)\,\delta+\mathrm{r}2_0(t))^2\Big)\text{^}(3/2)\Big)\Big)\Big/$$
$$\Big((\mathrm{r}2_2(t)\,\delta^2+\mathrm{r}2_1(t)\,\delta+\mathrm{r}2_0(t))^2\,\Big((\mathrm{r}1_2(t)\,\delta^2+\mathrm{r}1_1(t)\,\delta+\mathrm{r}1_0(t))^2-2\cos(\theta 1_2(t)\,\delta^2-$$
$$\theta 2_2(t)\,\delta^2+\theta 1_1(t)\,\delta-\theta 2_1(t)\,\delta+\theta 1_0(t)-\theta 2_0(t))\,(\mathrm{r}2_2(t)\,\delta^2+\mathrm{r}2_1(t)\,\delta+\mathrm{r}2_0(t))$$
$$(\mathrm{r}1_2(t)\,\delta^2+\mathrm{r}1_1(t)\,\delta+\mathrm{r}1_0(t))+(\mathrm{r}2_2(t)\,\delta^2+\mathrm{r}2_1(t)\,\delta+\mathrm{r}2_0(t))^2\Big)\text{^}(3/2)\Big)+$$
$$\mathrm{r}2_0''(t)==(\mathrm{r}2_2(t)\,\delta^2+\mathrm{r}2_1(t)\,\delta+\mathrm{r}2_0(t))\,(\theta 2_2'(t)\,\delta^2+\theta 2_1'(t)\,\delta+\theta 2_0'(t))^2\Big\}$$

We want to compute the series expansion of these equations. Plugging in gives the equatiosn we have already solved analytically!

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{New$\delta$Eq1$_\theta$[0], New$\delta$Eq1$_r$[0], New$\delta$Eq2$_\theta$[0], New$\delta$Eq2$_r$[0]}
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$$\left\{2\,\mathrm{r1}_0'(t)\,\theta1_0'(t)+\mathrm{r1}_0(t)\,\theta1_0''(t)==0,\ \frac{G\mathbb{M}}{\mathrm{r1}_0(t)^2}+\mathrm{r1}_0''(t)==\mathrm{r1}_0(t)\,\theta1_0'(t)^2,\right.$$
$$\left.2\,\mathrm{r2}_0'(t)\,\theta2_0'(t)+\mathrm{r2}_0(t)\,\theta2_0''(t)==0,\ \frac{G\mathbb{M}}{\mathrm{r2}_0(t)^2}+\mathrm{r2}_0''(t)==\mathrm{r2}_0(t)\,\theta2_0'(t)^2\right\}$$

Differentiating with respect to δ and plugging in $\delta = 0$ is a slick way to isloate the terms that are linear in δ

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TableForm[
 {New$\delta$Eq1$_\theta$'[0], New$\delta$Eq1$_r$'[0], New$\delta$Eq2$_\theta$'[0], New$\delta$Eq2$_r$'[0]}
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$$\frac{G\mathbb{M}\gamma\sin(\theta1_0(t)-\theta2_0(t))\,\mathrm{r2}_0(t)}{\left(\mathrm{r1}_0(t)^2-2\cos(\theta1_0(t)-\theta2_0(t))\,\mathrm{r2}_0(t)\,\mathrm{r1}_0(t)+\mathrm{r2}_0(t)^2\right)^{3/2}}+2\,\mathrm{r1}_1'(t)\,\theta1_0'(t)+2\,\mathrm{r1}_0'(t)\,\theta1_1'(t)+\mathrm{r1}_1(t)\,\theta1_0''(t)+\mathrm{r1}_0(t)\,\theta1_1''(t)==0$$

$$-\frac{2\,G\mathbb{M}\,\mathrm{r1}_1(t)}{\mathrm{r1}_0(t)^3}-\frac{3\,G\mathbb{M}\,(2\,\mathrm{r1}_0(t)\,\mathrm{r1}_1(t)-2\cos(\theta1_0(t)-\theta2_0(t))\,\mathrm{r2}_0(t)\,\mathrm{r1}_1(t)-2\cos(\theta1_0(t)-\theta2_0(t))\,\mathrm{r1}_0(t)\,\mathrm{r2}_1(t)+2\,\mathrm{r2}_0(t)\,\mathrm{r2}_1(t)-2\sin(\theta1_0(t)-\theta2_0(t))\,\mathrm{r1}_0(t)\,\mathrm{r}}{2\,\mathrm{r1}_0(t)^2\left(\mathrm{r1}_0(t)^2-2\cos(\theta1_0(t)-\theta2_0(t))\,\mathrm{r2}_0(t)\,\mathrm{r1}_0(t)+\mathrm{r2}_0(t)^2\right)}$$

$$2\,\mathrm{r2}_1'(t)\,\theta2_0'(t)+2\,\mathrm{r2}_0'(t)\,\theta2_1'(t)+\mathrm{r2}_1(t)\,\theta2_0''(t)+\mathrm{r2}_0(t)\,\theta2_1''(t)==\frac{G\mathbb{M}\sin(\theta1_0(t)-\theta2_0(t))\,\mathrm{r1}_0(t)}{\left(\mathrm{r1}_0(t)^2-2\cos(\theta1_0(t)-\theta2_0(t))\,\mathrm{r2}_0(t)\,\mathrm{r1}_0(t)+\mathrm{r2}_0(t)^2\right)^{3/2}}$$

$$-\frac{2\,G\mathbb{M}\,\mathrm{r2}_1(t)}{\mathrm{r2}_0(t)^3}-\frac{3\,G\mathbb{M}\,(2\,\mathrm{r1}_0(t)\,\mathrm{r1}_1(t)-2\cos(\theta1_0(t)-\theta2_0(t))\,\mathrm{r2}_0(t)\,\mathrm{r1}_1(t)-2\cos(\theta1_0(t)-\theta2_0(t))\,\mathrm{r1}_0(t)\,\mathrm{r2}_1(t)+2\,\mathrm{r2}_0(t)\,\mathrm{r2}_1(t)-2\sin(\theta1_0(t)-\theta2_0(t))\,\mathrm{r1}_0(t)\,\mathrm{r}}{2\,\mathrm{r2}_0(t)^2\left(\mathrm{r1}_0(t)^2-2\cos(\theta1_0(t)-\theta2_0(t))\,\mathrm{r2}_0(t)\,\mathrm{r1}_0(t)+\mathrm{r2}_0(t)^2\right)}$$

Although these look very formidable! They are not actually that bad because we already know all the terms with zero subscripts. In fact, they are linear ODEs for the functions with subscripts 1: if you look real hard. So they are explicitly solvable. Once you have those you can proceed to the next level with the quadratic corrections etc.