J. Soc. Indust. Appl. Math.
Vol. 13, No. 4, December, 1965
Printed in U.S.A.

GENERALIZED INVERSES OF PARTITIONED MATRICES*

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1. Summary. The well known formula for expressing the inverse of a partitioned matrix in terms of inverses of matrices of lower order is extended to generalized inverses of partitioned matrices.

2. Results. We define a generalized inverse of a matrix X to be a matrix $X^{(g)}$ such that

$$XX^{(g)}X = X.$$

It has been shown [5] that the general solution to the equations $X\mathbf{x} = \mathbf{y}$, if consistent, is given by

$$\mathbf{x} = X^{(g)}\mathbf{y} + (I - X^{(g)}X)\mathbf{z}$$

where $\mathbf{z}$ is an arbitrary vector.

Recent interest has focused on other variants of generalized inverses. We shall denote by $X^{(r)}$ a generalized inverse which also obeys the relation $X^{(r)}XX^{(r)} = X^{(r)}$. ($X^{(r)}$ is called a reflexive generalized inverse [6] or a semiinverse [3].) $X^{(N)}$ will denote a generalized inverse which obeys the relations $X^{(N)}XX^{(N)} = X^{(N)}$ and $XX^{(N)} = [XX^{(N)}]^H$. ($X^{(N)}$ is called a normalized generalized inverse [6] or a weak generalized inverse [7].) $X^\dagger$ will denote a generalized inverse which obeys the relations $X^\dagger XX^\dagger = X^\dagger$, $XX^\dagger = (XX^\dagger)^H$ and $(X^\dagger X) = (X^\dagger X)^H$. ($X^\dagger$ is called a pseudoinverse and is uniquely determined by X.) If X is square and nonsingular all the above types of generalized inverse reduce to X^{-1}. The names given here to the various types of generalized inverses are by no means standard. Penrose [5] calls the pseudoinverse the generalized inverse. An alternative nomenclature suggested by a referee would call $X^{(g)}$ a semiinverse, $X^{(r)}$ a reflexive semiinverse, $X^{(N)}$ a weak generalized inverse and $X^\dagger$ the generalized inverse.

Fundamental results in the theory of generalized inverses are the identity

$$X = X(X^HX)^{(g)}X^HX \tag{1}$$

and the conjugate transpose

$$X^H = (X^HX)(X^HX)^{(g)}X^H. \tag{2}$$

Proofs of these results can be found in [1], [4], [6].

* Received by the editors January 8, 1965, and in final revised form April 21, 1965

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If M is a nonnegative Hermitian matrix then we can write

$$M = [X_1 \mid X_2]^H [X_1 \mid X_2] = \left[\begin{array}{c|c} A & C \\ \hline C^H & B \end{array}\right],$$

where $A = X_1^H X_1$, $C = X_1^H X_2$, $B = X_2^H X_2$. Define

$$M^{(g)} = \left[\begin{array}{c|c} A^{(g)} + A^{(g)} C Q^{(g)} C^H A^{(g)} & -A^{(g)} C Q^{(g)} \\ \hline -Q^{(g)} C^H A^{(g)} & Q^{(g)} \end{array}\right], \tag{3}$$

where $Q = B - C^H A^{(g)} C$.

Using the identities (1) and (2) we find

$$MM^{(g)} = \left[\begin{array}{c|c} AA^{(g)} & 0 \\ \hline [I - QQ^{(g)}]\, C^H A^{(g)} & QQ^{(g)} \end{array}\right], \tag{4}$$

$$M^{(g)} M = \left[\begin{array}{c|c} A^{(g)} A & A^{(g)} C\, [I - Q^{(g)} Q] \\ \hline 0 & Q^{(g)} Q \end{array}\right], \tag{5}$$

$$MM^{(g)} M = \left[\begin{array}{c|c} A & C \\ \hline C^H & B \end{array}\right], \tag{6}$$

and

$$M^{(g)} M M^{(g)} = \left[\begin{array}{c|c} A^{(g)} [A + C Q^{(g)} Q Q^{(g)} C^H]\, A^{(g)} & -A^{(g)} C Q^{(g)} Q Q^{(g)} \\ \hline -Q^{(g)} Q Q^{(g)} C^H A^{(g)} & Q^{(g)} Q Q^{(g)} \end{array}\right]. \tag{7}$$

It is clear that $M^{(g)}$ given by (3) is a generalized inverse of M. Inspection of (7) also shows that replacing $A^{(g)}$ and $Q^{(g)}$ by $A^{(r)}$ and $Q^{(r)}$ yields $M^{(r)}$, a reflexive generalized inverse of M.

In order for (3) to yield an expression for a normalized generalized inverse [pseudoinverse] of M, (4) [(4) and (5)] must be Hermitian. A simple sufficient condition for this is nonsingularity of Q. A condition under which Q is nonsingular is given in the following lemma.

Lemma. *If the nonnegative $p \times p$ Hermitian matrix M is partitioned as*

$$M = \left[\begin{array}{c|c} A & C \\ \hline C^H & B \end{array}\right],$$

where A is $(p - q) \times (p - q)$ of rank r, B is $q \times q$ of rank q, and M is of rank $r + q$, then

$$Q = B - C^H A^{(g)} C$$

is nonsingular.

Proof. It suffices to show that Q is of rank q. The rank of M is the same as the rank of $Z = P_1 M P_2$, where

$$P_1 = \left[\begin{array}{c|c} I & 0 \\ \hline -C^H A^{(g)} & I \end{array}\right], \qquad P_2 = \left[\begin{array}{c|c} I & -A^{(g)} C \\ \hline 0 & I \end{array}\right].$$

Using (1) and (2) it is easily seen that

$$Z = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & Q \end{array}\right].$$

Hence rank M = rank $Z = r + q$ = rank A + rank Q, or rank $Q = q$ since rank $A = r$ by assumption.

We may summarize the above results in the following theorem.

THEOREM. *If a nonnegative Hermitian matrix M is partitioned in the form*

$$M = \left[\begin{array}{c|c} A & C \\ \hline C^H & B \end{array}\right],$$

then

(a) *a generalized inverse of M is given by* (3),

(b) *a reflexive generalized inverse of M is given by* (3) *with $A^{(g)}$ and $Q^{(g)}$ replaced by $A^{(r)}$ and $Q^{(r)}$.*

Further if rank M = rank A + rank B, *where B is nonsingular, then*

(c) *a normalized generalized inverse of M is given by* (3) *with $A^{(g)}$ and $Q^{(g)}$ replaced by $A^{(N)}$ and $Q^{(N)}$,*

(d) *a pseudoinverse of M is given by* (3) *with $A^{(g)}$ and $Q^{(g)}$ replaced by $A^\dagger$ and $Q^\dagger$.*

Expressions for pseudoinverses of partitioned matrices have recently been obtained in [2].

3. Remarks. Generalized inverses of the various types indicated above for an arbitrary matrix X can be computed in partitioned form by noting that

$$X^{(g)} = (X^H X)^{(g)} X^H$$

is a generalized inverse of X.

REFERENCES

[1] R. C. BOSE, *Analysis of Variance*, Unpublished lecture notes, University of North Carolina, Chapel Hill, 1959.

[2] R. E. CLINE, *Representations for the generalized inverse of a partitioned matrix*, this Journal, 12 (1964), pp. 588–601.

[3] J. S. FRAME, *Matrix operations and generalized inverses*, IEEE Spectrum, (1964), pp. 209–220.

[4] R. E. KALMAN, *New methods in Wiener filtering*, Proceedings of the First Symposium on Engineering Applications of Random Function Theory and Probability, John Wiley, New York, 1963.

[5] R. PENROSE, *A generalized inverse for matrices*, Proc. Cambridge Philos. Soc., 51(1955), pp. 406–413.

[6] C. A. ROHDE, *Contributions to the Theory, Computation, and Applications of Generalized Inverses*, Ph.D. thesis, University of North Carolina at Raleigh, 1964.

[7] M. ZELEN AND A. J. GOLDMAN, *Weak generalized inverses and minimum variance linear unbiased estimation*, Tech. Rep. 314, Mathematics Research Center, U. S. Army, Madison, Wisconsin, 1963.