FALLACIES, FLAWS, AND FLIMFLAM

EDITOR

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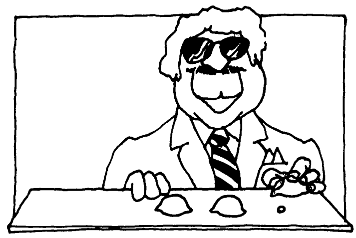

This column solicits mistakes, fallacies, howlers, anomalies and the like, that raise interesting mathematical issues and may be useful for teachers. Readers are invited to send submissions (which need not be original provided a full reference is given; photocopies would be appreciated), details about sources, and comments on material already published to Ed Barbeau.

The Problem of the Car and Goats

A year and a half ago (CMJ **22** (1991) 307–308), I drew attention to an analysis of four probability problems in *Mathematical Notes*, published by the Washington State University Mathematics Department. One of these became particularly notorious after receiving an airing in a *Parade* column of Marilyn Vos Savant. The resulting furore spilled over into the newspapers [see, for example, **4**, **14**, **28**, **37**, **40**, **41**, **45**]. Some of our colleagues were so moved by an ill-tempered response to Marilyn that one used about half his review of a book on trisections as a call for professional modesty [**52**], another took a page of MAA *Focus* to castigate ungallant and unprofessional behaviour [**23**] and a third noted that mathematicians themselves were not immune to innumeracy [**63**]. Undoubtedly, many editors along with those of the *CMJ* received manuscripts in the wake of Marilyn's column. However, as you might suspect, the problem is not new. As a service to the mathematical community, as well as to harried journal editors, I want to take note of a substantial literature on this and related problems. Some of the references were provided by others, including R. S. Lockhart and D. F. Andrews of my own institution. I am especially indebted to Domenico Rosa of the Teikyo Post University in Waterbury, CT. Rosa has made a hobby of collecting material on the problem and, as a tireless proponent of Bayes's theorem, has communicated his views in various professional and public outlets [**28**, **40**, **41**, **42**]. From him I received photocopies of all the newspaper articles and many of the earlier references. He also directed me to what is perhaps the first reference to Marilyn's problem, complete with a picture of Marilyn, in a textbook [**53**]. My list is surely incomplete and I welcome additions from readers.

The problem, known variously as **Marilyn's Problem, The Monty Hall Problem** and **The Car-and-Goats Problem** reads as follows:

M: A contestant in a game show is given a choice of three doors. Behind one is a car; behind each of the other two, a goat. She selects Door A. However, before the door is opened, the host opens Door C and reveals a goat. He then asks the contestant: "Do you want to switch your choice to Door B?" Is it to the advantage of the contestant (who wants the car) to switch? [**1–4**, **11–16**, **23**, **24**, **27–33**, **35–38**, **40–42**, **44–48**, **51–53**, **55**, **57–59**, **62–63**]

Here are some problems referred to in the literature as being equivalent or related. A good analysis of several of them appears in [**6**].

S: *The Shell Game*: A confidence man places a pea under one of three shells, out of sight of his mark. He then asks the mark to select the shell with the pea. After the mark picks a shell, the confidence man turns over one of the remaining shells, revealing no pea. He then asks the mark if he wishes to change his choice. Should the mark do so? [**22**, **37**, **44**]

P: *The Prisoner Paradox*: Two of three prisoners are to be executed, but no one of the prisoners knows which. One, A say, asks the guard: "Which of the other two is going to be executed? One of them will be and you will be giving me no information by telling me his name." The guard agrees and tells him that C is to be executed. A now thinks: "Before the guard said anything, my chances of being executed were 2 in 3. Now that I know it is either B or me, my chances are 1 in 2." Thus, the guard really has given information. [**6**, **21**, **22**, **34**, **39**, **44**, **50**, **54**]

B: *Bertrand Box Problem*: Each of three boxes has two drawers. Each drawer of the first has a gold coin; each drawer of the second has a silver coin; the third box has a gold coin in one drawer and a silver coin in the other. A box is chosen at random and a drawer opened to reveal a gold coin. What is the probability that the coin in the other drawer is silver? [**7**, **8**, **18**, **33**]

C: *Three Cards Problem*: In a hat are three cards. Both sides of one are black; both sides of a second are red; one side of the third is black while the other side is red. One card, selected at random, is placed on the table. A red side is showing. What is the probability that the other side is black? [**6**, **10**, **43**]

F: *Second Sibling Paradox*: A family has two children, at least one of which is a boy. What is the probability that one is a girl? Does the answer change if it is given that the *eldest* child is a boy? [**6**, **21**, **22**, **39**, **61**]

A: *Paradox of the Second Ace*: What is the probability that a hand of two cards dealt from a deck consisting of the aces of hearts and spades and the jacks of hearts and spades contains two aces given that it contains (a) at least one ace; (b) the ace of spades. Answers: (a) $1/5$; (b) $1/3$. [**5**, **6**, **9**, **17**, **19–22**, **24–26**, **49**, **60**]

R: *Restricted Choice*: You are South and declarer. North (dummy) holds ♠K10 × × while you hold ♠A × × × ×. The remaining spades ♠QJ × × are in the other two hands. A low spade is led from the North hand; East produces the Queen which you take with the Ace, West following with a low spade. It is now your lead. Should you lead towards the King in the hope that one opponent has the singleton Jack, or should you expect West to have ♠J × and plan to finesse (*i.e.*, play the Ten from the dummy if West plays low)? [**24**, **25**]

An instinctive response to Problem **M** is that the probability of having a car behind Door A rises to $1/2$, so that there is no point in switching. Marilyn's assertion that a switch would succeed with probability $2/3$ was vehemently challenged by many of her readers, but she eventually managed to win over most of her critics.

Many solvers felt that some clarification was necessary. For example, if one explicitly assumes that (1) the host will always open an unselected door concealing a goat, and (2) the contestant will always be offered a chance to switch, then the initial choice of the contestant can be regarded as a "parking" choice whose effect is (possibly) to close off one of the options available to the host. Thus, in this way, the host will in effect be giving positive information to the contestant in the event (with probability $2/3$) that the contestant initially chooses a door concealing a goat. This intuitively suggests that the odds for doors A and B concealing the car may not be the same.

Some analysts [**23**, **24**] distinguish among different games. Thus, the contestant may decide before the game begins whether to switch when given the opportunity. Alternatively, one might consider the possibility of a win if the contestant switches given that the host opens Door C, thus opening the discussion to conditional probability and Bayes's theorem. In this formulation, the probability q that the host opens Door C when the car is behind Door A becomes material. In an extreme case, if $q = 0$, the contestant always wins by switching.

It may happen that the host is free to decide not to open any door at all. Indeed, he may be more likely to open a losing door when the contestant has already selected a winning door. Tierney [**55**] consulted Monty Hall himself, who emphasized the psychological factor involved in the game. In a simulation, Hall awarded a goat right away if this were the contestant's first choice. Only if the contestant chose a door with the car did he open a second door and try to lure the contestant away by a cash inducement. (He called this the Henry James treatment, in recognition of James' story, *The Turn of the Screw*.) In this situation, the contestant would be foolish to switch.

Finally, one might consider the possibility that the host himself is ignorant of the location of the prize. *He* may reveal the car and the contestant loses right away.

In the literature, a number of ways of tackling the problem were used. Some of these were critically examined in [**33**], with a response given by vos Savant in [**58**].

(1) Analogy: One would surely switch if there were 1,000,000 doors and, after the contestant's initial selection, the host opened all the remaining doors but one to reveal a goat behind each. [**57**] See [**13**, **32**] for a generalization in which there are n doors, one of which the host opens after the initial choice of the contestant.

(2) How to switch and win: The car is behind either Door B or Door C with probability $2/3$. The host's action allows the contestant to rule out Door C if she decides against A.

(3) Use of conditional probability and Bayes's formula $(P(A)P(B|A) = P(B)P(A|B))$. [**11**, **24**, **29**, **31**, **33**, **42**, **44**]

(4) Analysis of sample space; enumeration of equally likely events. [**31**, **46**, **47**, **57**]

(5) Use of tree diagram. [**11**, **48**]

(6) Simulation. [**27**, **48**, **57**]

The problems **S** and **P** are isomorphic to **M**. In [**37**] is recounted how Charles E. Ford of St. Louis University used the shell game to raise money at a church picnic. While Ford was not looking, the contestant was to slip a quarter under one of three inverted cups. Ford put his finger on one cup but did not pick it up; he told the contestant to pick up one of the other two which did not cover the quarter. Ford always switched his choice, and more often than not chose the cup with the quarter, raising about eight dollars for the church. Problems **B** and **C** are also the same, but raise the same need for care in specifying the sample space. To heighten the similarity to problem **M**, it is of interest to look at Bertrand's own treatment of **B** [**7**]:

> Three boxes have identical appearance. Each has two drawers; each drawer contains a coin. The coins in the first box are of gold; those in the second are of silver; the third box contains a gold and a silver coin. You select a box. What is the probability of finding a gold and a silver coin in its drawers? There are three equally likely possibilities since the three boxes appear identical. Only one is favorable. The probability is $1/3$.
> The box is selected. You open a drawer. Whatever coin you find, only two cases remain possible. The second drawer could contain a coin whose substance may or may not differ from that in the first drawer.
> Of the two cases, only one is favorable to the box having different coins. The probability of having taken this box is $1/2$. How can we accept, however, that it suffices to open a drawer to raise the probability from $1/3$ to $1/2$? The reasoning is not valid. This is not in fact so. After one box is opened, two cases are possible. Of the two cases, only one is favorable, it is true, but the two cases are not equally likely. If a gold coin is seen, the other could be of silver, but one would have a stronger bet that it would be of gold.
> Suppose to make the argument more graphic, that in place of three boxes there are three hundred. One hundred contain two gold coins, one hundred two silver coins, and one hundred a gold and a silver coin. You open a drawer of each box; you see, therefore, three hundred coins, one hundred of gold, one hundred of silver, for sure; the other hundred are indeterminant; they belong to boxes for which the coins are not the same; chance determines the number. We should expect, on opening three hundred drawers, to see fewer than two hundred gold coins. The probability that the first one we see belongs to one of the one hundred boxes for which the other is gold is thus greater than $1/2$.

Len Gillman, with whom I shared an earlier draft of this note, gave me reference [**50**]. He also sent me Alan Truscott's response to a query, in which Truscott confirmed having introduced the idea of restricted choice in the Contract Bridge Journal in 1953 or 1954, and noted that it was discussed in Terence Reese' book, *The expert game*, in 1958 ("one of the best books ever written on card play").

Lest anyone think that Marilyn's days of trial are over, note that problems **A** and **F** have reared their heads in her column [**60, 61**].

Incidentally, FFF #13 (CMJ 21 (1990) 35; 22 (1991) 308–309) concerning a switch that may earn double or half an amount of money makes a brief appearance in the response of Shaughnessy and Dick to the letter of Hecht [**27**].

References

1. 'Let's make a deal' revisited, (On the lighter side), *The Actuary*, March, 1991; April, 1991 **M**
2. Cecil Adams, The straight dope, (Response to query of Robert E. Johanson), *Chicago Reader* (Isthmus, The Culture), Dec. 7, 1990 **M**
3. Igor Alexeff, Ask Marilyn, (Letter), *Mensa Bulletin*, July/August, 1991. Page 13. **M**
4. Karl V. Amatneek, Letter to the editor, *New York Times*, Aug. 11, 1991 **M**
5. W. W. Rouse Ball & H. S. M. Coxeter, *Mathematical Recreations and Essays*, 12th edition, University of Toronto, 1974, Page 44. **A**

6. Maya Bar-Hillel & Ruma Falk, Some teasers concerning conditional probabilities, *Cognition* 11 (1982), 109–122 **PCFA**
7. J. Bertrand, *Calcul des Probabilités*, Gauthier-Villars, 1888. Page 2. **B**
8. D. Brunk, *An Introduction to Mathematical Statistics*, Ginn-Blaisdell, 1965. Page 31. **B**
9. Donald C. Butler, Letter to the editor, *American Statistician* 20 (No. 5, Dec., 1966), 36–37 **A**
10. Stephen K. Campbell, *Flaws and Fallacies in Statistical Thinking*, Prentice-Hall, 1974. Page 132. **C**
11. Young H. Chun, Game show problem, (Letters to the editor), *OR/MS Today* (June, 1991), 9 **M**
12. Darwin DeLapp, The three boxes, Letters to the editor, *Bostonia* (July/August, 1991), 10 **M**
13. Robert Frankel, Monty's return, (Reader reflections), *Mathematics Teacher* 85 (1992), 176 **M**
14. Cory Franklin, Letter to the editor, *New York Times*, Aug. 11, 1991 **M**
15. Kendrick Frazier, 'Three door' problem provokes letters, controversy, John Geohegan, SI readers show their stuff, (Follow-up and letters), *Skeptical Inquirer* 16 (1992), 192–199 **M**
16. Sidney Freeman, The three boxes, (Letters to the editor), *Bostonia* (July/August, 1991), 10 **M**
17. John E. Freund, Puzzle or paradox?, *American Statistician* 19 (No. 4, Oct., 1965), 29, 44 **A**
18. John E. Freund, *Modern Elementary Statistics*, Seventh edition, Prentice-Hall, (1952), 1988. Page 153. **B**
19. George Gamow & Marvin Stern, *Puzzle Math*, Viking, NY, 1958 **A**
20. Martin Gardner, *Scientific American* 196 (#4, April, 1957), 166. Reprinted in *Mathematical Puzzles and Diversions*, Simon & Schuster, 1959. Pages 49–51. **A**
21. Martin Gardner, *Scientific American* 200 (#5, May, 1959), 166; 200 (#6, June, 1959), 164; 201 (#4, Oct., 1959), 180, 182; 201 (#5 Nov. 1959), 188. Reprinted in *Second Scientific American Book of Mathematical Puzzles and Diversions* Simon & Schuster, 1961 **PFA**
22. Martin Gardner, Probability paradoxes, (Notes of a fringe-watcher), *Skeptical Inquirer* 16 (1992), 129–132 **SPFA**
23. Leonard Gillman, The car-and-goat fiasco, *MAA Focus*, Vol. 11, No. 3, June-July, 1991. Page 8 **M**
24. Leonard Gillman, The car and the goats, *American Mathematical Monthly* 99 (1992), 3–7 **MAR**
25. N. T. Gridgeman, Letter to the editor, *American Statistician* 21 (No. 3, June, 1967), 38–39 **A**
26. Ralph Haertel, Letter to the editor, *American Statistician* 20 (No. 1, Feb., 1966), 34–36 **A**
27. Deborah Hecht, Simple dilemma, (Reader Reflections), *Mathematics Teacher* 85 (1992), 90–91 **M**
28. David Jacobson, Read this! It's got math and a bit of everything, *Hartford Courant*, March 1, 1991 **M**
29. Jeremy Kahan, Alternative solution for Monty, (Reader Reflections), *Mathematics Teacher* 85 (1992), 90 **M**
30. S. Knight, Let's make a deal, (Reader Reflections) *Mathematics Teacher* 85 (1992), 250, 252 **M**
31. Anthony Lo Bello, Ask Marilyn: the mathematical controversy in *Parade Magazine*, *Mathematical Gazette* 75 (Oct., 1991), 275–277 **M**
32. Janet S. Milton & David L. Albig, To switch or ..., (Reader Reflections), *Mathematics Teacher* 85 (1992), 8, 10 **M**
33. J. P. Morgan, N. R. Chaganty, R. C. Dahiya & M. J. Doviak, Let's make a deal, (Comment by Richard G. Seymann), *American Statistician* 45 (1991), 284–289 **MB**
34. F. Mosteller, Fifty Challenging Problems in Probability with Solutions, Addison Wesley, 1965. Problem 13, page 28. **P**
35. Richard Pearson, The three boxes, Letters to the editor, *Bostonia* (July/August, 1991), 9–10 **M**
36. Massimo Piattelli-Palmarini, Probability blindness: neither rational nor capricious, *Bostonia* (Boston University) (March/April, 1991), 28–35. Sidebar C: The three boxes. Pages 33–34. **M**
37. E. F. Porter, Mathematicians are the goat when they challenge Marilyn vos Savant on probability. *St. Louis Post Dispatch*, Tues., March 19, 1991 **MS**
38. Gary P. Posner, Nation's mathematicians guilty of 'innumeracy', *Skeptical Inquirer* 15 (1991), 342–345 **M**
39. Anatol Rapoport, *Decision Theory and Decision Behaviour*, Kluwer, 1989. Pages 72–75. **PF**
40. Michael Rapoport, Brain boggler no big deal to professor, *Waterbury Republican-American*, Feb. 20, 1991 **M**
41. Domenico Rosa, Letter to the editor, *New York Times*, Aug. 11, 1991 **M**
42. Domenico Rosa, Letter to the editor, *American Statistician* (submitted) **M**
43. Sheldon Ross, *A First Course in Probability*. (2nd ed., 1984), 70 **C**
44. Sam C. Saunders, The shell game, prisoner paradox and other paradoxes of conditional probability, *Mathematical Notes* from Washington State University, Vol. 33, No. 2, April, 1990 **MSPF**
45. Lore Segal, Letter to the editor, *New York Times*, Aug. 11, 1991 **M**
46. Steve Selvin, A problem in probability, (Letters to the editor), *American Statistician* 29 (1975), 67 **M**

47. Steve Selvin, On the Monty Hall problem, (Letters to the editor), *American Statistician* 29 (1975), 134 **M**
48. J. Michael Shaughnessy & Thomas Dick, Monty's dilemma: should you stick or switch?, *Mathematics Teacher* 84 (1991), 252–256 **M**
49. H. Fairfield Smith, "Puzzle or paradox?"-and Bayes, (Letter to the editor), *American Statistician* (No. 2, April, 1967), 42–44 **A**
50. Norton Starr, A paradox in probability theory, *Mathematics Teacher* 66 (1973), 166–168 **P**
51. Ira S. Steinberg, The three boxes, (Letters to the editor), *Bostonia* (July/August, 1991), 8–9 **M**
52. Ian Stewart, A budget of trisections, (Book review), *Mathematical Intelligencer* 14 (1992), 73–74 **M**
53. James Stewart, Lothar Redlin & Saleem Watson, *College Algebra*, Brooks/Cole, 1992. Pages 469–471. **M**
54. Gábor J. Székely, *Paradoxes in Probability Theory and Mathematical Statistics*, Akademiai Kiadó, Budapest; D. Reidel, Dordrecht, Holland, 1986. Pages 68–69. **P**
55. John Tierney, Behind Monty Hall's doors: puzzle, debate and answer?, *New York Times*, July 21, 1991 **M**
56. Alan Truscott, Understanding restricted choice is a guide to choosing right door, and the right play, Syndicated bridge column, Aug. 4, 1991 **R**
57. Marilyn vos Savant, Ask Marilyn, *Parade Magazine*, Sept. 9, 1990; Dec. 2, 1990; Feb. 17, 1991; July 7, 1991 **M**
58. Marilyn vos Savant, Marilyn vos Savant's reply, (Letters to the editor; response by J. P. Morgan, et al.), *American Statistician* 45 (1991), 347 **M**
59. Marilyn vos Savant, Ask Marilyn–the last word, (Letter), *Mensa Bulletin*, September, 1991 **M**
60. Marilyn vos Savant, Ask Marilyn, *Parade Magazine*, Sept. 8, 1991; Jan. 26, 1992 **A**
61. Marilyn vos Savant, Ask Marilyn, *Parade Magazine*, Oct. 13, 1991; Jan 5, 1992 **F**
62. Michael R. Wise, Letter to Marilyn, March 26, 1991. Reprinted in the *Prudential Newsletter*. **M**
63. Joseph Woolfson, Letter to the editor, *New York Times*, Aug. 11, 1991 **M**

The Car and the Gloats

...Insight is better without an accompanying gloat than with.

Paul R. Halmos, *I Want to Be a Mathematician*, MAA Spectrum, 1985, p.4.

Pólya Award Winners

The recipients of the George Pólya Award for mathematical exposition in the 1991 *College Mathematics Journal* have been announced:

William Dunham
Euler and the Fundamental Theorem of Algebra
College Mathematics Journal 22 (1991) 282-293

Howard Eves
Two Surprising Theorems on Cavalieri Congruence
College Mathematics Journal 22 (1991) 118-124