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UNIQUENESS OF GAUSS–BIRKHOFF QUADRATURE FORMULAS*

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Dedicated to Professor George G. Lorentz on the occasion of his 75th birthday.

Abstract. We show that the Gauss-Birkhoff quadrature formulas introduced in [4] are uniquely determined. This will be verified using techniques of continuous deformation. The methods of proof also yield an interlacing condition for the knots as the number of them increases.

Key words. Birkhoff interpolation, Birkhoff quadrature, Gaussian quadrature

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1. Introduction. This is a continuation of the discussion on Gauss-Birkhoff quadrature formulas which was started in [4]. There we have shown the existence of formulas of double precision

$$\int_0^1 f(x)\,dg(x) = \sum_{i=1}^m a_i f^{(k_i)}(x_i), \qquad f \in P_{2m-1}, \tag{1.1}$$

if g is absolutely continuous and strictly increasing (see also [4, Remark V.2]) and $(k_1, \cdots, k_m)$ is pyramidal. The latter means that the natural numbers k_i satisfy

$$\begin{aligned} &k_i + 1 \geqq k_{i-1} \geqq k_i, \qquad i = 2, \cdots, I, \\ &k_i = 0, \qquad i = I, \cdots, J, \\ &k_i + 1 \geqq k_{i+1} \geqq k_i, \qquad i = J, \cdots, m-1, \end{aligned} \tag{1.2}$$

for some indices I and J, $1 \leqq I \leqq J \leqq m$. We always assume that the knots of the formula are ordered according to

$$0 < x_1 < x_2 < \cdots < x_m < 1, \tag{1.3}$$

and P_k denotes polynomials of degree k or less.

In this paper we shall show that there is one and only one formula of type (1.1), for each pyramidal set $(k_1, \cdots, k_m)$. The methods of proof are based on techniques of continuous deformation (elements of Brouwer's degree theory) via the implicit function theorem, combined with an induction argument. So the methods of proof are related to the methods of Bojanov et al. [2], who have extended the basic idea in Barrow's paper [1].

It is of interest that our methods of proof also yield an interlacing result for the knots of formulas (1.1) as m increases. This might be helpful for the computation of the knots and the weights of this class of formulas.

Although our methods used here could be extended to the multipoint Gaussian-Birkhoff formulas (see [4, Thm. 5.1]), we have decided to treat formulas with simple knots only. This is a compromise in order to keep the technical details at a low level. We may also note that the error constants for the multipoint formulas are always worse than the error constants of the simple point formulas having the same degree of exactness. This is an easy consequence of the method of proof in [4]; see also [5]. Anyway, amongst all formulas (1.1) with m fixed, the classical Gaussian formula (which is characterized by $k_i = 0$, $i = 1, \cdots, m$) always has the smallest error constant.

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2. Preliminaries. In order to prove the uniqueness of formula (1.1), we have to extend the main result of [4] slightly. Given the pyramidal sequence $(k_1, \cdots, k_m)$, we call $(k_{-m_0+1}, \cdots, k_{m+m_1})$, $m_0 \geqq 0$, $m_1 \geqq 0$ an *extension* of $(k_1, \cdots, k_m)$, provided that the natural numbers k_j satisfy

$$\begin{aligned} &k_{i-1} \geqq k_i, \quad i = -m_0+2, \cdots, 1, \quad \text{and} \\ &k_{i+1} \geqq k_i, \quad i = m, \cdots, m+m_1-1. \end{aligned} \tag{2.1}$$

According to this, we shall deal with formulas

$$\begin{aligned} &\int_0^1 f(x)\, dg(x) = \sum_{i=-m_0+1}^{m+m_1} [a_i f^{(k_i)}(x_i) + b_i f^{(k_i+1)}(x_i)], \qquad f \in P_{2M-1}, \\ &x_{-m_0+1} < \cdots < x_0 < x_1 < \cdots < x_m < x_{m+1} < \cdots < x_{m+m_1}, \\ &M := m_0 + m + m_1. \end{aligned} \tag{2.2}$$

Later on, knots $x_{-m_0+1}, \cdots, x_0$ and $x_{m+1}, \cdots, x_{m+m_1}$ will be located outside the open interval $(0, 1)$ while the knots $x_1, \cdots, x_m$ will satisfy $0 < x_1 < \cdots < x_m < 1$. But for the moment let us deal with the general case.

In the following we shall often refer to notions introduced in [4]; these we do not repeat here in order to keep the paper condensed (see also [3]). Formula (2.2) will not necessarily exist, but it does exist for Pólya extensions:

DEFINITION. $(k_{-m_0+1}, \cdots, k_{m+m_1})$ is called a *Pólya extension* of the pyramidal sequence $(k_1, \cdots, k_m)$ if the incidence matrix $E = (e_{ik})_{i=-m_0+1}^{m+m_1}{}_{k=0}^{2M-1}$ with $e_{ik} = 1$ for $k = k_i$ and $k = k_i + 1$, and $e_{ik} = 0$ otherwise, satisfies the Pólya condition [3, § 1.4]

$$\sum_{i=-m_0+1}^{m+m_1} \sum_{k=0}^{r} e_{ik} \geqq r+1, \qquad r = 0, \cdots, 2M-1.$$

Henceforth, let us assume that $(k_{-m_0+1}, \cdots, k_{m+m_1})$ is a Pólya extension. Then the weights a_i, b_i of formula (2.2) are functions of $X = (x_{-m_0+1}, \cdots, x_{m+m_1})$, given implicitly by the solution of the linear system of equations

$$\int_0^1 \frac{x^\nu}{\nu!}\, dg(x) = \sum_{i=-m_0+1}^{m+m_1} \left[a_i(X) \frac{x_i^{\nu-k_i}}{(\nu-k_i)!} + b_i(X) \frac{x_i^{\nu-k_i-1}}{(\nu-k_i-1)!} \right], \qquad \nu = 0, \cdots, 2M-1; \tag{2.3}$$

here, $1/r! := 0$ for $r = -1, -2, \cdots$. Since the matrix of the system (2.3) is the (transpose of the) matrix of the Birkhoff interpolation problem (E, X), it turns out that the weights are continuously differentiable functions on $R^M_< := \{(\xi_1, \cdots, \xi_M);\ \xi_1 < \xi_2 < \cdots < \xi_M\}$.

Using the continuity of interpolation for the conservative matrix E [3, § 5.6], we can extend formula (2.2) to the boundary of $R^M_<$ by applying the method of coalescence. If $E^1 = (e^1_{ik})_{i=1}^{j}{}_{k=0}^{2M-1}$ is an incidence matrix gained from E by coalescence of subsequent rows, and if $y_1 < y_2 < \cdots < y_j$, then we still have a unique formula

$$\int_0^1 f(x)\, dg(x) = \sum_{e^1_{i,k}=1} a_{ik}(Y) f^{(k)}(y_i), \qquad f \in P_{2M-1}. \tag{2.3'}$$

This formula is the limit of (2.3) in the following sense. If $X \to (y_1, \cdots, y_1, y_2, \cdots, y_2, \cdots, y_j, \cdots, y_j)$ and the knots $x_{i_1}, \cdots, x_{i_2}$ are converging to y_{i_0}, then $\sum_{i=i_1}^{i_2} [a_i(X) f^{(k_i)}(x_i) + b_i(X) f^{(k_i+1)}(x_i)]$ converges to $\sum_{e^1_{i_0,k}=1} a_{i_0,k}(Y) f^{(k)}(y_{i_0})$, for any polynomial $f \in P_{2M-1}$.

Now let us assume that

$$x_{-m_0+1} < \cdots < x_0 \leqq 0 \quad \text{and} \quad 1 \leqq x_{m+1} < \cdots < x_{m+m_1} \tag{2.4}$$

are fixed knots while

$$0 < x_1 < \cdots < x_m < 1 \tag{2.5}$$

are variable knots. We then have

THEOREM 2.1. *Given the Pólya extension* $(k_{-m_0+1}, \cdots, k_{m+m_1})$ *of the pyramidal sequence* $(k_1, \cdots, k_m)$ *and the fixed knots* (2.4), *then the minimization problem*

$$\min_{0<x_1<\cdots<x_m<1} \int_0^1 \Omega_X(t)\, dg(t),$$

$$\Omega_X(t) = \frac{t^{2M}}{(2M)!} + \alpha_{2M-1} t^{2M-1} + \cdots + \alpha_0, \tag{2.6}$$

$$\Omega_X^{(k_i)}(x_i) = \Omega_X^{(k_i+1)}(x_i) = 0, \qquad i = -m_0+1, \cdots, m+m_1,$$

has a solution $0 < x_1^* < \cdots < x_m^* < 1$. *Corresponding to this, formula* (2.3) *is of Gaussian type,*

$$b_i(X^*) = 0, \qquad i = 1, \cdots, m,$$
$$X^* = (x_{-m_0+1}, \cdots, x_0, x_1^*, \cdots, x_m^*, x_{m+1}, \cdots, x_{m+m_1}). \tag{2.7}$$

The proof of this theorem is just a copy of the proof of Theorem 4.1 in [4] (which deals with the case $m_0 = m_1 = 0$), and we omit the details.

3. Uniqueness. We shall now verify the following:

THEOREM 3.1. *Given the Pólya extension* $(k_{-m_0+1}, \cdots, k_{m+m_1})$ *of the pyramidal sequence* $(k_1, \cdots, k_m)$, *and the fixed knots* (2.4), *then there is one and only one set of knots* $0 < x_1 < \cdots < x_m < 1$ *such that* $X = (x_{-m_0+1}, \cdots, x_{m+m_1})$ *solves the system of equations*

$$b_i(X) = 0, \qquad i = 1, \cdots, m. \tag{3.1}$$

Equivalently, we may say that the extremal problem (2.6) has a unique solution and, equivalently, for given knots (2.4), there is a unique formula (2.2) of Gaussian type. In particular, specializing for $m_0 = m_1 = 0$, the Gaussian formula of [4] is unique.

Our proof of Theorem 3.1 uses induction with respect to m, the number of knots interior to the interval (0, 1). Thus there will be some natural analogies to methods used in [2].

In order to start the induction, let $m = 1$ and $k_1 = 0$ the corresponding pyramidal "sequence." For m_0, m_1 arbitrary, the existence of formula (2.3) with $b_1(X) = 0$ is equivalent to the existence of a one-point Gaussian formula for the system

$$T = \{p \in P_{2M-1};\ p^{(k_i)}(x_i) = p^{(k_i+1)}(x_i) = 0,\ i \neq 1\}.$$

Now T is a two-dimensional Chebyshev space on the open interval (0, 1) and Krein's theorem [6] yields the unique one-point Gaussian formula.

In the induction step, let $m \geqq 2$. We may assume that $(k_1, \cdots, k_{m-1})$ is pyramidal (since otherwise the sequence $(k_2, \cdots, k_m)$ is pyramidal and we can apply an apparent substitution). We fix the knots $x_{-m_0+1} < \cdots < x_0 < 0$ and $1 < x_{m+1} < \cdots < x_{m+m_1}$, and start with some Gaussian knots $0 < x_1^* < \cdots < x_m^* < 1$ according to Theorem 2.1.

We let $\xi = x_m$ take on values in an interval $]x_m^* - \varepsilon, 1 + \varepsilon[$. This will let us define curves $0 < x_1(\xi) < x_2(\xi) < \cdots < x_{m-1}(\xi) < \xi$ with $x_i(x_m^*) = x_i^*$, $i = 1, \cdots, m-1$, in the way that

$$X(\xi) = (x_{-m_0+1}, \cdots, x_0, x_1(\xi), \cdots, x_{m-1}(\xi), \xi, x_{m+1}, \cdots, x_{m+m_1}) \tag{3.2}$$

solves the (reduced) system of equations

$$b_i(X(\xi)) = 0, \qquad i = 1, \cdots, m-1. \tag{3.3}$$

For $\xi > 1$, (3.3) has a unique solution, due to the induction hypothesis. Assuming that there are two different Gaussian formulas with m Gaussian points means that the curves have points of bifurcation, a contradiction as we shall see later.

At the solutions $X(\xi)$ of (3.3) where

$$0 < x_1(\xi) < x_2(\xi) < \cdots < x_{m-1}(\xi) < \xi \tag{3.4}$$

and $\xi \leqq 1$ we shall apply the implicit function theorem. According to (2.3), we have for $j = 1, \cdots, m$:

(3.5)
$$0 = \sum_{i=-m_0+1}^{m+m_1} \left\{ \frac{\partial a_i(X)}{\partial x_j} \frac{x_i^{\nu-k_i}}{(\nu-k_i)!} + \left[a_i(X)\delta_{ij} + \frac{\partial b_i(X)}{\partial x_j} \right] \frac{x_i^{\nu-k_i-1}}{(\nu-k_i-1)!} + b_i(X)\delta_{ij} \frac{x_i^{\nu-k_i-2}}{(\nu-k_i-2)!} \right\},$$
$$\nu = 0, \cdots, 2M-1.$$

At the solutions of (3.3) this yields for $j = 1, \cdots, m-1$:

$$\sum_{i=-m_0+1}^{m+m_1} \left\{ \frac{\partial a_i(X(\xi))}{\partial x_j} \frac{x_i^{\nu-k_i}}{(\nu-k_i)!} + \left[a_i(X(\xi))\delta_{ij} + \frac{\partial b_i(X(\xi))}{\partial x_j} \right] \frac{x_i^{\nu-k_i-1}}{(\nu-k_i-1)!} \right\} = 0, \tag{3.6}$$
$$\nu = 0, \cdots, 2M-1.$$

The matrix of this linear system is identical with the matrix of (2.3); thus at the solution $X(\xi)$,

$$a_i(X(\xi))\delta_{ij} + \frac{\partial b_i(X(\xi))}{\partial x_j} = 0, \qquad i, j = 1, \cdots, m-1. \tag{3.7}$$

This shows that the Jacobian $J(X)$ of $(b_1(X), \cdots, b_{m-1}(X))$ at $X(\xi)$ with respect to $x_1, \cdots, x_{m-1}$ is a diagonal matrix with ith diagonal entry

$$\frac{\partial b_i(X(\xi))}{\partial x_i} = -a_i(X(\xi)), \qquad i = 1, \cdots, m-1. \tag{3.8}$$

Now sign $(a_i(X(\xi))) = (-1)^{k_i}$ if k_i is in the descending part of the pyramidal sequence $(k_1, \cdots, k_{m-1})$ and sign $(a_i(X(\xi))) = +1$ otherwise (see the argument in [4], in particular formula (5.1) therein). So the determinant of the Jacobian has a constant sign, at every solution $X(\xi)$.

In this way we find an open interval $]\alpha, \beta[$ with $\alpha < x_m^* < \beta$ and the continuously differentiable functions (3.4) defined on this interval via the solutions of (3.3) with $x_i(x_m^*) = x_i^*$, $i = 1, \cdots, m-1$. We may assume that $]\alpha, \beta[$ is chosen to be maximal, and we next show that $\beta > 1$.

Let us assume on the contrary that $\beta \leqq 1$. This implies that in the limit as $\xi \to \beta$ either the Jacobian $J(X(\xi))$ becomes singular, or

$$\lim_{\xi \to \beta} (x_j(\xi) - x_{j+1}(\xi)) = 0 \tag{3.9}$$

for some $j = 0, \cdots, m-1$ (with $x_0(\xi) \equiv 0$, $x_m(\xi) = \xi$). It is sufficient to prove that (3.9) cannot hold, for then $\lim_{\xi \to \beta} X(\xi) = X(\beta)$ is of type (3.4), and $X(\beta)$ is a solution of (3.3), to which we may apply our arguments. Thus $J(X(\beta))$ is regular, and β is not maximal.

For $\alpha < \xi < \beta$ and the set $X(\xi)$ of knots, we have the Gaussian type quadrature formula

$$\begin{aligned} \int_0^1 f(x)\, dg(x) = & \sum_{i=1}^{m-1} a_i(X(\xi)) f^{(k_i)}(x_i(\xi)) \\ & + a_m(X(\xi)) f^{(k_m)}(\xi) + b_m(X(\xi)) f^{(k_m+1)}(\xi) \\ & + \sum_{i \in I} [a_i(X(\xi)) f^{(k_i)}(x_i) + b_i(X(\xi)) f^{(k_i+1)}(x_i)] \end{aligned} \tag{3.10}$$

for all $f \in P_{2M-1}$ with $I := \{-m_0+1, \cdots, 0, m+1, \cdots, m+m_1\}$.

In the limit as $\xi \to \beta$, if

$$(x_1(\xi), x_2(\xi), \cdots, x_{m-1}(\xi), \xi, \xi) \to (\underbrace{y_1, \cdots, y_1}_{\nu_1}, \cdots, \underbrace{y_\mu, \cdots, y_\mu}_{\nu_\mu})$$

with $0 \leqq y_1 < y_2 \cdots < y_\mu = \beta$ we can apply the continuity argument of deriving (2.3′) from (2.3) in order to get a formula

$$\int_0^1 f(x)\,dg(x) = \sum_{i=1}^{\mu} \sum_{j=0}^{\nu_i - 1} a_{ij} f^{(\kappa_i + j)}(y_i)$$

$$(3.10') \qquad + \sum_{i \in I} [a_i(X(\beta)) f^{(k_i)}(x_i) + b_i(X(\beta)) f^{(k_i+1)}(x_i)]$$

$$\text{for } f \in P_{2M-1} \text{ with } \kappa_i = \min\{k_j;\, x_j(\xi) \to y_i\}.$$

Here we rely on the fact that $(k_i, \cdots, k_m)$ is pyramidal. We then have to show that formula (3.10′) can be exact for $f \in P_{2M-1}$ only in case $\mu = m$ and $0 < y_1 < \cdots < y_{m-1} < y_m = \beta$. Then (3.9) will not hold and we are done.

If (3.9) is true for some $j = 0, \cdots, m-2$, then we find a polynomial $0 \neq p \in P_{2M-1}$ satisfying

$$p(x) \geqq 0, \qquad x \in [0, 1],$$

$$p^{(k_i)}(x_i) = p^{(k_i+1)}(x_i) = 0, \qquad i \in I,$$

$$p^{(j)}(y_i) = 0, \quad i = 1, \cdots, \mu, \quad j = \kappa_i, \cdots, \kappa_i + \nu_i + \sigma_i - 1,$$

with $\sigma_i = 1$ if ν_i is odd, and $\sigma_i = 0$, otherwise (and $\sigma_1 = 0$ if $y_1 = 0$). This yields the desired contradiction since (3.10′) will not hold for $f = p$. Thus $\mu \geqq m - 1$,

$$0 < y_1 < \cdots < y_{m-1} \leqq y_m = \beta$$

and we have to show that $y_{m-1} < \beta$. If we assume that $y_{m-1} = \beta$ we get from the formulas (3.10) and (3.10′) that

$$(3.11) \qquad \begin{aligned} &a_{m-1}(X(\xi)) f^{(k_{m-1})}(x_{m-1}(\xi)) + a_m(X(\xi)) f^{(k_m)}(\xi) + b_m(X(\xi)) f^{(k_m+1)}(\xi) \\ &\to a_{m-1,0} f^{(k_{m-1})}(\beta) + a_{m-1,1} f^{(k_{m-1}+1)}(\beta) + a_{m-1,2} f^{(k_{m-1}+2)}(\beta) \end{aligned}$$

as $\xi \to \beta$. So, β will be a Gaussian knot of order 3 in formula (3.10′), and this implies that $a_{m-1,2}$ has to be positive.

Now, we know that

$$(3.12) \qquad b_m(X(\xi)) < 0, \qquad x_m^* < \xi < \beta,$$

due to the following argument: If $b_m(X(\xi)) = 0$ for some $\alpha < \xi < \beta$ then, putting $j = m$ and $X = X(\xi)$ in (3.5) we get (3.6) for $j = m$, and (3.7) for $i, j = 1, \cdots, m$. Therefore

$$\frac{db_m(X(\xi))}{d\xi} = \frac{\partial b_m(X(\xi))}{\partial x_m} = -a_m(X(\xi)) < 0.$$

Since $b_m(X(\xi))$ is a continuously differentiable function of ξ on the interval $]\alpha, \beta[$ it turns out that $b_m(X(\xi))$ vanishes at only one point, namely $\xi = x_m^*$.

This is the desired contradiction. In case $k_{m-1} < k_m$ we would have $0 < a_{m-1,2} = \lim_{\xi \to \beta} b_m(X(\xi)) \leqq 0$. In case $k_{m-1} = k_m$ we have to rewrite (3.11) using divided differences in order to find that

$$a_{m-1,2} = \lim_{\xi \to \beta} \gamma(X(\xi))$$

with

$$\gamma(X(\xi)) = (\xi - x_{m-1}(\xi)) b_m(X(\xi)),$$

which again is impossible.

This proves that $\beta > 1$ and hence, the curves $x_i(\xi)$ with $x_i(x_m^*) = x_i^*$, $i = 1, \cdots, m-1$, are locally unique on some interval $]\alpha, \beta[$ with

$$\alpha < x_m^* < 1 < \beta.$$

We are now ready to complete the proof of Theorem 3.1 for the fixed knots (2.4) satisfying the additional assumption

$$x_0<0 \quad \text{and} \quad x_{m+1}>1. \tag{3.13}$$

(Note that this was an assumption in our arguments!) Assume that $m\geqq 2$ and $0<x_1^*<\cdots<x_m^*<1$, $0<\tilde{x}_1^*<\cdots<\tilde{x}_m^*<1$ are two different sets such that the corresponding X^* and $\tilde{X}^*$ are solutions of (3.1). We find the locally unique curves

$$\begin{matrix} x_i(\xi) & \text{with } x_i(x_m^*)=x_i^*, \\ \tilde{x}_i(\xi) & \text{with } \tilde{x}_i(\tilde{x}_m^*)=\tilde{x}_i^*, \end{matrix} \qquad i=1,\cdots,m-1,$$

which are defined on some intervals $]\alpha,\beta[$ and $]\tilde{\alpha},\tilde{\beta}[$, respectively. We may assume that $x_m^*\leqq \tilde{x}_m^*$. According to the induction hypothesis we have

$$x_i(\xi)=\tilde{x}_i(\xi) \quad \text{for } 1<\xi<\min(\beta,\tilde{\beta}), \quad i=1,\cdots,m-1,$$

and there must be a point of bifurcation in $[\tilde{x}_m^*, 1]$, a contradiction. This proves Theorem 3.1 subject to (3.13).

It is easy to get rid of the additional assumption (3.13). Namely, if $x_0<0$ and $x_{m+1}=1$ we put $\xi=x_{m+1}$, move ξ to a right neighborhood of 1 and use the previous arguments. By symmetry, this also proves the theorem for $x_0=0$ and $x_{m+1}>1$, and now the third possibility, $x_0=0$ and $x_{m+1}=1$, may be considered by moving $\xi=x_{m+1}$ away from 1. This concludes the proof of Theorem 3.1 in full generality.

4. Interlacing property. It is well known that the knots of the classical Gaussian formulas interlace as the number of them increases. We have a similar property for the Gaussian-Birkhoff formulas. This we shall state for the case $m_0=m_1=0$ only.

THEOREM 4.1. *Let* $(k_1,\cdots,k_{m-1})$ *and* $(k_1,\cdots,k_m)$ *be pyramidal and let*

$$\int_0^1 f(x)\,dg(x)=\sum_{i=1}^{m-1}\alpha_i f^{(k_i)}(\xi_i), \qquad f\in P_{2m-3}, \tag{4.1}$$

and

$$\int_0^1 f(x)\,dg(x)=\sum_{i=1}^{m} a_i f^{(k_i)}(x_i), \qquad f\in P_{2m-1}, \tag{4.2}$$

be the corresponding Gaussian-Birkhoff formulas with $0<\xi_1<\cdots<\xi_{m-1}<1$ *and* $0<x_1<\cdots<x_m<1$. *Then*

$$x_i<\xi_i, \qquad i=1,\cdots,m-1. \tag{4.3}$$

We call (4.3) the *interlacing property* of Gaussian-Birkhoff knots. Due to arguments of symmetry we also have: If $(k_0,k_1,\cdots,k_{m-1})$ is pyramidal, too, and

$$\int_0^1 f(x)\,dg(x)=\sum_{i=0}^{m-1} a_i' f^{(k_i)}(x_i'), \qquad f\in P_{2m-1}, \tag{4.2$'$}$$

then

$$\xi_i<x_i', \qquad i=1,\cdots,m-1. \tag{4.3$'$}$$

Combining both (4.3) and (4.3') in case $(k_0,\cdots,k_{m-1})=(k_1,\cdots,k_m)$, i.e., $(k_1,\cdots,k_m)=(0,\cdots,0)$, we arrive at the interlacing property for the classical Gaussian formulas, namely $x_1<\xi_1<x_2<\xi_2<\cdots<x_{m-1}<\xi_{m-1}<x_m$.

Actually we shall prove more than (4.3). We shall find formulas

$$\begin{aligned} &\int_0^1 f(x)\,dg(x)=\sum_{i=1}^{m} a_i(X(\xi))f^{(k_i)}(x_i(\xi))+b_m(X(\xi))f^{(k_m+1)}(\xi) \quad \text{for } f\in P_{2m-1} \\ &\qquad \text{with } 0<x_1(\xi)<x_2(\xi)<\cdots<x_{m-1}(\xi)<\min(1,\xi), \qquad x_m(\xi)=\xi, \end{aligned} \tag{4.2$''$}$$

where $\alpha<\xi<\infty$ for some $\alpha<x_m$ (we use the notation of Theorem 4.1). Moreover, $x_i(\xi)$, $i=1,\cdots,m-1$, will be continuously differentiable,

$$x_i(x_m)=x_i,$$

$$\frac{dx_i(\xi)}{d\xi}>0, \qquad x_m<\xi<\infty, \quad \text{and} \tag{4.4}$$

$$\lim_{\xi\to\infty} x_i(\xi)=\xi_i, \qquad i=1,\cdots,m-1.$$

From this, (4.3) follows immediately.

In order to prove this, we follow the proof in § 3, in order to find that the curves $x_i(\xi)$, $i=1,\cdots,m-1$, are defined in some interval $]\alpha,\beta[$ with $\alpha<x_m<\beta$. If β is chosen maximal, then $\beta=\infty$ and

$$b_m(X(\xi))<0, \qquad x_m<\xi<\infty. \tag{4.5}$$

Now let us differentiate (3.3) with respect to ξ. This gives, using (3.7),

$$\frac{\partial b_i(X(\xi))}{\partial \xi}-a_i(X(\xi))\frac{dx_i(\xi)}{d\xi}=0, \qquad i=1,\cdots,m-1. \tag{4.6}$$

The partial derivative of b_i with respect to $x_m(\xi)=\xi$ can be taken from (3.5) which we may write as

$$0=\sum_{i=1}^{m}\left\{\frac{\partial a_i(X)}{\partial x_m}p^{(k_i)}(x_i)+\left[a_i(X)\delta_{im}+\frac{\partial b_i(X)}{\partial x_m}\right]p^{(k_i+1)}(x_i)+b_i(X)\delta_{im}p^{(k_i+2)}(x_i)\right\}$$

for all $p\in P_{2m-1}$.

Specializing to the fundamental polynomials $p=p_l$ with

$$p_l^{(k_i)}(x_i)=0, \qquad p_l^{(k_i+1)}(x_i)=\delta_{il}, \qquad i=1,\cdots,m, \tag{4.7}$$

it turns out that

$$0=\frac{\partial b_l(X)}{\partial x_m}+b_m(X)p_l^{(k_m+2)}(x_m), \qquad l=1,\cdots,m-1,$$

and (4.6) now reads (since $x_m(\xi)=\xi$)

$$a_l(X(\xi))\frac{dx_l(\xi)}{d\xi}=-b_m(X(\xi))p_l^{(k_m+2)}(\xi), \qquad l=1,\cdots,m-1. \tag{4.8}$$

According to the arguments used in [4], we already know that (with I and J as in (1.2))

$$\operatorname{sign} a_l(X(\xi))=\begin{cases}(-1)^{k_l} & \text{for } l=1,\cdots,I-1,\\ +1 & \text{for } l=I,\cdots,m-1,\end{cases} \qquad x_m<\xi<\infty.$$

Let us verify that

$$a_l(X(\xi))p_l^{(k_m+2)}(\xi)>0, \qquad x_m<\xi<\infty, \tag{4.9}$$

for $l=1,\cdots,m-1$. First let $l=1,\cdots,I-1$ (i.e. k_l is from the "descending" part of the sequence $k_1,\cdots,k_m$). Since $p_l^{(k_l)}(x_l)=0$ and $p_l^{(k_l+1)}(x_l)=1$ we have

$$p_l^{(k_l)}(x_l+\varepsilon)>0. \tag{4.10}$$

Now $p_l^{(k_l)}$ cannot have a zero in the open interval $]x_l,x_{l+1}[$ and $p_l^{(k_{l+1})}$ has a zero of exact multiplicity 2 at x_{l+1}; for otherwise, using the Rolle type argument as in [4], all derivatives of p_l would have a zero and p_l would be the zero function, which is apparently impossible. We find in case $k_{l+1}=k_l$:

$$\operatorname{sign} p_l^{(k_l)}(x_l+\varepsilon)=\operatorname{sign} p_l^{(k_l)}(x_{l+1}-\varepsilon)=\operatorname{sign} p_l^{(k_{l+1})}(x_{l+1}-\varepsilon)$$
$$=\operatorname{sign} p_l^{(k_{l+1})}(x_{l+1}+\varepsilon),$$

and in case $k_{l+1}=k_l-1$:

$$\operatorname{sign} p_l^{(k_l)}(x_l+\varepsilon)=\operatorname{sign} p_l^{(k_l)}(x_{l+1}-\varepsilon)=-\operatorname{sign} p_l^{(k_{l+1})}(x_{l+1}-\varepsilon)$$
$$=-\operatorname{sign} p_l^{(k_{l+1})}(x_{l+1}+\varepsilon).$$

Repeating the argument yields

$$(4.11)\qquad (-1)^{k_I}\operatorname{sign} p_l^{(k_I)}(x_I+\varepsilon)=\operatorname{sign} p_l(x_I-\varepsilon).$$

Since for $j=I,\cdots,m$ all zeros of $p_l^{(k_j)}$ on $[x_j,\infty[$ appear in the equations (4.7), we have

$$\operatorname{sign} p_l(x_I-\varepsilon)=\operatorname{sign} p_l(x_I+\varepsilon)=\cdots=\operatorname{sign} p_l(x_J+\varepsilon)$$
$$=\cdots=\operatorname{sign} p_l^{(k_m)}(x_m+\varepsilon)=\operatorname{sign} p_l^{(k_m+2)}(x_m+\varepsilon).$$

Finally, $p_l^{(k_m+2)}$ cannot have a zero on $[x_m,\infty[$ and

$$(4.12)\qquad \operatorname{sign} p_l(x_I-\varepsilon)=\operatorname{sign} p_l^{(k_m+2)}(\xi),\qquad x_m<\xi<\infty.$$

Combining (4.10)-(4.12) proves (4.9) for $l=1,\cdots,I-1$. Similarly, if $l=I,\cdots,m-1$ we directly conclude that

$$1=\operatorname{sign} p_l^{(k_l)}(x_l+\varepsilon)=\operatorname{sign} p_l^{(k_{l+1})}(x_{l+1}+\varepsilon)$$
$$=\cdots=\operatorname{sign} p_l^{(k_m)}(x_m+\varepsilon)=\operatorname{sign} p_l^{(k_m+2)}(\xi)$$

for $x_m<\xi<\infty$. Now (4.5) and (4.9) yield the second statement of (4.4).

Let us look at formula (4.2″) as $\xi\to\infty$. It is easy to see that

$$\lim_{\xi\to\infty} a_m(X(\xi))=\lim_{\xi\to\infty} b_m(X(\xi))=0.$$

For example, $a_m(X(\xi))=\int_0^1 p_\xi(x)\,dg(x)$ for the fundamental polynomial $p_\xi\in P_{2m-1}$ satisfying $p_\xi^{(k_i)}(x_i(\xi))=p_\xi^{(k_i+1)}(x_i(\xi))=0$, $i=1,\cdots,m-1$, and $p_\xi^{(k_m)}(\xi)=1$, $p_\xi^{(k_m+1)}(\xi)=0$. If we put $\tilde p_\xi(t)=p_\xi(\xi t)$, we have

$$\xi^{-k_m}\tilde p_\xi(t)\to\tilde p_\infty(t)=-\frac{(m-1-k_m)!}{(m-1)!}t^{m-2}((m-2-k_m)t-(m-1))$$

as $\xi\to\infty$ (using the continuity of Birkhoff interpolation, [3, § 5.6]). From this we find that

$$\max\{|p_\xi(x)|;\,0\leqq x\leqq 1\}\to 0\quad\text{as } \xi\to\infty.$$

Finally, the space

$$T(\xi):=\{f\in P_{2m-1};\,f^{(k_m)}(\xi)=f^{(k_m+1)}(\xi)=0\}$$

will be P_{2m-3} in the limit. In order to see this for $k_m=0$, we map P_{2m-3} onto $T(\xi)$ by $f\to f_\xi$ with $f_\xi(x)=f(x)((x-\xi)/\xi)^2$, whence $\lim_{\xi\to\infty} f_\xi=f$; for $k_m>0$ we may consider $f^{(k_m)}$ in place of f. Therefore, formula (4.2″) for $f\in T(\xi)$ will be formula (4.1) in the limit.

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