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FURTHER RESULTS ON GENERALIZED INVERSES OF PARTITIONED MATRICES*

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Abstract. Necessary and sufficient conditions are given for the blocks in generalized inverses of the partitioned matrices $[A \quad B]$ and $\begin{bmatrix} A \\ C \end{bmatrix}$ to be independent of each other. Some of these conditions are then incorporated into conditions suitable for finding the Moore–Penrose inverse of a particular bordered matrix.

1. Introduction. In [5], the general bordered matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{1.1}$$

was considered, and necessary and sufficient conditions for the blocks in the (1)-, (1, 3)-, and (1, 4)-inverses of M to be independent of each other were given. The present paper is divided into two parts. In the first part, independence of blocks of generalized inverses of the partitioned matrices

$$[A \quad B] \quad \text{and} \quad \begin{bmatrix} A \\ C \end{bmatrix} \tag{1.2}$$

is considered. We first give conditions analogous to the ones given in [5] for the matrix in (1.1). Then we take one of these conditions and find equivalent conditions concerning the individual blocks of the matrices in (1.2). These latter results have partly been given by Cline [3] and Ben-Israel and Greville [2, p. 210].

In the second part of the paper, we incorporate the conditions on the individual blocks of the matrices given in (1.2) into conditions suitable for finding the Moore–Penrose inverse of the special bordered matrix

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}. \tag{1.3}$$

We make use of the results in [2] on intersections of manifolds in finding this inverse under these conditions. The particular method used is also applicable for finding the Moore–Penrose inverse of the matrix given in (1.1) when A, B, or C is zero (under the corresponding assumptions). It is generally desirable to find expressions for the Moore–Penrose inverse of the matrix (1.1) or at least for special cases such as (1.3); these expressions may reduce the sizes of the matrices involved in numerical work, dealing with differential equations and eigenvalue computation [2, p. 228].

All matrices of this paper are over the complex field. If A is a complex matrix, $R(A)$ denotes the range of A, A^* the conjugate transpose of A, $N(A)$ the

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nullspace of A, and $P_{N(A)}$ the orthogonal projection onto $N(A)$. The Moore–Penrose inverse $A^\dagger$ of A is the unique matrix X which satisfies the equations:

$$(1)\ AXA = A, \quad (2)\ XAX = X, \quad (3)\ (AX)^* = AX, \quad (4)\ (XA)^* = XA.$$

In general, if a matrix X satisfies equations (i), (j), and (k), then X is called an (i, j, k)-inverse of A. For properties of these various inverses, the reader is referred to [2].

What is meant by independence of blocks of generalized inverses of the matrix in (1.1) was given in [5]. We shall give here the corresponding definitions for the partitioned matrices (1.2). Let

$$G_1 = \begin{bmatrix} Q_1 \\ L_1 \end{bmatrix} \quad \text{and} \quad G_2 = \begin{bmatrix} Q_2 \\ L_2 \end{bmatrix}$$

be two, possibly different (1)-inverses of $M = [A \quad B]$. Then the blocks of all (1)-inverses of M are said to be independent whenever

$$G = \begin{bmatrix} Q_1 \\ L_2 \end{bmatrix}$$

is a (1)-inverse of M for every possible choice of G_1 and G_2. Independence of blocks of (1, 3)- and (1, 4)-inverses of $[A \quad B]$ and of generalized inverses of $\begin{bmatrix} A \\ C \end{bmatrix}$ is defined similarly.

2. Results on independence. The first theorem gives results similar to the ones obtained in [5] for the matrix (1.1). The proof of the theorem follows along the same lines as the proofs of the theorems in [5] and hence is omitted.

THEOREM 2.1. *For the partitioned matrix*

$$M = [A \quad B],$$

the following statements are equivalent:

(i) *There exists a* (1)-*inverse* M^- *for M such that*

$$M^-M = \begin{bmatrix} *** & 0 \\ 0 & *** \end{bmatrix}.$$

(ii) *The blocks in the* (1)-*inverses for M are independent of each other.*

(iii) *If* M^- *is a* (1)-*inverse for M and we let*

$$M^-M = \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix},$$

then

$$Q_3Q_2 = 0 \quad or \quad Q_2Q_3 = 0.$$

(*The "or" can be replaced by "and".*)

(iv) $$M^\dagger M = \begin{bmatrix} *** & 0 \\ 0 & *** \end{bmatrix}.$$

(v) *The blocks in the* $(1,3)$-*inverses for* M *are independent of each other.*

(vi) *If* $a \in R(M)$ *and* $M^\dagger a = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$, *then* $\|x_0\| \leqq \|x\|$ *and* $\|y_0\| \leqq \|y\|$ *for all* x, y *such that* $M\begin{bmatrix} x \\ y \end{bmatrix} = a$.

There is an analogous set of conditions for the partitioned matrix $\begin{bmatrix} A \\ C \end{bmatrix}$ (excluding the extra equivalent statement (vi) of Theorem 2.1).

The above results are mainly in terms of the forms of $M^- M$ and MM^-. We now give conditions which involve the individual blocks of the partitioned matrices.

THEOREM 2.2. *For the partitioned matrix*

$$M = [A \quad B],$$

the following statements are equivalent:

(i)
$$M^\dagger M = \begin{bmatrix} *** & 0 \\ 0 & *** \end{bmatrix}.$$

(ii) $R(A) \cap R(B) = \{0\}$.
(iii) $R(B^*) = R(B^*(I - AA^\dagger))$.
(iv) $B[(I - AA^\dagger)B]^\dagger B = B$.
(v) $BB^\dagger(AA^\dagger + BB^\dagger)^\dagger B = B$.
(vi) *If* $M^\dagger = \begin{bmatrix} Q \\ * \end{bmatrix}$, $AQA = A$.
(vii) *If* $M^\dagger = \begin{bmatrix} * \\ U \end{bmatrix}$, $BUB = B$.

Proof. Assuming $R(A)$, $R(B) \subseteq C^m$, we have $R(A) \cap R(B) = \{0\} \Leftrightarrow [R(A) \cap R(B)]^\perp = C^m \Leftrightarrow R(A)^\perp + R(B)^\perp = C^m \Leftrightarrow N(A^*) + N(B^*) = C^m \Rightarrow R(B^*) = B^*(N(A^*)) \Leftrightarrow R(B^*B) = B^*(N(A^*)) \Rightarrow R(B) \subseteq N(A^*) + N(B^*) \Rightarrow N(A^*) + N(B^*) = C^m$, and hence (ii)$\Leftrightarrow$(iii).

Now, following Cline [3], let $C = (I - AA^\dagger)B$ and observe that $C^\dagger C = C^\dagger B$. We then have

$$BC^\dagger B = B \Leftrightarrow BC^\dagger C = B \Leftrightarrow C^\dagger CB^* = B^* \Leftrightarrow R(C^*) = R(B^*),$$

and hence (iii)$\Leftrightarrow$(iv).

We next show that (ii)$\Leftrightarrow$(v). From Anderson and Duffin [1], $P_{R(A)\cap R(B)} = 2AA^\dagger(AA^\dagger + BB^\dagger)^\dagger BB^\dagger$, and so

$$R(A) \cap R(B) = \{0\} \Leftrightarrow AA^\dagger(AA^\dagger + BB^\dagger)^\dagger BB^\dagger = 0.$$

Now, $R(AA^\dagger + BB^\dagger) = R(AA^\dagger) + R(BB^\dagger)$, and hence

$$(AA^\dagger + BB^\dagger)(AA^\dagger + BB^\dagger)^\dagger BB^\dagger = BB^\dagger$$

or

$$AA^\dagger(AA^\dagger + BB^\dagger)^\dagger BB^\dagger + BB^\dagger(AA^\dagger + BB^\dagger)^\dagger BB^\dagger = BB^\dagger.$$

Thus

$$R(A)\cap R(B)=\{0\}\Leftrightarrow BB^{\dagger}(AA^{\dagger}+BB^{\dagger})^{\dagger}BB^{\dagger}=BB^{\dagger}\Leftrightarrow BB^{\dagger}(AA^{\dagger}+BB^{\dagger})^{\dagger}B=B,$$

and hence (ii)$\Leftrightarrow$(v).

To finish the proof of the theorem, we show that (i)$\Rightarrow$(ii)$\Rightarrow$(vi)$\Rightarrow$(i). That (i)$\Rightarrow$(ii)$\Rightarrow$(vii)$\Rightarrow$(i) follows similarly. Letting $M^{\dagger}=\begin{bmatrix} Q \\ U \end{bmatrix}$, we have

$$M^{\dagger}M=\begin{bmatrix} QA & QB \\ UA & UB \end{bmatrix}.$$

Now assuming (i) is true, $UA=0$ and $QB=0$. Hence, if $x\in R(A)\cap R(B)$, there exist v, w such that $x=Av=Bw$, and so $Qx=0$ and $Ux=0$. Hence $x\in N(M^{\dagger})=N(M^*)=N(A^*)\cap N(B^*)$, and since $R(A)\cap N(A^*)=\{0\}$, $x=0$. Thus, (i)$\Rightarrow$(ii).

Since $MM^{\dagger}M=M$,

$$AQA+BUA=A \quad \text{or} \quad BUA=A-AQA.$$

Assuming (ii) holds, $A-AQA=0$, and thus (ii)$\Rightarrow$(vi). On the other hand, if we assume (vi), then $BUA=0$. But $(M^{\dagger}M)^*=M^{\dagger}M$ implies $UA=B^*Q^*$ and so $BB^*Q^*=0$. Thus, $B^*Q^*=0$ and hence (vi)$\Rightarrow$(i) follows. The proof of the theorem is now complete.

It should be clear that (iii) is equivalent to rank $B^*=\operatorname{rank}(B^*(I-AA^{\dagger}))$ and that because of condition (ii), conditions (iii–v) are equivalent to

$$R(A^*)=R(A^*(I-BB^{\dagger})),$$

$$A[(I-BB^{\dagger})A]^{\dagger}A=A,$$

$$AA^{\dagger}(AA^{\dagger}+BB^{\dagger})^{\dagger}A=A,$$

respectively. We can also observe that

$$\operatorname{rank} M=\operatorname{rank} A+\operatorname{rank} B\Leftrightarrow R(A)\cap R(B)=\{0\}$$

$$\Leftrightarrow N(M)=\left\{\begin{bmatrix} u \\ v \end{bmatrix}\middle| u\in N(A),\ v\in N(B)\right\}.$$

We should point out that (i)$\Leftrightarrow$(iv) also follows from Corollaries 3.2 and 3.2(a) of Cline [3]. Moreover, forms for $M^{\dagger}$ are given in [3] and in [2] under conditions (iv) and (ii), respectively. It also follows from the proofs in [2] and [3] that

$$M^{\dagger}M=\begin{bmatrix} A^{\dagger}A & 0 \\ 0 & B^{\dagger}B \end{bmatrix}$$

under these conditions.

There is an analogous set of conditions for the partitioned matrix $M=\begin{bmatrix} A \\ C \end{bmatrix}$. In particular, $MM^{\dagger}$ is block-diagonal if and only if $R(A^*)\cap R(C^*)=\{0\}$.

If we consider the bordered matrix

$$M=\begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

then from Theorem 2.2 and Theorem 3 in [5] we have

$$(2.1)\qquad R\left(\begin{bmatrix} A \\ C \end{bmatrix}\right) \cap R\left(\begin{bmatrix} B \\ D \end{bmatrix}\right) = \{0\} \Leftrightarrow M^{\dagger}M \text{ is block-diagonal}$$

$\Leftrightarrow$ the blocks in the (1, 3)-inverses of M are independent of each other. Similarly,

$$R\left(\begin{bmatrix} A^* \\ B^* \end{bmatrix}\right) \cap R\left(\begin{bmatrix} C^* \\ D^* \end{bmatrix}\right) = \{0\} \Leftrightarrow MM^{\dagger} \text{ is block-diagonal}$$

$\Leftrightarrow$ the blocks in the (1, 4)-inverses of M are independent of each other.

These results, however, do not fully answer the questions raised in [5] for this general bordered matrix. But, one consequence of these results is the following. It is known from [4] that for the bordered matrix

$$M = \begin{bmatrix} A^*A & C \\ C^* & 0 \end{bmatrix},$$

$M^{\dagger}M$ is block-diagonal. Thus

$$R\left(\begin{bmatrix} A^*A \\ C^* \end{bmatrix}\right) \cap R\left(\begin{bmatrix} C \\ 0 \end{bmatrix}\right) = \{0\}.$$

This fact can also be proved directly.

3. The Moore–Penrose inverse of a bordered matrix. We now consider the particular bordered matrix in (1.3) and find its Moore–Penrose inverse under certain conditions. We explore the "intersection of manifolds" idea given in [2, p. 201] and used to find the Moore–Penrose inverse of the partitioned matrices given in (1.2).

THEOREM 3.1. *Let*

$$M = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}.$$

Then, if $R(A(I - C^{\dagger}C)) \cap R(B) = \{0\}$, *the matrix*

$$(3.1)\qquad Y = \left[\begin{array}{c|c} Q(P+Q)^{\dagger}[(I-BB^{\dagger})A]^{\dagger} & C^{\dagger} - Q(P+Q)^{\dagger}C^{\dagger} \\ \hline B^{\dagger} - B^{\dagger}AQ(P+Q)^{\dagger}[(I-BB^{\dagger})A]^{\dagger} & -B^{\dagger}AC^{\dagger} + B^{\dagger}AQ(P+Q)^{\dagger}C^{\dagger} \end{array}\right]$$

is a (1, 2, 4)*-inverse for M, where* $P = P_{N((I-BB^{\dagger})A)}$ *and* $Q = P_{N(C)}$. *If we further assume that* $R(A^*(I - BB^{\dagger})) \cap R(C^*) = \{0\}$, *then* $Y = M^{\dagger}$.

Proof. First observe that

$$R\left(\begin{bmatrix} A \\ C \end{bmatrix}\right) \cap R\left(\begin{bmatrix} B \\ 0 \end{bmatrix}\right) = \{0\} \Leftrightarrow R(A(I - C^{\dagger}C)) \cap R(B) = \{0\}$$

and hence from (2.1),

$$(3.2)\qquad M^{\dagger}M \text{ is block-diagonal} \Leftrightarrow R(A(I - C^{\dagger}C)) \cap R(B) = \{0\}.$$

Now suppose $\begin{bmatrix} a \\ b \end{bmatrix} \in R(M)$. Then it is straightforward that

$$M\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{for some } y$$

$$\Leftrightarrow \begin{Bmatrix} Ax + BB^\dagger(a - Ax) = a \\ Cx = b \end{Bmatrix}$$

$$\Leftrightarrow \begin{Bmatrix} (I - BB^\dagger)Ax = (I - BB^\dagger)a \\ Cx = b \end{Bmatrix}$$

$$\Leftrightarrow \begin{Bmatrix} x = [(I - BB^\dagger)A]^\dagger (I - BB^\dagger)a + z,\ z \in N((I - BB^\dagger)A) \\ x = C^\dagger b + w,\ w \in N(C) \end{Bmatrix}$$

$$\Leftrightarrow \begin{Bmatrix} x = [(I - BB^\dagger)A]^\dagger a + z,\ z \in N((I - BB^\dagger)A) \\ x = C^\dagger b + w,\ w \in N(C) \end{Bmatrix}$$

as

$$[(I - BB^\dagger)A]^\dagger (I - BB^\dagger) = [(I - BB^\dagger)A]^\dagger. \tag{3.3}$$

Thus, using (a′) of Corollary 3 in [2, Chap. 5], we then have

$$M\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{for some } y \Leftrightarrow x = C^\dagger b - Q(P+Q)^\dagger (C^\dagger b - [(I - BB^\dagger)A]^\dagger a) + z,$$
$$z \in N((I - BB^\dagger)A) \cap N(C).$$

Moreover, from [2],

$$\bar{x} = C^\dagger b - Q(P+Q)^\dagger (C^\dagger b - [(I - BB^\dagger)A]^\dagger a)$$

is the vector of minimum norm in this set.

Next, we consider the vectors y. We have

$$M\begin{bmatrix} \bar{x} \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \Leftrightarrow By = a - A\bar{x} \Leftrightarrow y = B^\dagger (a - A\bar{x}) + z, \qquad z \in N(B),$$

and

$$\bar{y} = B^\dagger a - B^\dagger A \bar{x}$$

is the vector of minimum norm in this set.

We now assume that $R(A(I - C^\dagger C)) \cap R(B) = \{0\}$. Then it follows from (3.2) and Theorem 2.1 (or Theorem 3 in [5]) that $M^\dagger \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix}$. But from (3.1), $Y\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix}$, and hence Y is a $(1,4)$-inverse for M.

To verify that Y is a (2)-inverse of M, observe that $[(I - BB^\dagger)A]^\dagger B = 0$ from (3.3) and hence

$$YM = \left[\begin{array}{c|c} Q(P+Q)^\dagger [(I - BB^\dagger)A]^\dagger A + C^\dagger C - Q(P+Q)^\dagger C^\dagger C & 0 \\ \hline 0 & B^\dagger B \end{array}\right].$$

Now,

$$\begin{aligned}&Q(P+Q)^{\dagger}[(I-BB^{\dagger})A]^{\dagger}A+C^{\dagger}C-Q(P+Q)^{\dagger}C^{\dagger}C\\&\quad=Q(P+Q)^{\dagger}[(I-BB^{\dagger})A]^{\dagger}(I-BB^{\dagger})A+C^{\dagger}C\\&\qquad-Q(P+Q)^{\dagger}C^{\dagger}C\\&\quad=C^{\dagger}C+Q(P+Q)^{\dagger}(-C^{\dagger}C+[(I-BB^{\dagger})A]^{\dagger}(I-BB^{\dagger})A),\end{aligned}$$

and the proof that Y is a (2)-inverse of M then follows as in the proof of Theorem 6 in [2, Chap. 5]. The reader can see [2] for the details.

By direct multiplication,

$$MY=\left[\begin{array}{c|c}AQ(P+Q)^{\dagger}[(I-BB^{\dagger})A]^{\dagger}+BB^{\dagger}-BB^{\dagger}AQ(P+Q)^{\dagger}[(I-BB^{\dagger})A]^{\dagger} & AC^{\dagger}-AQ(P+Q)^{\dagger}C^{\dagger}-BB^{\dagger}AC^{\dagger}+BB^{\dagger}AQ(P+Q)^{\dagger}C^{\dagger}\\\hline 0 & CC^{\dagger}\end{array}\right]$$

or

$$MY=\left[\begin{array}{c|c}BB^{\dagger}+(I-BB^{\dagger})AQ(P+Q)^{\dagger}[(I-BB^{\dagger})A]^{\dagger} & (I-BB^{\dagger})AC^{\dagger}-(I-BB^{\dagger})AQ(P+Q)^{\dagger}C^{\dagger}\\\hline 0 & CC^{\dagger}\end{array}\right].$$

We now make the further assumption that $R(A^*(I-BB^{\dagger}))\cap R(C^*)=\{0\}$ (or equivalently that $MM^{\dagger}$ is block-diagonal) and so from [2] again, $(P+Q)$ $P+Q)^{\dagger}=I$. Hence

$$(I-BB^{\dagger})AQ(P+Q)^{\dagger}=(I-BB^{\dagger})A(-P+P+Q)(P+Q)^{\dagger}=(I-BB^{\dagger})A,$$

and we have

$$MY=\left[\begin{array}{c|c}BB^{\dagger}+(I-BB^{\dagger})A[(I-BB^{\dagger})A]^{\dagger} & 0\\\hline 0 & CC^{\dagger}\end{array}\right].$$

Thus $(MY)^*=MY$, and the theorem is now proved.

Using Theorem 2.2 we can give other conditions equivalent to $R(A(I-C^{\dagger}C))\cap R(B)=\{0\}$ and $R(A^*(I-BB^{\dagger}))\cap R(C^*)=\{0\}$. Moreover, it is easy to verify that

$$R\left(\begin{bmatrix}A\\C\end{bmatrix}\right)\cap R\left(\begin{bmatrix}B\\0\end{bmatrix}\right)=\{0\}\Leftrightarrow N(A)\cap N(C)=N((I-BB^{\dagger})A)\cap N(C)$$

and thus

$$R(A(I-C^{\dagger}C))\cap R(B)=\{0\}\Leftrightarrow N(A)\cap N(C)=N((I-BB^{\dagger})A)\cap N(C).$$

Similarly,

$$R(A^*(I-BB^{\dagger}))\cap R(C^*)=\{0\}\Leftrightarrow N(A^*)\cap N(B^*)=N((I-C^{\dagger}C)A^*)\cap N(B^*).$$

If we use forms (a) and (b) of Corollary 3 of [2, Chap. 5] in the proof of Theorem 3.1, we obtain two other expressions for $\bar{x}$. These are

$$\bar{x}=[(I-BB^{\dagger})A]^{\dagger}a+P(P+Q)^{\dagger}(C^{\dagger}b-[(I-BB^{\dagger})A]^{\dagger}a)$$

and

$$\bar{x}=([(I-BB^{\dagger})A]^{\dagger}(I-BB^{\dagger})A+C^{\dagger}C)^{\dagger}([(I-BB^{\dagger})A]^{\dagger}a+C^{\dagger}b),$$

respectively. From these expressions, two other forms for Y can directly be written down.

The fact that the lower-right block of M is zero plays no special role. If instead, A, B or C is zero, then $M^{\dagger}$ can be derived under corresponding conditions, either by using the above method or by "flipping" the blocks in M, using the unitary matrix $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$. The method given in [2] to obtain generalized inverses of bordered matrices has also been used by L. Mihalyffy [6].

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