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A Short Survey on Preconditioning Techniques for Large Scale Dense Complex Linear Systems in Electromagnetics

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In solving systems of linear equations arising from practical scientific and engineering modeling and simulations such as electromagnetics applications, it is critical to choose a fast and robust solver. Due to the large scale of those problems, preconditioned Krylov subspace methods are most suitable. In electromagnetics simulations, the use of preconditioned Krylov subspace methods in the context of multilevel fast multipole algorithms (MLFMA) is particularly attractive. In this paper, we present a short survey on a few preconditioning techniques in this application. We also compare several preconditioning techniques combined with the Krylov subspace methods to solve large dense linear systems arising from electromagnetic scattering problems and present some numerical results.

1 Introduction

In computational electromagnetics (CEM), we frequently need to numerically solve large complex-valued system of linear equations of the form

$$Ax = b, \tag{1}$$

which arises from electromagnetics modeling, such as from electromagnetic wave propagation, scattering, and inverse scattering. The coefficient matrix A is usually a dense matrix if the integral equation formulation is employed.

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The electromagnetic wave scattering by three-dimensional (3D) arbitrarily shaped dielectric and conducting objects can be obtained by finding the solution of a hybrid integral equation [1]. The solution to electromagnetic wave interaction with material coated objects has applications in radar cross section prediction for coated targets, printed circuit, and microstrip antenna analysis [2–4]. In the applications of our interest, the linear equation (1) is generated from the hybrid integral equation by using the Method of Moments (MoM) [1, 5, 6], where the coefficient matrix A is large and dense, for electrically large targets in electromagnetic scattering. For some scattering structures, such as those from the hybrid integral equation formulations, the matrix A can be poorly conditioned.

There are other formulations to generate dense linear systems from electromagnetics simulations, such as that used by Carpentieri et al. [7, 8]. One can use Boundary Element Method (BEM) to generate symmetric non-Hermitian or nonsymmetric coefficient matrices from boundary integral equations in electromagnetic wave propagation phenomena. BEM is different from MoM. They both can result in dense coefficient matrices. Another reason to mention BEM here is that it is particularly useful for applications with moving objects because BEM does not need to discretize the space regions [7–10].

The matrix equation (1) can be solved by using direct solution methods or iterative solution methods. Direct solution methods, of which the Gauss elimination is representative, are widely used, when the problem size is not too large, and are considered more robust than iterative methods. Unfortunately, direct methods scale poorly with the problem size when the memory space and CPU time of the computing machine become an issue [11, 12], especially for problems arising from the discretization of integral equations in electromagnetic scattering or wave propagation. For standard Cholesky factorization or Gauss elimination, the floating-point operation count is on the order of $O(N^3)$, where N is the number of unknowns (columns) of the coefficient matrix. In contrary, the iterative methods, especially the Krylov subspace type iterative methods (or Krylov methods for short), can present an attractive alternative since they can achieve a computational complexity in the order of $O(N^{iter}N^2)$, where N^{iter} indicates the number of iterations needed to reach convergence and N^2 is the computational cost of a matrix-vector product operation, which accounts for the major cost of the Krylov methods. In electromagnetics simulations, the fast multipole method (FMM) approach is popularly utilized to reduce the computational complexity of a matrix-vector product from $O(N^2)$ to $O(N^{1.5})$ [13, 14]. With the multilevel fast multipole algorithm (MLFMA) the computational complexity can be further reduced to $O(N\log N)$ [15–18].

Due to their favorable storage and floating-point operation requirement, iterative methods are very popular not only in CEM applications, but also in nuclear and petroleum industry, and other application areas [19–21]. When

iterative methods are used for the solution of linear systems from practical applications, the lack of robustness is a widely recognized weakness, compared with the direction solution methods [22]. In some applications, when the coefficient matrix is poorly conditioned, we cannot get the convergence of an iterative solution process in a reasonable amount of time. This lack of robustness presents serious doubts on the acceptance of iterative methods in industrial application packages.

The robustness of the iterative methods can be enhanced by using preconditioning techniques. Most efficient preconditioning techniques are usually application specific and general-purpose preconditioners may not work well for particular applications. So, for using Krylov subspace methods to solve problems in CEM applications, we need to develop specialized preconditioning techniques in the context of the MLFMA to improve their performance and reliability.

It is well-known that, by using a preconditioner, the linear system (1) can be transformed into another equivalent system

$$M^{-1}Ax = M^{-1}b \tag{2}$$

with more favorable matrix spectrum properties, e.g., smaller condition number or more tightly clustered eigenvalues, for iterative solution methods [22,23]. Here M is a non-singular matrix and is called the preconditioner to take the effect of the transformation. A good preconditioner should be constructed inexpensively and should be a good approximation to the inverse of the coefficient matrix A. This means that the preconditioned iteration (Krylov methods applied on equation (2)) should converge faster than the unpreconditioned one and each preconditioned iteration should not be too expensive. Generally, we need M (or M^{-1}) to be a sparse matrix to maintain the efficiency and low storage need of the original features of the iterative methods.

How to construct an efficient preconditioner for a given coefficient matrix is not an easy work. Finding a good preconditioner to solve a given linear system can be considered as a combination of art and science [22], especially to solve large dense linear systems. The main purpose of this paper is to discuss some robust preconditioning techniques which can efficiently solve dense linear systems arising from some electromagnetics applications.

This short survey focuses more on the previous work of the authors, but we also discuss relevant works of the others in the paper. Due to the time and space constraints, it is unavoidable that some researchers' work might not be included in this survey. If that is the case, the omissions are unintentional.

2 Preconditioned iterative methods

Recently, the development and practice of efficient preconditioning techniques in iteratively solving dense matrices have become the subject of growing interest [9, 24–30]. For sparse matrices, polynomial preconditioners for Krylov subspace methods were popular in for some applications in early stage of preconditioning studies [31–33]. The incomplete factorization methods were first introduced by Varga [19, 34, 35], then Meijerink and van der Vorst recognized that they could be used as a preconditioner for the conjugate gradient method [36]. Another class of more recent alternative general-purpose preconditioning techniques for sparse matrices is sparse approximate inverse [37, 38].

After a brief review of basic preconditioning techniques, the following sections will present some recent research work for constructing efficient and robust preconditioners for large dense linear systems in electromagnetics.

2.1 *Incomplete factorization*

In this section we will focus on two factorization algorithms to construct preconditioners. One is using incomplete Cholesky factorization for symmetric dense coefficient matrices, and another is using incomplete LU (ILU) factorization for nonsymmetric dense coefficient matrices.

2.1.1 ***Incomplete Cholesky factorization.*** In the paper of Carpentieri et al. [9], the coefficient matrix $A = [a_{ij}]$ is dense, complex, symmetric and non-Hermitian, and arises from the discretization of boundary integral equations in electromagnetism. So, they use $\tilde{A}$, a sparse approximation of the dense coefficient matrix A, to construct the preconditioner.

There are several means to generate the sparse matrix $\tilde{A}$. It can be obtained by directly sparsifying the dense matrix A (removing small size entries) [39], or by utilizing the underlying physical properties during the process of building the dense matrix A [40].

The existence of the incomplete Cholesky factorization for symmetric positive definite matrices was proved in [36]. Some related research work can be found in [41–43]. The incomplete Cholesky factorization is normally denoted by IC for short. The constructed preconditioner M is in the form of $M = LDL^T$, where D is a diagonal matrix and L is a unit lower triangular matrix. The detailed algorithm for computing a preconditioner M by the IC factorization and some other strategies for discarding fill-in entries can be found in [22]. However, the result of this type of preconditioners seems not always good in many practical applications because the IC factorization of A may produce some triangular factors that are unstable or ill-conditioned and

result in a poor preconditioner. This happens especially when the matrix in question is only symmetric, but not positive definite. One can also find some related discussions in the paper of Chow and Saad [44].

2.1.2 ***Incomplete LU preconditioner.*** Incomplete LU (ILU) preconditioning techniques are widely used in solving sparse linear systems, and some good results were obtained in the solution of complex symmetric indefinite systems arising from the Helmholtz equation [45, 46]. But until recently, preconditioning techniques for solving dense matrices are less extensively studied [25, 47–49], compared to that for the sparse matrices. There are some strategies related to the sparsity pattern based ILU factorization preconditioning [50]. Many of these preconditioning techniques, such as ILU(0), rely on a fixed sparsity pattern, obtained from a sparsified coefficient matrix by dropping small magnitude entries. The ILU(0) preconditioner, which is constructed from a sparsified matrix, is shown to be inefficient to solve dense linear systems for some electromagnetic scattering problem [9, 51].

The following is the basic idea to construct an efficient ILU preconditioner from a specific dense matrix arising from the wave scattering problem in electromagnetism by Lee, Zhang, and Lu [51].

In the electromagnetic scattering simulation field, the popularity of the fast multipole methods (FMM) and the multilevel fast multipole algorithms (MLFMA) demands specialized preconditioning techniques that can be constructed with the special features of the FMM and MLFMA in mind. The diagonal and block diagonal preconditioners are considered by a few authors [15–17, 52] in the context of the FMM and MLFMA. Using the addition theorem for the free-space scalar Green's function, for an N-dimensional vector x, the matrix-vector product Ax can be written as [13, 15]

$$Ax = (A_D + A_N)x + V_f \Lambda V_s \, x,$$

where V_f, Λ, and V_s are sparse matrices. In fact, the dense matrix A can be structurally divided into three parts A_D, A_N and $A_F = V_f \Lambda V_s$. A_D is the block diagonal part of A, A_N is the block near-diagonal part of A, and A_F is the far part of A. Here the terms "near" and "far" refer to the distance between groups of elements defined in the FMM and MLFMA. An illustration of these submatrix sparsity patterns can be found in [51].

With robustness in consideration, Lee et al. [51] construct a preconditioner M from the ILU factorization of the near-part matrix $(A_D + A_N)$, and apply the preconditioner M to transfer the linear system (1) into the linear system (2). Note that although the preconditioner is constructed from the near-part matrix $(A_D + A_N)$, it is applied to precondition the full dense matrix A, which

is only implicitly available in the MLFMA [51].

There exist some high accuracy ILU type preconditioners to solve sparse matrices [53–55]. Many of them are constructed using either an enlarged sparsity pattern or using some threshold value based drop tolerance to allow more fill-in in the construction phase. Since memory cost is one of the major concerns in large scale electromagnetic scattering simulations, a preconditioner using a huge amount of memory space is not desirable. To avoid this, Lee et al. [51] use a dual dropping strategy (ILUT) of Saad [54] to control both the computational cost and the memory cost.

When ILU preconditioners are used in combination with the Krylov subspace methods, the preconditioned iteration tends to converge much faster than the unpreconditioned one. Since an ILUT preconditioner may be reused several times by solving the same matrix with several different right-hand side vectors in electromagnetics applications, it may be worthwhile to invest a little bit more computing resources to construct a high accuracy ILU type preconditioner with a slightly larger amount of fill-in, compared with the ILU(0) preconditioner with no fill-in. However, we need to guarantee that the computational and storage complexity of the whole implementations does not exceed that dictated by the FMM or MLFMA.

2.2 *Sparse approximate inverse preconditioner*

The common idea for constructing a sparse approximate inverse (SAI) preconditioner is that a sparse matrix $M \approx A^{-1}$ can be explicitly computed and used as a preconditioner for the Krylov subspace methods [23, 38]. There are two classes of methods to construct sparse approximate inverse preconditioners, one is the Frobenius-norm minimization method, another is the factorized sparse inverse method, also known as the AINV algorithm.

2.2.1 ***Frobenius-norm minimization.*** This class of sparse approximate inverse techniques was first introduced in [37], some other related work can be found in [56]. The basic idea is the attempt to find a sparse matrix M which minimizes the Frobenius norm of the residual matrix $I - AM$ as

$$\min_{M \in G} \|I - AM\|_F^2 = \sum_{j=1}^{n} \min_{m_j \in G_j} \|e_j - Am_j\|_2^2, \tag{3}$$

where G is a certain sparsity pattern constraint, e_j and m_j are the column vectors of the identity matrix I and the preconditioner matrix M. Each of the sub-minimization problems in the right-hand side of (3) can be solved

independently, which leads to the inherent parallelism claim in this class of SAI construction techniques.

The choice of the sparsity pattern constraint G will play the key role in constructing the sparse approximate inverse preconditioner. There are two classes of sparsity pattern selection strategies. They are commonly referred to as static and dynamic sparsity pattern selection strategies [57, 58]. For a given amount of storage space, the dynamic pattern selection strategy is more accurate than the static pattern selection strategy in general, but it needs more setup time to construct [59].

For the dense linear systems in electromagnetics problems, we can find applications of some sparse approximate inverse preconditioning techniques in [7,8,26,27,60,61]. Lee et al. [26] make use of the near-part matrix (A_D+A_N), as in the previous section for the ILU preconditioning, to construct a sparse approximate inverse preconditioner. The basic algorithm is as follow:

Algorithm *1* Computing an SAI matrix M from the near-part matrix $(A_D + A_N)$

1: Obtain a sparsified matrix $\tilde{B}_N$ from $(A_D + A_N)$ with respect to ϵ_1
2: Further sparsify $\tilde{B}_N$ with respect to ϵ_2 to obtain another sparse matrix $\tilde{C}_N$
3: Compute the sparse approximate inverse matrix M^* of the matrix $\tilde{B}_N$ based on the sparsity pattern of $\tilde{C}_N$
4: Further sparsify the matrix M^* with respect to ϵ_3 to obtain the preconditioner M

Here ϵ_1, ϵ_2 and ϵ_3 are three user input threshold parameters to sparsify the matrices at different stages of the construction process.

Another way to construct a sparse approximate inverse preconditioner from a dense matrix is provided by Carpentieri et al. [9]. They use $\tilde{A}$ (the approximation of the coefficient matrix) to compute the SAI preconditioner M. Again, the information of the underlying physical meshes is used in the construction of the sparse matrix $\tilde{A}$.

2.2.2 ***Factorized sparse approximate preconditioner.*** Factorized approximate inverse preconditioners for general sparse matrices can be efficiently constructed by using a biconjugation process [62, 63]. The basic idea is that if A can be factorized as $A = LDU$, where L is unit lower triangular, D is diagonal, and U is unit upper triangular, then $A^{-1} = U^{-1}D^{-1}L^{-1}$. Here, the matrices U^{-1} and L^{-1} may be dense triangular matrices, so we compute sparse approximate matrices $\tilde{Z} \approx U^{-1}$ and $\tilde{W} \approx L^{-1}$. Then the sparse approximate inverse preconditioner for the original matrix A can be written as

$M = \tilde{Z} D^{-1} \tilde{W} \approx A^{-1}$.

For the dense matrix, similar to the sparsification strategy used for the Frobenius-norm minimization SAI preconditioner, Carpentieri et al. [9] use the approximation of the coefficient matrix A to construct the preconditioner. For completeness, we mention that there is also another approach to decomposing the coefficient matrix using factorized sparse approximate inverse (FSAI) algorithm. The related research work can be found in the papers by Kolotilina and Yeremin [64, 65].

Based on our own experience with both sparse approximate inverse preconditioners and the ILU preconditioners, in the electromagnetics applications, we think that the sparse approximate inverse preconditioners can be more expensive to compute and are slightly less efficient than the ILU preconditioners, in terms of reducing the number of iterations of the Krylov subspace methods. However, there are exceptions. The biggest advantages of the sparse approximate inverse preconditioners are that they can achieve almost the same preconditioning effect of the ILU preconditioners, but use much less storage space, and they have good potential for implementations on high performance parallel computers.

2.3 *Block diagonal and SVD stabilized block diagonal preconditioner*

We can construct a standard block diagonal preconditioner A_D^{-1} from the block diagonal submatrix A_D, and apply the block diagonal preconditioner A_D^{-1} to the linear system (1) [16–18]. That is, $A_D^{-1} A x = A_D^{-1} b$. Here, A_D is a block diagonal submatrix consisting of several block submatrices such as $A_1, A_2, \cdots, A_m$. Each individual block can be decomposed independently by an LU factorization like $A_i = L_i U_i$, and the small size linear system for each block A_i can be solved as $L_i U_i x_i = b_i$. The block diagonal preconditioner A_D^{-1} is of m independently inverted submatrix $A_1^{-1}, A_2^{-1}, \cdots, A_m^{-1}$, where $A_i^{-1} = (L_i U_i)^{-1}$. The block diagonal preconditioner, although is not the most efficient one, provides a good structure for potential parallelism [16, 71].

It is possible that the block diagonal preconditioner sometimes fails to improve the convergence of the Krylov methods for solving ill-conditioned linear systems in the electromagnetic scattering applications. It has been reported that the direct application of standard block diagonal preconditioners does not accelerate the convergence of the Krylov subspace methods in the context of hybrid integral equation formulations for simulating electromagnetics problems with coated materials [51]. We suspect that some of these individual blocks may be ill-conditioned or close to singular. Their LU factorizations may be unstable, in the sense that large size entries are generated in the inverse LU factors.

Singular value decomposition (SVD) is known to be a very powerful technique for dealing with matrices that are either singular or else numerically very close to singular [67,68]. In many cases when an LU factorization fails to give satisfactory results, SVD can tell us what the problem is or how to solve it. The computational and memory cost of computing SVD of large size matrices may be expensive. To avoid this potential drawback of SVD, we apply SVD to each small block submatrix A_i $(1 \leq i \leq m)$ [69]. A recent paper by Wang et al. [66] describes how to use SVD to construct an efficient block diagonal type preconditioner to solve dense linear systems arising from the electromagnetic scattering problems.

By using SVD, A_i can be decomposed into the product of three matrices as following [72]

$$A_i = U_i \Sigma_i V_i^H,$$

where U_i and V_i are unitary matrices, and all the singular values in a descending order are stored in the diagonal of Σ_i as $\Sigma_i = \text{diag}[\sigma_1, \sigma_2, \cdots, \sigma_j, \cdots, \sigma_k]$, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k \geq 0$, where k is the size of the block submatrix A_i.

The inverse of A_i is computed as $A_i^{-1} = V_i \Sigma_i^{-1} U_i^H$. When some of the singular values are close to or equal to zero, the submatrix A_i is close to singular or numerically singular. In such cases, A_i^{-1} is either very large in size or does not exist. To stabilize the inverse, we replace some small singular values of Σ_i by a larger value. This can be done with a threshold strategy. Given a threshold value ϵ, if there exists an integer j such that $\sigma_j > \epsilon$, but $\sigma_{j+1} \leq \epsilon$ $(1 \leq j \leq k)$, then for every l $(j < l \leq k)$, we set $\sigma_l = \epsilon$.

Here we have the modified diagonal matrix as $\bar{\Sigma}_i = \text{diag}[\sigma_1, \sigma_2, \cdots, \sigma_j, \epsilon, \cdots, \epsilon]$, and the stabilized inverse of A_i is computed as $\bar{A}_i^{-1} = V_i \bar{\Sigma}_i^{-1} U_i^H$.

So, our SVD stabilized block diagonal preconditioner is like

$$M^{-1} = \begin{pmatrix} \bar{A}_1^{-1} & & & \\ & \bar{A}_2^{-1} & & \\ & & \ddots & \\ & & & \bar{A}_k^{-1} \end{pmatrix}$$

We now explain how to choose a suitable threshold value to modify the singular values of each block. We can use a static strategy to modify the singular values, which means to choose a global threshold value for every block. Alternatively, we can also compute some characteristics for each and every block during the SVD decomposition, such as the mean value of all the singular values

for each block, the smallest singular value, or choosing the threshold $\epsilon_i = (\sigma_1/\sigma_k) \cdot error\_bound$, ($\epsilon_i$ is the threshold value for the ith block and the *error_bound* is the stopping criterion for the iterative method used). This dynamic procedure can make the threshold value more reasonable when the singular values are quite different for different blocks. Both methods have the same purpose, which is to make the SVD stabilized block inverse better conditioned than the standard block diagonal one. We know that

$$\text{cond}(A_i) = \sigma_1/\sigma_k \quad (k \text{ is the column size of block } A_i)$$

where σ_1 is the largest singular value of the block A_i and σ_k is the smallest one. By using the modification strategies, the condition number of each block will be reduced to σ_1/ϵ. This means the inverse of that block will be more stable. We expect that this stabilization will make the new preconditioner more effective than the standard block diagonal preconditioner.

Of course, there is always a trade-off between stability and accuracy of the computed preconditioner. Choosing a large threshold value produces a more stable, but less accurate inverse, while choosing a smaller threshold value doing the opposite thing.

2.4 *Some other preconditioners*

There are also some other more complicated preconditioners such as those involving matrix reordering techniques before the factorization process, which have been used to reduce the fill-in and improve the stability of the incomplete factorization. A lot of works have been done to reorder the coefficient matrix for constructing better preconditioners [70]. We have experimented a few standard matrix reordering schemes in constructing preconditioners for solving the dense matrices arising from the electromagnetic scattering problems. Unfortunately, the majority of our experimental results have turned out to be disappointing.

Another class of efficient ILU type preconditioners is the multilevel preconditioners, which can also be combined with the Krylov subspace methods. One can find research papers for the multilevel ILU based preconditioners in [71] and multilevel sparse approximate inverse techniques in [72]. We would like to mention that there are also some work done by Lee et al. [73] and Carpentieri et al. [47] to use two-level preconditioners to solve the dense linear systems arising from electromagnetics simulations.

cases	unknowns	matrices	nonzeros	target size and description
P1C	1,416	A	2,005,056	Dielectric plate over conducting
		A_D	66,384	plate $2.98824 \times 2 \times 0.1$
		$A_D + A_N$	155,616	Frequency=0.2GHZ
C1C	20,176	A	407,070,976	Conducting pipe with 4
		A_D	1,565,032	dielectric coating rings inside,
		$A_D + A_N$	3,728,842	$36 \times 3.86236 \times 3.87$, Frequency=5GHZ
P3C	100,800	A	10,160,640,000	Antenna array
		A_D	3,571,808	Array size:22.25×22.25
		$A_D + A_N$	7,211,632	Frequency=0.3GHZ

Table 1. Information about the matrices used in the experiments (from [27])

case	setup	#it	total	case	setup	#it	total
C1C**0N**		812	1431.1	P3C**0N**		347	2499.9
C1C**1N**	–	751	1310.3	P3C**1N**	–	201	1450.9
C1C**2N**		509	893.2	P3C**2N**		216	1529.1
C1C**0I**		367	1028.4	P3C**0I**		37	414.5
C1C**1I**	67.5	112	327.6	P3C**1I**	111.5	12	205.7
C1C**2I**		179	577.5	P3C**2I**		20	275.9
C1C**0S**		322	714.9	P3C**0S**		41	376.0
C1C**1S**	110.9	120	326.2	P3C**1S**	64.2	17	189.6
C1C**2S**		199	476.4	P3C**2S**		26	254.7

Table 2. Numerical results with different Krylov methods and preconditioners for solving different test cases (from [27])

3 Numerical results

The following numerical results are in part extracted from the authors' previous papers [27,66] to demonstrate the efficiency and robustness of the preconditioned Krylov subspace methods with different preconditioners to solve the dense linear systems arising from the electromagnetic scattering problems.

Table 1 contains information about the test cases and the test matrices used in the numerical experiments. The mesh sizes for all the test structures are about one tenth of a wavelength.

All cases were tested on one processor of an HP superdome supercluster at the University of Kentucky. The processor had 2 GB local memory and ran at 1.5 GHz. The code was written in Fortran 77 and was run in single precision. The iteration process was terminated when the 2-norm of the residual vector was reduced by 10^{-3}, or when the number of iterations exceeded 2000.

In Table 2, the notations used are: 0=BiCG, 1=BiCGSTAB, 2=TFQMR, N=NONE, I=ILUT, and S=Sparse Approximate Inverse. The numerical numbers in the columns "setup" and "total" are the CPU time in seconds for constructing the preconditioners in question and for the total solution time (preconditioner construction and preconditioned iterative solution process), respectively. The columns "#it" record the number of iterations needed for convergence in each case.

It can be seen that the number of iterations is reduced substantially by

preconditioner	ϵ	#it	it_{CPU}
NONE	–	1013	45.75
BLOCK	–	1319	59.94
SVDB	10.0	557	25.82
	15.0	467	21.52
	19.5	400	18.05

Table 3. Numerical results for the P1C case using block diagonal and SVD stabilized block diagonal preconditioners (from [66])

using either the ILUT or SAI preconditioners, compared with the no preconditioner cases, in all three test problems. The differences in using different Krylov iterative methods are slightly varied in most test cases, but the use of preconditioners resulted in sharp decrease in the number of iterations needed for convergence. Furthermore, the total CPU time for the simulations is also reduced greatly with the use of the preconditioners.

Table 3 shows the results of running the P1C case with the BiCG solver using the standard block diagonal preconditioner and the stabilized SVD block diagonal preconditioner. For the SVD block diagonal preconditioner, we use the static threshold strategy to update the singular values which means to use a global threshold for each and every block. ϵ is the threshold value, and it_{CPU} is the CPU time for the iteration part. In this case, the BiCG solver converged in 1013 iterations without using any preconditioner. It converged in 1319 iterations when the standard block diagonal preconditioner was used. Once again, this result demonstrates that the use of standard block diagonal preconditioner may hamper the convergence of the Krylov subspace iterations in certain cases in electromagnetics simulations. However, by using the SVD stabilized block diagonal preconditioner, the number of iterations was reduced to about a half of that needed in the unpreconditioned case. The CPU time needed for the entire simulations was also reduced accordingly.

4 Summary

We have discussed a few preconditioning techniques for iteratively solving dense linear systems arising from the discretization of integral equation formulations of electromagnetic scattering problems. These preconditioning techniques are best suitable in the context of the multilevel fast multipole algorithms. When carefully implemented, they can accelerate the convergence of the Krylov iteration process substantially. In many cases, they can turn a non-converging process into a fast converging process.

However, not all preconditioning techniques discussed in this paper are suitable for all electromagnetic scattering applications. In fact, the geometry of

the underlying problems sometimes affects the quality and the suitability of the constructed preconditioners a lot. For example, the cavity type simulation problems are known to be difficult to solve in electromagnetic scattering applications. It is expected that more efficient preconditioner constructions must take more physical information into consideration in the preconditioner construction process. This requires to construct the preconditioner during the coefficient matrix build-up process in the fast multipole algorithms. A much deeper understanding of the fast multipole algorithms is imperative for constructing the next generation preconditioning techniques in this very interesting application field.

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