Newton Method for Solution of Nonlinear Equations

(Dr. Tom Co 9/6/2008)

Problem: Given function $f(x)$, we want to find a root x_* such that $f(x_*) = 0$.

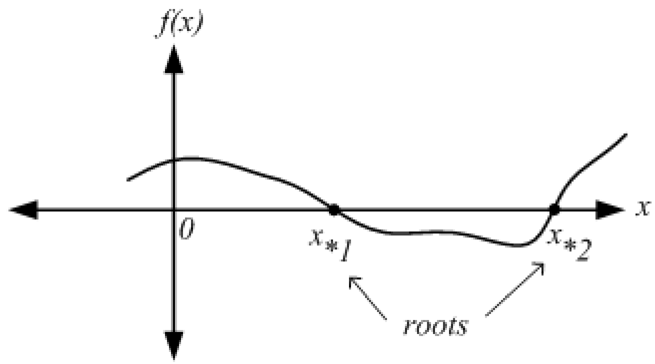


Figure 1. Roots of *f(x)*.

Main Idea: Assume that the initial guess x_{guess} is close to a root such that the function $f(x)$ can be approximated by a line as shown in Figure 2.

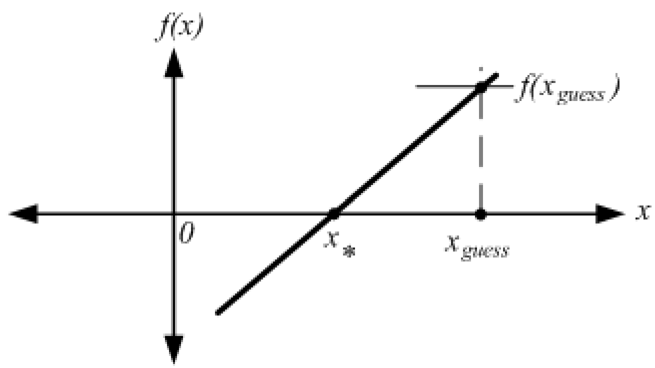


Figure 2. Approximate line.

then

$$\left.\frac{df}{dx}\right|_{x=x_{guess}} = \frac{\text{rise}}{\text{run}} = \frac{f(x_{\text{guess}}) - 0}{x_{\text{guess}} - x_*}$$

Rearranging,

$$x_* = x_{\text{guess}} - \frac{f(x_{\text{guess}})}{\left.\frac{df}{dx}\right|_{x=x_{guess}}}$$

Since the assumption may not be true, multiple iterations are needed. The Newton method is then given by

$$x_{k+1} = x_k - \frac{f(x_k)}{\left.\frac{df}{dx}\right|_{x=x_k}}$$

with $x_0 = x_{\text{guess}}$ and the iterations ends when $|x_{k+1} - x_k| < \text{tolerance}$.

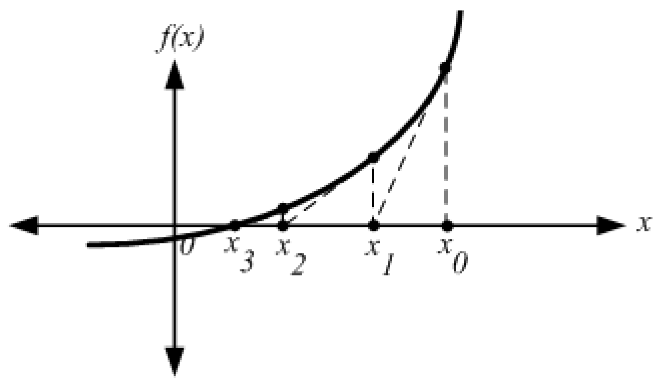
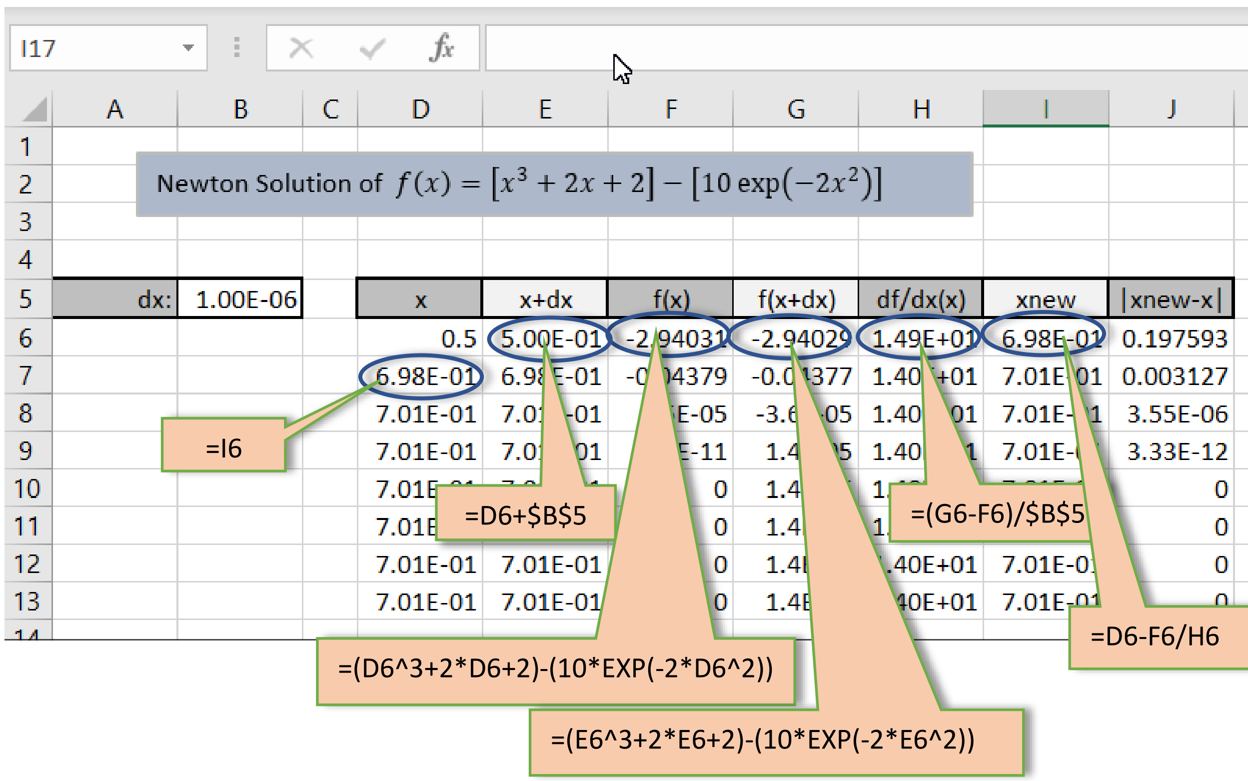


Figure 3. Newton method.

Remarks:

1. If the function is too complicated, an approximate derivative is often used.
2. If the derivative is close to zero, then the method may diverge.
3. If the function is highly nonlinear, the method may diverge unless the guess is very close to the root.
4. Multiple roots are possible in general. One needs to choose initial guesses carefully to obtain the desired roots.

Example:



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Newton Solution of $f(x) = [x^3 + 2x + 2] - [10 \exp(-2x^2)]$
dx:
1.00E-06
x
x+dx
f(x)
f(x+dx)
df/dx(x)
xnew
|xnew-x|
=I6
=D6+$B$5
=(G6-F6)/$B$5
=D6-F6/H6
=(D6^3+2*D6+2)-(10*EXP(-2*D6^2))
=(E6^3+2*E6+2)-(10*EXP(-2*E6^2))