Geometric models of the card game SET

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Abstract

The card game SET can be modeled by four-dimensional vectors over $\mathbb{Z}_3$. These vectors correspond to points in the affine four-space of order three ($AG(4,3)$), where lines correspond to SETs, and in the affine plane of order nine ($AG(2,9)$). SETless collections and other aspects of the game of SET will be explored through caps in $AG(4,3)$ and conics in $AG(2,9)$.

1 Introduction

The game of SET is played using a deck of 81 cards, each of which is assigned color, number, shape, and shading. There are three different possibilities for each of the four attributes. The possible colors are red, green, and purple, the numbers are 1, 2, and 3, the shapes are diamond, oval, and squiggle, and the shadings are solid, striped, and open. To play the game, 12 cards are placed face up on a table, and players compete to find a SET of three cards such that each of the attributes are either all the same on the three cards, or all different. After a SET is found, the three cards are replaced, and play continues until the cards are partitioned into SETs, or until there are no SETs left.

Many mathophiles are attracted to the game of SET due to the pattern recognition skills needed to play it. The game of SET is much more than simple pattern recognition, however. It can be analyzed using mathematical models. This paper attempts to delve into the structure of the game of SET using various finite geometries.

2 Modeling SET in $AG(4,3)$

Let V be a four dimensional vector space over the field $\mathbb{Z}_3$. Each card may be represented by a vector in V, with entries from $\mathbb{Z}_3$ according to the card's attributes. Each coordinate represents a different attribute. Up to isomorphism, assume the first coordinate represents color, the second represents number, the third represents shape, and the fourth represents shading. Also up to isomorphism, coordinate values are assigned as seen in the following table. For example, a card with one green open squiggle would be represented as the vector $(0,-1,1,1)$.

	-1	0	1
Color	Red	Green	Purple
Number	One	Two	Three
Shape	Oval	Diamond	Squiggle
Shading	Solid	Striped	Open

From this vector space viewpoint, a SET is defined as three vectors that add to give $\vec{0}$. This satisfies the rules of the game of SET because if an attribute is the same on the three cards, then that coordinate position of the vector sum will be a multiple of three, which is congruent to 0 (mod 3), and if an attribute is different on every card, then the sum of the three corresponding entries will be zero.

This paper is primarily interested in interpreting the game of SET geometrically. To that end, we will begin with the model set forth by Davis and MacLagan in [2]. Let $AG(4,3)$ denote the four dimensional affine space of order three. In $AG(4,3)$, points are the 3^4 elements of V from above. Lines are cosets of one dimensional subspaces of V; that is, they have the form $\{\vec{v}+k\vec{w} : k \in \mathbb{Z}_3\}$, for any $\vec{v},\vec{w} \in V$ with $\vec{w} \neq \vec{0}$. Each line consists of 3 points, and there is a total of $\frac{3^4(3^4-1)}{3(3-1)} = 1080$ lines, since there are 3^4 possibilities for $\vec{v}$, 3^4-1 possibilities for $\vec{w}$, and $3(3-1)$ ways of describing each distinct coset due to scalar multiples. The connection between geometry and the game of SET is based on the Affine Collinearity Rule found in [2].

Theorem 2.1. *Three cards form a SET if and only if their corresponding points in $AG(4,3)$ are collinear.*

Proof. Three vectors, $\vec{v_1}, \vec{v_2}$, and $\vec{v_3}$, in $AG(4,3)$ form a SET if and only if $\vec{v_1}+\vec{v_2}+\vec{v_3} = \vec{0}$, which is equivalent to $\vec{v_1}+(\vec{v_1}+(\vec{v_2}-\vec{v_1}))+(\vec{v_1}-(\vec{v_2}-\vec{v_1})) = \vec{0}$, as $\vec{v_3} = -\vec{v_1} - \vec{v_2} = 2\vec{v_1} - \vec{v_2}$. Thus, vectors $\vec{v_1}, \vec{v_2}$, and $\vec{v_3}$ compose the set $\{\vec{v_1} + k(\vec{v_2} - \vec{v_1}) : k \in \mathbb{Z}_3\}$, which is a line of $AG(4,3)$. $\square$

Note that since there are 1080 lines in $AG(4,3)$, there are 1080 possible SETs in the deck of 81 SET cards.

3 Modeling SET in $AG(2,9)$

Let $GF(9)$ denote the finite field of order 9, which is a vector space over $\mathbb{Z}_3$. Let $\{1, \epsilon\}$ be a basis for $GF(9)$ over $\mathbb{Z}_3$, where $\epsilon \in GF(9) \setminus \mathbb{Z}_3$. Thus for any $x \in GF(9)$, x can be written uniquely as $x = x_1 + x_2\epsilon$, for $x_1, x_2 \in \mathbb{Z}_3$. As every point in $AG(2,9)$ can be represented by (x, y) for $x, y \in GF(9)$, there is a natural bijection between points of $AG(4,3)$ and points of $AG(2,9)$. Specifically, the point (x_1, x_2, x_3, x_4) corresponds to $(x_1 + x_2\epsilon, x_3 + x_4\epsilon)$. The points of $AG(2,9)$ will also be viewed as vectors in the two dimensional vector space over $GF(9)$, denoted $VS(2,9)$. Therefore, the 81 points of $AG(2,9)$ can be used to uniquely represent SET cards, just as the points of $AG(4,3)$ represent SET cards. When considering the point $(x_1 + x_2\epsilon, x_3 + x_4\epsilon)$ in $AG(2,9)$, the values of x_1, x_2, x_3, x_4 may be assigned as in the table on page 2 with each x_i representing a different SET attribute. Just as with $AG(4,3)$, a SET is defined as three points, $\vec{u}, \vec{v}$, and $\vec{w}$ in $AG(2,9)$, such that $\vec{u}+\vec{v}+\vec{w} = \vec{0}$, where $\vec{u}, \vec{v}$, and $\vec{w}$ are viewed as ordered pairs over $GF(9)$. Lines in $AG(2,9)$ are cosets of one-dimensional subspaces of $VS(2,9)$; therefore, they have the form $\{\vec{v} + k\vec{w} : k \in GF(9)\}$ for any $\vec{v}, \vec{w} \in VS(2,9)$ with $\vec{w} \neq \vec{0}$. Notice that each line contains 9 points, and there are 90 total lines. We have a similar collinearity rule in $AG(2,9)$ as was considered in $AG(4,3)$.

Theorem 3.1. *If three cards form a SET, then their representative points in $AG(2,9)$ are collinear.*

Proof. The proof of Theorem 2.1 immediately shows that if three points $\vec{v_1}, \vec{v_2}$, and $\vec{v_3}$ in $AG(2,9)$ form a SET, then they form the set $\{\vec{v_1}+k(\vec{v_2}-\vec{v_1}) : k \in \mathbb{Z}_3\}$. Thus, $\vec{v_1}, \vec{v_2}$, and $\vec{v_3}$ belong to the set $\{\vec{v_1}+k(\vec{v_2}-\vec{v_1}) : k \in GF(9)\}$, which is a line of $AG(2,9)$. $\square$

Notice that three points of $AG(2,9)$ can lie on the same line yet not sum to zero, hence not form a SET. However, three points of $AG(2,9)$ form a SET if and only if those points form an affine subline; that is, they have the form $\{\vec{v} + k\vec{w} : k \in \mathbb{Z}_3\}$ for $\vec{v}, \vec{w} \in AG(2,9)$ with $\vec{w} \neq \vec{0}$.

4 SETless Collections

One area of interest in analyzing the game of SET is the notion of a SETless collection of cards. A collection of cards is *SETless* if no three cards form a SET. Geometrically, SETless collections are modeled using caps. A *cap* is defined as a set of points, no three of which are collinear. From Theorem 2.1, we know that three representative vectors form a SET if and only if their points in $AG(4,3)$ are collinear. Thus, a cap would necessarily be a SETless collection. Using known theory of caps in $AG(4,3)$, Davis and MacLagan, in [2], noted that the largest possible SETless collection consists of 20 cards. These caps of maximal size 20 were discovered by Pellegrino in [5], which are further described by Hill in [3]. Therefore, we know that it is possible to find 20 SET cards such that no three of them form a SET. We also know that any collection of 21 cards is guaranteed to have at least one SET in it.

4.1 Conic partition of $AG(2,9)$

There are other SETless collections that are of interest as well. Using $AG(2,9)$ to model SET, we may partition the 81 SET cards into nine SETless collections, each of size nine. The partition arises in the projective plane $PG(2,9)$, which is the projective completion of $AG(2,9)$ obtained by adding points at infinity which correspond to the parallel classes of lines in the affine plane $AG(2,9)$. The affine points of $PG(2,9)$ have the form $(x,y,1)$ for $x,y \in GF(9)$ and naturally correspond to (x,y) in $AG(2,9)$; the points at infinity have the form $(x,1,0)$ and $(1,0,0)$ for $x \in GF(9)$.

A *conic* in $PG(2,9)$ is a set of 10 points satisfying a nondegenerate quadratic form. The conics that we will use contain one point at infinity, hence consist of 9 points of $AG(2,9)$. Since no three points of a conic are collinear, a conic in $AG(2,9)$ is a two-dimensional cap. Using the contrapositive of Theorem 3.1, a conic is a SETless collection.

For $\delta \in GF(9)$, define $C_\delta = \{(y^2+\delta, y, 1) : y \in GF(9)\} \cup \{(1,0,0)\}$. The set C_δ satisfies the nondegenerate quadratic form $xz = y^2 + \delta z^2$, hence forms a conic in $PG(2,9)$. It is not difficult to see that $C_\delta \cap C_{\delta'}$ is the point at infinity $(1,0,0)$ for $\delta \neq \delta'$. Let γ be a primitive element of $GF(9)$, and for a fixed $r \in \mathbb{Z}_3$, let $\mathcal{F}_r = \{C_{\gamma t+r} : t \in \mathbb{Z}_3\}$. It is shown in [1] that the 28 points covered by $\mathcal{F}_r$ form a configuration in $PG(2,9)$ known as a "unital." The three unitals obtained from $\mathcal{F}_0, \mathcal{F}_1$, and $\mathcal{F}_2$ pairwise meet at the point at infinity $(1,0,0)$. Moreover, each $\mathcal{F}_r$ is a family of three conics which also pairwise meet at $(1,0,0)$. Therefore, $\{\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2\}$ is a collection of three families of three conics each. These nine conics (with the point at infinity $(1,0,0)$ deleted from each) partition the affine plane $AG(2,9)$. In terms of the game of SET, we can partition the 81 SET cards into nine SETless collections, each of size nine.

4.2 Maximal SETless collection in $AG(2,9)$

The field $GF(9)$ is an extension field of $\mathbb{Z}_3$; therefore, it can be modeled by the set $\{a + b\alpha : a, b \in \mathbb{Z}_3\}$, where $\alpha^2 = -1$. Likewise, the field $GF(81)$ is a quadratic extension of $GF(9)$; therefore, it can be modeled by the set $\{c+d\beta : c, d \in GF(9)\}$, where $\beta^2 = \alpha + 1$. Thus the elements of $GF(81)$ are of the form $a+b\alpha+(c+d\alpha)\beta = a+b\alpha+c\beta+d\alpha\beta$ with $a, b, c, d \in \mathbb{Z}_3$. This can easily be applied to our model of SET, with a, b, c, and d representing the attributes of our card. Melissa Mills in [4] showed that the set $\{x \in GF(81) : x^{20} = 1\}$ is a maximal SETless collection. Thus, when these elements of $GF(81)$ are viewed as a collection of points $\{(a, b, c, d) \in AG(4,3) : (a+b\alpha+c\beta+d\alpha\beta)^{20} = 1\}$, they form Pellegrino's cap of size 20. When these elements are viewed as a collection of points $\{(a+b\alpha, c+d\alpha) \in AG(2,9) : (a+b\alpha+c\beta+d\alpha\beta)^{20} = 1\}$, however, they form an interesting configuration.

Using the computer-algebra software MAGMA, as an alternative to tedious algebraic manipulations, this set of 20 points in $AG(2,9)$ was shown to consist of two disjoint conics. The conics consist of ten points each (they contain no points at infinity), and each is in the interior of the other. Note that a point is in the interior of a conic if it lies on no tangent lines of that conic. The conics can be described by the equations $x^2 + \gamma^3 y^2 = 1$ and $-x^2 - \gamma^3 y^2 = 1$, where γ is a primitive element of $GF(9)$ such that $\gamma^2 = \alpha$. The interior points of the conics are $\{(x, y) : x^2 + \gamma^2 y^2 - 1$ is a square in $GF(9)\}$ and $\{(x, y) : -x^2 - \gamma^2 y^2 - 1$ is a square in $GF(9)\}$, respectively.

5 Leftover Collections

Define a *leftover* collection to be a nonempty collection of SET cards remaining after a game of SET has been completed. Note that a leftover collection will be SETless since a game of SET only ends when there are no SETs remaining. One important fact needed in analyzing leftover collections is that the sum of all vectors of the four dimensional vector space over $\mathbb{Z}_3$ is $\vec{0}$. This is the case because there are 27 vectors with 0 in the first coordinate, 27 with 1 in the first coordinate, and 27 with -1 in the first coordinate, and similarly with the other coordinates. Thus, when added, the sum is zero in each coordinate position. Since all of the SET cards can be represented by vectors of the four dimensional vector space over $\mathbb{Z}_3$, the sum of all SET cards is $\vec{0}$. This result can be used to analyze leftover collections.

Lemma 5.1. *The vectors in a leftover collection sum to* $\vec{0}$.

Proof. The sum of the cards in each individual SET that has been found throughout the game is $\vec{0}$. Therefore, the sum of the remaining cards after all SETs have been found is the sum of all cards minus the sum of each SET that was found, which is $\vec{0}$. □

This result and previous results can help to determine the possible sizes of a leftover collection.

Theorem 5.2. *The size L of a leftover collection is a multiple of three for which* $6 \leq L \leq 18$.

Proof. Note that the size of a leftover collection must be divisible by three, since cards are removed three at a time in the game of SET. Suppose that a leftover collection contains three cards. By Lemma 5.1, these three cards must sum to $\vec{0}$. Thus they form a SET, which is a contradiction since a leftover collection does not contain a SET. Therefore, since the size of a leftover collection must be divisible by three, a leftover collection must contain at least six cards.

Suppose that a leftover collection contains 21 cards. Then there is guaranteed to be a SET in those 21 cards since the largest possible SETless collection is 20. This is a contradiction since a leftover collection does not contain a SET. Therefore, since the size of a leftover collection must be divisible by three, a leftover collection can contain at most 18 cards. □

6 Conclusion

The game of SET is remarkable in that it can be completely described using mathematical structures. The vector space model is entirely sufficient in describing how to play the game of SET. Affine geometries, which arise from the vector space model, help us to better understand and visualize different aspects of the game of SET, such as SETless collections and leftover collections. These aspects might be explored even further by examining partial line spreads of $AG(4,3)$.

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