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GENERALIZED INVERSES OF A BORDERED MATRIX OF OPERATORS*

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Abstract. The results of Hall and Meyer in [2] are extended to (1)-inverses of a matrix of bounded linear operators. The blocks of (1, 3)- and (1, 4)-inverses of the matrix are completely characterized and are shown to be independent of each other. A form for the Moore–Penrose inverse of the matrix and related results are also obtained.

1. Introduction. In [2] Hall and Meyer considered the bordered matrix

$$\begin{bmatrix} A'A & X \\ X' & 0 \end{bmatrix}$$

and completely characterized the blocks of (1)-inverses for this matrix. Furthermore, it was shown that these blocks are entirely independent of each other.

In this paper the blocks of the above matrix are replaced by bounded linear operators on Hilbert spaces and the results in [2] are extended to the infinite dimensional setting. The (1, 3)- and (1, 4)-inverses of the matrix are considered in detail, and the operators which are blocks in these inverses are characterized. Various conditions and forms for these operators are obtained, and it is shown that these operators are also independent of each other.

A form of the Moore–Penrose inverse of the matrix is obtained, and the connection between this inverse and the restricted pseudoinverse defined by Minamide and Nakamura in [3] is indicated. The results in this paper and the results in [2] point to some interesting relationships between (1)-, (1, 3)-, and (1, 4)-inverses of the matrix. Some related results are also given.

2. Notation and preliminaries. For a bounded linear operator A from a Hilbert space H_1 into a Hilbert space H_2, A^*, $N(A)$, and $R(A)$ denote the adjoint, null space, and range of A, respectively; $P_{R(A)}$ denotes the orthogonal projection on $R(A)$. If A has closed range, $A^\dagger$ denotes the unique bounded linear operator X from H_2 into H_1 which satisfies

(i) $$AXA = A,$$

(ii) $$XAX = X,$$

(iii) $$(AX)^* = AX,$$

and

(iv) $$(XA)^* = XA.$$

More generally, if a bounded linear operator X satisfies conditions (i), (j), and (k), X is called an (i, j, k) inverse of A. For a given (1)-inverse A^- of A, F_A denotes $I - A^-A$ and E_A denotes $I - AA^-$.

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Another characterization [1] of $A^\dagger$ is the following: $A^\dagger$ is the unique bounded linear operator X which satisfies

(i) $$XAv = v \quad \text{for all } v \in R(A^*),$$

and

(ii) $$Xw = 0 \quad \text{for all } w \in N(A^*).$$

For properties of $A^\dagger$ see [1] or [5].

In this paper, B always denotes the matrix

$$B = \begin{bmatrix} T & C \\ C^* & 0 \end{bmatrix}$$

where $T = A^*A$, A is a bounded linear operator from Hilbert space H_1 to Hilbert space H_2, and C^* is a bounded linear operator from H_1 to Hilbert space H_3. It is easy to check that B defines a bounded linear operator from $H_1 \oplus H_3$ into $H_1 \oplus H_3$, where $H_1 \oplus H_3$ denotes the direct sum of H_1 and H_3 and is equipped with the usual inner product. It will be assumed that $A(N(C^*))$ is closed and that C has closed range (except in § 6).

Generalizing the definitions given in [2], we have the following definitions. Every bounded linear operator which appears as an upper left block in some (1)-inverse for B will be called a *C_{11}-operator.* Similarly, every bounded linear operator which appears as an upper left block in some (1, 3)-inverse ((1, 4)-inverse) for B will be called a *C_{11}^3-operator* (*C_{11}^4-operator*). Likewise, those bounded linear operators which appear as upper right blocks in (1, 3)-inverses ((1, 4)-inverses) for B will be called *C_{12}^3-operators* (*C_{12}^4-operators*). The C_{21}^3-, C_{21}^4-, C_{22}^3-, and C_{22}^4-operators are defined analogously. For example if Q, U, L, and R are bounded linear operators and

$$\begin{bmatrix} Q & U \\ L & R \end{bmatrix}$$

is a (1, 4)-inverse for B, then Q, U, L, and R are C_{11}^4-, C_{12}^4-, C_{21}^4-, and C_{22}^4-operators, respectively. By a matrix inverse for B, we will mean an inverse which can be expressed as a matrix of bounded linear operators (for example, the (1, 4)-inverse above). Thus, the operators defined in the above classes are blocks in matrix inverses for B.

3. (1)-inverses of B. Our first objective is to show that (1)-inverses of B exist in this infinite-dimensional setting. Equivalently, we are showing that B has closed range. We first establish the following lemma.

LEMMA 3.1. *The bounded linear operator*

$$E_C T F_{C^*}$$

has closed range.

Proof. Since $R(F_{C*}) = N(C^*)$,

$$R(AF_{C*}) = A(N(C^*)). \tag{3.1}$$

Now $R((E_C A^*)^*) = R(AE_C^*)$. But C^{-*} is a (1)-inverse of C^* and so $R(E_C^*) = N(C^*)$. Hence

$$R((E_C A^*)^*) = (A(N(C^*)). \tag{3.2}$$

Thus, from (3.1)–(3.2), since $A(N(C^*))$ is closed we have

$$R(E_C T F_{C*}) = R(E_C A^* A F_{C*}) = R(E_C A^*). \tag{3.3}$$

But since $R((E_C A^*)^*)$ is closed, so is $R(E_C A^*)$. Thus from (3.3), $R(E_C T F_{C*})$ is closed.

In view of the lemma there are (1)-inverses for $E_C T F_{C*}$ ($(E_C T F_{C*})^\dagger$ is a (1)-inverse). It is then easy to see that the proofs of the theorems in [2] are valid for infinite-dimensional analogs. From [2], therefore, (1)-inverses of B do exist and we have a complete characterization of all matrix (1)-inverses for B. In particular the following theorem gives a (1)-inverse for B.

THEOREM 3.1. *Let C^-, C^{*-}, and $(E_C T F_{C*})^-$ be any (1)-inverses for C, C^*, and $E_C T F_{C*}$ respectively. Then, the matrix of bounded linear operators*

$$\left[\begin{array}{c|c} Q & C^{*-} - QTC^{*-} \\ \hline C^- - C^-TQ & -C^-TC^{*-} + C^-TQTC^{*-} \end{array}\right],$$

where $Q = F_{C*}(E_C T F_{C*})^- E_C$ is a (1)-inverse for B.

Proof. The proof of this theorem follows from Theorem 3.1 of [2]. Note that the above matrix defines a continuous transformation since the individual blocks are continuous.

4. $B^\dagger$ and particular (1, 3)-, (1, 4)-inverses of B. Letting $F = I - CC^\dagger$, $F^* = F$ and F is the orthogonal projection onto $N(C^*)$. Thus, since $A(N(C^*))$ is closed, $(AF)^\dagger$ is a well-defined bounded linear operator. Also, since $R(FA^*AF)$ is closed by Lemma 3.1, $(FA^*AF)^\dagger$ is well-defined. Now observe that

$$\begin{aligned} F(FA^*AF)^\dagger F &= F[(AF)^*(AF)]^\dagger F = F(AF)^\dagger (AF)^{*\dagger} F \\ &= [F(AF)^\dagger][F(AF)^\dagger]^* = (AF)^\dagger (AF)^{\dagger *} \end{aligned}$$

since $R((AF)^\dagger) = R((AF)^*) \subseteq R(F)$ and F is a projection. Hence

$$F(FA^*AF)^\dagger F = (FA^*AF)^\dagger. \tag{4.1}$$

Similarly,

$$(FA^*AF)^\dagger F = (FA^*AF)^\dagger = F(FA^*AF)^\dagger. \tag{4.2}$$

Also, from (4.1),

$$\begin{aligned} A(FA^*AF)^\dagger A^* &= AF(FA^*AF)^\dagger FA^* = (AF)(AF)^\dagger (AF)^{*\dagger}(AF)^* \\ &= (AF)(AF)^\dagger (AF)(AF)^\dagger = (AF)(AF)^\dagger = P_{R(AF)}. \end{aligned} \tag{4.3}$$

These observations will be useful in the following theorem.

THEOREM 4.1. *Let C^- and C^{*-} be any (1, 3)-inverses of C and C^* respectively.*

Then, if $Q = (FTF)^\dagger$, *where* $F = I - CC^\dagger$, *the matrix of bounded linear operators*

$$Y = \left[\begin{array}{c|c} Q & C^{*-} - QTC^{*-} \\ \hline C^- - C^-TQ & -C^-TC^{*-} + C^-TQTC^{*-} \end{array}\right]$$

is a (1, 3)-*inverse of* B.

Proof. Since C^- is a (1, 3)-inverse of C,

$$E_C = I - CC^- = I - (CC^-)^* = I - C^{-*}C^*.$$

But C^{-*} is a (1)-inverse of C^* and so $R(E_C) = N(C^*)$. Thus, E_C is the orthogonal projection onto $N(C^*)$ and hence

$$E_C = F. \tag{4.4}$$

By direct multiplication,

$$BY = \left[\begin{array}{c|c} CC^- + E_CTQ & E_CTC^{*-} - E_CTQTC^{*-} \\ \hline 0 & C^*C^{*-} \end{array}\right]$$

and hence from (4.4),

$$BY = \left[\begin{array}{c|c} CC^- + FTQ & FTC^{*-} - FTQTC^{*-} \\ \hline 0 & C^*C^{*-} \end{array}\right].$$

Now from (4.2),

$$FTQ = FTF(FTF)^\dagger,$$

and since C^- is a (1, 3)-inverse of C,

$$(CC^- + FTQ)^* = CC^- + FTQ. \tag{4.5}$$

Also, from (4.3),

$$(FTQT)^* = TQTF = A^*P_{R(AF)}AF,$$

and so

$$(FTQT)^* = A^*AF = TF,$$

and hence $FTQT = FT$. Thus

$$FTC^{*-} - FTQTC^{*-} = 0. \tag{4.6}$$

Finally, since C^{*-} is a (1, 3)-inverse of C^*,

$$(C^*C^{*-})^* = C^*C^{*-} \tag{4.7}$$

and thus (4.5)–(4.7) imply that $(BY)^* = BY$.

That Y is a (1)-inverse of B follows from Theorems 3.1, 4.1, and 4.2 of [2] (the continuity of Y is clear).

In a similar manner one can prove an analogous theorem for (1, 4)-inverses.

THEOREM 4.2. *Let* C^- *and* C^{*-} *be any* (1, 4)-*inverses of* C *and* C^* *respectively. Then, the matrix* Y *given in the previous theorem is a* (1, 4)-*inverse of* B.

The following form for $B^\dagger$ is almost a direct consequence of the previous two theorems.

THEOREM 4.3. *For the matrix*

$$B = \begin{bmatrix} A^*A & C \\ C^* & 0 \end{bmatrix}$$

of bounded linear operators,

$$B^\dagger = \left[\begin{array}{c|c} Q & C^{\dagger *} - QTC^{\dagger *} \\ \hline C^\dagger - C^\dagger TQ & -C^\dagger TC^{\dagger *} + C^\dagger TQTC^{\dagger *} \end{array}\right], \tag{4.8}$$

where $T = A^*A$, $Q = (FTF)^\dagger$ *and* $F = I - CC^\dagger$.

Proof. It can be verified directly that the given matrix is a (2)-inverse of B. The result then follows from the previous two theorems.

In [3] Minamide and Nakamura define the *restricted pseudoinverse* $A^\dagger_{C^*}$ of A with respect to $N(C^*)$ and show that it has the constrained best approximate solution property. It is straightforward to verify that

$$(FA^*AF)^\dagger A^* = (AF)^\dagger = A^\dagger_{C^*}. \tag{4.9}$$

Now using (4.8)–(4.9) we obtain

$$B^\dagger \begin{bmatrix} A^*b \\ f \end{bmatrix} = \begin{bmatrix} (FTF)^\dagger A^*b + C^{*\dagger}f - (FTF)^\dagger TC^{*\dagger}f \\ ************* \end{bmatrix}$$

or

$$B^\dagger \begin{bmatrix} A^*b \\ f \end{bmatrix} = \begin{bmatrix} A^\dagger_{C^*}b + C^{*\dagger}f - A^\dagger_{C^*}AC^{*\dagger}f \\ ********** \end{bmatrix}.$$

But, from [3], $A^\dagger_{C^*}b + C^{*\dagger}f - A^\dagger_{C^*}AC^{*\dagger}f$ is the constrained best approximate solution of the system $Av = b$, subject to $C^*v = f$. Thus, we obtain this particular solution by using $B^\dagger$.

Now consider the class of (1, 4)-inverses of B. In general, if M is a bounded linear operator and if M^- is a (1, 4)-inverse of M, then M^-y is the unique minimum norm solution of $Mx = y$ for each $y \in R(M)$ (the proof on p. 44 of [6] holds in the infinite-dimensional case). But, $B^\dagger$ is a (1, 4)-inverse of B and $\begin{bmatrix} A^*b \\ f \end{bmatrix}$ is in $R(B)$ if $f \in R(C^*)$. Thus, if B^- is any (1, 4)-inverse for B and if $f \in R(C^*)$, and

$$B^- \begin{bmatrix} A^*b \\ f \end{bmatrix} = \begin{bmatrix} \hat{u} \\ * \end{bmatrix},$$

then $\hat{u}$ is also the constrained best approximate solution of the system $Av = b$. Hence, we also obtain this best solution by using only (1, 4)-inverses of B. We should remark, however, that using inverses of B might involve conditioning problems in numerical computations.

5. Characterization of the blocks of (1, 3)-, (1, 4)-inverses of *B*. In this section the matrix (1, 3)- and (1, 4)-inverses of B are completely characterized through a characterization of the blocks of these inverses. In the following we again let

$F = I - CC^\dagger$, the orthogonal projection onto $N(C^*)$. Our discussion will focus on (1, 4)-inverses of B, but the development for (1, 3)-inverses is similar.

Recall that the bounded linear operator X is a (1, 4)-inverse for B if and only if $XB = B^\dagger B$. Now, it is easy to see that any bounded linear operator of the form

$$B^\dagger + M(I - BB^\dagger), \tag{5.1}$$

where M is a bounded linear operator, is a (1, 4)-inverse for B, as

$$[B^\dagger + M(I - BB^\dagger)]B = B^\dagger B + M(B - BB^\dagger B) = B^\dagger B.$$

Conversely, if B^- is any (1, 4)-inverse for B,

$$B^\dagger + B^-(I - BB^\dagger) = B^\dagger + B^- - B^-BB^\dagger = B^\dagger + B^- - B^\dagger = B^-,$$

and hence we can choose $M = B^-$ in (5.1) to obtain B^-. As M ranges over all possible matrices of bounded linear operators, the form (5.1) gives all matrix (1, 4)-inverses for B.

Now it is clear from the proof of Theorem 4.1 that

$$BB^\dagger = \left[\begin{array}{c|c} CC^\dagger + FTF(FTF)^\dagger & 0 \\ \hline 0 & C^\dagger C \end{array}\right]. \tag{5.2}$$

Hence, with the above observations we can exhibit the following forms for the blocks of (1, 4)-inverses of B.

THEOREM 5.1. *Let*

$$G_1 = F - (FTF)^\dagger(FTF) \quad \textit{and} \quad G_2 = I - C^\dagger C.$$

Then,

(i) *If Q is a C^4_{11}-operator, then*

$$Q = (FTF)^\dagger + ZG_1 \tag{5.3}$$

for some bounded linear operator Z. In fact, we can choose $Z = Q$. Conversely, every operator of the form (5.3) *is a C^4_{11}-operator.*

(ii) *If U is a C^4_{12}-operator, then*

$$U = C^{\dagger *} - (FTF)^\dagger TC^{\dagger *} + ZG_2 \tag{5.4}$$

for some bounded linear operator Z. In fact, we can choose $Z = U$. Conversely, every operator of the form (5.4) *is a C^4_{12}-operator.*

(iii) *If L is a C^4_{21}-operator, then*

$$L = C^\dagger - C^\dagger T(FTF)^\dagger + ZG_1 \tag{5.5}$$

for some bounded linear operator Z. In fact, we can choose $Z = L$. Conversely, every operator of the form (5.5) *is a C^4_{21}-operator.*

(iv) *If R is a C^4_{22}-operator, then*

$$R = -C^\dagger TC^{\dagger *} + C^\dagger T(FTF)^\dagger TC^{\dagger *} + ZG_2 \tag{5.6}$$

for some bounded linear operator Z. In fact, we can choose $Z = R$. Conversely, every operator of the form (5.6) *is a C^4_{22}-operator.*

Moreover, if Q, U, L, and R are any C_{11}^{4}-, C_{12}^{4}-, C_{21}^{4}-, and C_{22}^{4}-operators, respectively, then Q, U, L, and R are independent in the sense that the composite matrix

$$H = \begin{bmatrix} Q & U \\ L & R \end{bmatrix}$$

is a $(1, 4)$*-inverse for* B.

Proof. The results (i)–(iv) follow from the above observations by using (5.1)–(5.2). Now, if Q, U, L, and R are any C_{11}^{4}-, C_{12}^{4}-, C_{21}^{4}-, and C_{22}^{4}-operators respectively, then (5.3)–(5.6) hold, where the variable operator Z is Q, U, L, and R, respectively. Hence, with $M = H$, the matrix (5.1) becomes H, and thus H is a $(1, 4)$-inverse for B.

The above theorem shows that the blocks of $(1, 4)$-inverses of B are independent of each other. It should be emphasized that even though Q, U, L and R may possibly come from four different $(1, 4)$-inverses of B, Theorem 5.1 asserts that the new composite matrix H is a $(1, 4)$-inverse for B. Needless to say, this independence does not generally occur for an arbitrary bordered matrix. It does occur, however, whenever $BB^{\dagger}$ is of the general form (5.2) with a 0 for the $(1, 2)$- and $(2, 1)$-blocks.

It is easy to verify that for $T = A^*A$,

$$(5.7) \qquad (FTF)^{\dagger}(FTF) = (AF)^{\dagger}(AF) = (FTF)(FTF)^{\dagger},$$

so that any of these three expressions can be used in G_1 in the above theorem.

In [2] it was shown that the product TQT is invariant for all C_{11}-matrices Q. For $(1, 4)$-inverses of B we have the following invariance.

COROLLARY 5.1. *The product QT is invariant for all C_{11}^{4}-operators Q. In fact, for $T = A^*A$,*

$$QT = (FTF)^{\dagger}T = (AF)^{\dagger}A$$

for all C_{11}^{4}-operators Q.

Proof. From (5.7), we have

$$(FTF)^{\dagger}(FTF)T = (AF)^{\dagger}(AF)T = (AF)^{\dagger}(AF)(A^*A)$$

and since $(AF)^{\dagger}(AF) = P_{R(AF)^*}$ we obtain

$$(5.8) \qquad (FTF)^{\dagger}(FTF)T = FA^*A = FT.$$

Thus, from (5.3),

$$QT = (FTF)^{\dagger}T = (FTF)^{\dagger}FA^*A = [(AF)^*(AF)]^{\dagger}(AF)^*A = (AF)^{\dagger}A.$$

Note that if we use the invariance of the above corollary for a C_{11}^{4}-operator Q,

$$TQT = T(AF)^{\dagger}A = A^*AF(AF)^{\dagger}A = A^*P_{R(AF)}A,$$

which is the invariant term mentioned above (see Theorem 4.1 in [2]).

In the next theorem we will use the forms of Theorem 5.1 to give general conditions which characterize the blocks of $(1, 4)$-inverses of B. This theorem also exhibits some more invariant expressions. In the theorem it is assumed that Q, U, L, and R are all bounded linear operators.

THEOREM 5.2. *Let D be the invariant expression $T - TQT$, where Q is any C_{11}-operator. Each of the following statements is true*:

(5.9) $\quad Q$ *is a* C_{11}^4-*operator iff* $QC = 0$ *and* $Q(FTF) = (FTF)^{\dagger}(FTF)$.

(5.10) $\quad U$ *is a* C_{12}^4-*operator iff* $UC^* = [I - (FTF)^{\dagger}T]CC^{\dagger}$.

(5.11) $\quad L$ *is a* C_{21}^4-*operator iff* $LT = C^{\dagger}D$ *and* $LC = C^{\dagger}C$.

(5.12) $\quad R$ *is a* C_{22}^4-*operator iff* $RC^* = -C^{\dagger}D$.

Proof. We will first prove (5.9). If Q is a C_{11}^4-operator, then Q is of the form (5.3). It is then trivial that $QC = 0$ and $Q(FTF) = (FTF)^{\dagger}(FTF)$. Conversely, if we assume Q satisfies the equations of (5.9) and let $Z = Q$ in (5.3) we have

$$\begin{aligned}(FTF)^{\dagger} + ZF - Z(FTF)^{\dagger}(FTF) &= (FTF)^{\dagger} + QF - Q(FTF)^{\dagger}(FTF)\\ &= (FTF)^{\dagger} + Q - QCC^{\dagger} - Q(FTF)(FTF)^{\dagger}\\ &= (FTF)^{\dagger} + Q - (FTF)^{\dagger}(FTF)(FTF)^{\dagger}\\ &= Q,\end{aligned}$$

so that Q is a C_{11}^4-operator.

The proofs of (5.10) and (5.12) are also straightforward. To finish the proof of the theorem, we prove (5.11). First, if L is a C_{21}^4-operator, then L is of the form (5.5), and so it is easily seen that $LT = C^{\dagger}D$ and $LC = C^{\dagger}C$. Conversely, if we assume L satisfies the equations in (5.11) and we let $Z = L$ in (5.5) we have

$$\begin{aligned}&C^{\dagger} - C^{\dagger}T(FTF)^{\dagger} + ZF - Z(FTF)^{\dagger}(FTF)\\ &\quad = C^{\dagger} - C^{\dagger}T(FTF)^{\dagger} + L - LCC^{\dagger} - L(FTF)(FTF)^{\dagger}\\ &\quad = C^{\dagger} - C^{\dagger}T(FTF)^{\dagger} + L - C^{\dagger}CC^{\dagger} - LTF(FTF)^{\dagger} + LCC^{\dagger}TF(FTF)^{\dagger}\\ &\quad = -C^{\dagger}T(FTF)^{\dagger} + L - C^{\dagger}DF(FTF)^{\dagger} + C^{\dagger}CC^{\dagger}TF(FTF)^{\dagger}\\ &\quad = L - C^{\dagger}DF(FTF)^{\dagger},\end{aligned}$$

and since $DF = 0$ from (5.8), this last expression is L. Thus, L is a C_{21}^4-operator, and the proof of (5.11) is complete.

Note that (5.9) and (5.11) say that Q and L are (1, 4)-inverses of FTF and C, respectively.

It is interesting to compare (5.10)–(5.12) with (4.5)–(4.7) (conditions which characterize the blocks of (1)-inverses of B) of [2]. Since $DF = 0$ it follows that

$$CC^{\dagger}D = D \quad \text{and} \quad DCC^{\dagger} = D.$$

Using these two equations, it is immediate that (5.10)–(5.12) imply (4.5)–(4.7), respectively, of [2].

For each of the preceding statements on (1, 4)-inverses, there is an analogous statement for (1, 3)-inverses. Since

$$\begin{aligned}W \text{ is a } (1,4)\text{-inverse for } B &\Leftrightarrow WB = B^{\dagger}B\\ &\Leftrightarrow BW^* = B^{\dagger}B = BB^{\dagger} \Leftrightarrow W^* \text{ is a } (1,3)\text{-inverse for } B,\end{aligned}$$

these analogies are easy to find.

The blocks of (1, 3)-inverses of B are also independent of each other. For (1, 3)-inverses, the forms (5.3)–(5.6) are replaced by

(i) $$Q = (FTF)^{\dagger} + G_1 Z,$$

(ii) $$U = C^{\dagger *} - (FTF)^{\dagger} TC^{\dagger *} + G_1 Z,$$

(iii) $$L = C^{\dagger} - C^{\dagger} T(FTF)^{\dagger} + G_2 Z,$$

and

(iv) $$R = -C^{\dagger} TC^{\dagger *} + C^{\dagger} T(FTF)^{\dagger} TC^{\dagger *} + G_2 Z.$$

In this case the invariance QT is replaced by the invariance

$$TQ = T(FTF)^{\dagger} = A^*[(AF)^*]^{\dagger}.$$

For (1, 3)-inverses of B, the conditions (5.9)–(5.12) are replaced by

(i) Q is a C_{11}^3-operator iff $C^*Q = 0$ and $(FTF)Q = (FTF)(FTF)^{\dagger}$,

(ii) U is a C_{12}^3-operator iff $TU = DC^{\dagger *}$ and $C^*U = C^{\dagger}C$,

(iii) L is a C_{21}^3-operator iff $CL = CC^{\dagger}[I - T(FTF)^{\dagger}]$,

and

(iv) R is a C_{22}^3-operator iff $CR = -DC^{\dagger *}$.

6. Results using nonclosed ranges. In the previous sections it has been assumed that $A(N(C^*))$ and $R(C)$ are closed. We now consider the possibilities when these conditions are relaxed. In doing so we recall (see [4]) the definition of $S^{\dagger}$ if S is a bounded linear operator with a possibly nonclosed range. The domain of $S^{\dagger}$ is $R(S) \oplus N(S^*)$ and $S^{\dagger}$ is characterized as the unique linear transformation X such that

$$X(Sv + w) = v$$

for all $v \in \overline{R(S^*)}$ and $w \in N(S^*)$. Here it is the case that $S^{\dagger}S = P_{\overline{R(S^*)}}$.

First let us assume only that $A(N(C^*))$ is closed and let $F = I - C^{*\dagger}C^*$. In this general situation we still have

$$R(F) = N(C^*) \quad \text{and} \quad F = P_{N(C^*)}.$$

We also observe that the definition of $A_{C^*}^{\dagger}$ requires only that $A(N(C^*))$ be closed. Now

$$R(FA^*AF) = R[(AF)^*(AF)] = R((AF)^*)$$

and thus $R(FA^*AF)$ is closed since $R(AF)$ is closed by assumption. It is then easy to see that

$$(FA^*AF)^{\dagger}A^* = (AF)^{\dagger} = A_{C^*}^{\dagger}$$

is still true in this situation. Moreover, the expression given in § 4 for the constrained best approximate solution is still valid if we stipulate that $f \in R(C^*) \oplus N(C)$. The proof of Theorem 3.1 in [3] holds in this case also.

What can be said when $A(N(C^*))$ is also not necessarily closed? In this case the domain of $(FA^*AF)^\dagger$ is

$$R(FA^*AF) \oplus N(FA^*AF)$$

and hence $(FA^*AF)^\dagger \circ FA^*$ is defined on

$$R(AF) \oplus N(FA^*).$$

Now observe that

$$\overline{R(FA^*AF)} = \overline{R[(AF)^*(AF)]} = \overline{R((AF)^*)}$$

and suppose $AFx \in R(AF)$, where $x \in \overline{R((AF)^*)}$. Then, since $x \in \overline{R(FA^*AF)}$,

$$(FA^*AF)^\dagger \circ FA^*AFx = x.$$

But because

$$(FA^*)(\overline{R(AF)} \setminus R(AF)) \subseteq \overline{R(FA^*AF)} \setminus R(FA^*AF,,$$

$(FA^*AF)^\dagger \circ FA^*$ is not defined on $\overline{R(AF)} \setminus R(AF)$. Thus

$$(FA^*AF)^\dagger \circ FA^* = (AF)^\dagger$$

even in this most general situation. Also notice that

$$R((AF)^\dagger) = \overline{R(FA^*)} \subseteq \overline{R(F)} = R(F) = N(C^*)$$

and so $R((AF)^\dagger) \subseteq N(C^*)$.

Now if $b \in R(AF) \oplus N(FA^*)$, then $(AF)^\dagger b$ is the constrained best approximate solution of $Av = b$, subject to $v \in N(C^*)$. This follows from a modified proof of Theorem 3.1 of [3]. But from Theorem 2.2 of [4], it follows that $R(AF) \oplus N(FA^*)$ is precisely the set of vectors b for which the system $Av = b$ possesses a constrained approximate solution. Thus $(AF)^\dagger$ serves as a "restricted pseudoinverse" even when $A(N(C^*))$ is not closed.

Unfortunately, obtaining $B^\dagger$ and matrix (1, 3)- and (1, 4)-inverses of B whenever $A(N(C^*))$ and $R(C)$ are possibly nonclosed presents a more difficult task. This is a possible place for future research.

7. Some related results. The following theorem gives equivalent conditions in order that $B^\dagger$ be of the same form as B. It also generalizes Theorem 4.3 of [2].

THEOREM 7.1. *The following are equivalent:*

(i) *If*

$$\begin{bmatrix} Q & * \\ * & * \end{bmatrix}$$

is a (1)*-inverse of* B, *then* $TQT = T$.

(ii) *There exists a matrix of bounded linear operators of the form*

$$\begin{bmatrix} * & * \\ * & 0 \end{bmatrix}$$

which is a (1)*-inverse of* B.

(iii) $R(T) = R(TF_{C^*})$.
(iv) $B^\dagger$ *is of the form*

$$B^\dagger = \begin{bmatrix} * & * \\ * & 0 \end{bmatrix}.$$

Proof. The first three statements are equivalent by Theorem 4.3 in [2]. Now, if (4) holds, (2) clearly holds. Conversely, if (1) holds,

$$-C^\dagger TC^{\dagger *} + C^\dagger TQTC^{\dagger *} = 0,$$

and hence $B^\dagger$ is of the desired form.

The final results that we now present give further relationships between B and $B^\dagger$. These results may possibly be an aid in the computation of the invariant term $D = T - TQT$ discussed in [2].

THEOREM 7.2. *Assume*

$$B^\dagger = \begin{bmatrix} M & U \\ L & * \end{bmatrix}.$$

Then

(i) $$(TM)^2 = TM$$

and thus $I - TM$ is also a projection.

(ii) $$C^\dagger = L(I - TM)^\dagger$$

and

(iii) $$L = C^\dagger \Leftrightarrow TP_C = P_C T \Leftrightarrow U = C^{*\dagger}.$$

Proof. In the proof we again let $F = I - CC^\dagger$. To prove (i) observe that from (4.2) and (4.8), $M = F(FA^*AF)^\dagger$, and so $TM = A^*(AF)(FA^*AF)^\dagger$. Thus

$$TMTM = A^*(AF)(FA^*AF)^\dagger FA^*(AF)(FA^*AF)^\dagger = A^*(AF)(AF)^\dagger(AF)(FA^*AF)^\dagger$$
$$= A^*(AF)(FA^*AF)^\dagger = TM.$$

Clearly this equality implies that $I - TM$ is a projection (and thus has closed range).

To show that (ii) holds we observe from (4.8) that $L = C^\dagger(I - TM)$ and hence

$$C^\dagger(I - TM)(I - TM)^\dagger = L(I - TM)^\dagger$$

or

$$(I - TM)(I - TM)^\dagger C^{\dagger *} = [L(I - TM)^\dagger]^*. \tag{7.1}$$

Now, if $v \in N(I - MT)$ and $x \in H_3$,

$$\langle Cx, v\rangle = \langle Cx, MTv\rangle = \langle x, C^*MTv\rangle = \langle x, C^*F(FTF)^\dagger Tv\rangle = 0$$

since $R(F) = N(C^*)$. Thus

$$R(C^{\dagger *}) = R(C) \perp N(I - MT) = N((I - TM)^*)$$

and so

$$R(C^{\dagger *}) \subseteq R(I - TM). \tag{7.2}$$

The result (ii) now follows from (7.1)–(7.2).

To verify (iii),

$$L = C^{\dagger} \Leftrightarrow C^{\dagger}TM = 0 \Leftrightarrow C^{\dagger}A^*(AF)(FA^*AF)^{\dagger} = 0 \Leftrightarrow C^{\dagger}A^*[(AF)^*]^{\dagger} = 0$$
$$\Leftrightarrow C^{\dagger}A^*(AF) = 0 \Leftrightarrow CC^{\dagger}TF = 0 \Leftrightarrow P_CT(I - P_C) = 0$$
$$\Leftrightarrow P_CT = P_CTP_C \Leftrightarrow P_CT = TP_C.$$

That $P_CT = TP_C \Leftrightarrow U = C^{*\dagger}$ follows in a similar manner.

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