Obtaining the Roots of a Cubic Equations

Given a cubic equation,

$$z^3 + A\cdot z^2 + B\cdot z + C = 0$$

Let

$$z = x - \frac{A}{3}$$

then

$$x^3 + \left(\frac{-1}{3}\cdot A^2 + B\right)\cdot x + \frac{2}{27}\cdot A^3 + C - \frac{1}{3}\cdot B\cdot A = 0$$

or

$$x^3 + p\cdot x = q$$

where,

$$p = \left(\frac{-1}{3}\cdot A^2 + B\right)$$

$$q = -\left(\frac{2}{27}\cdot A^3 + C - \frac{1}{3}\cdot B\cdot A\right)$$

Further, let

$$x = y - \frac{p}{3y}$$

then we obtain a 6th order polynomial equation in y given by

$$y^6 - q\cdot y^3 - \frac{1}{27}\cdot p^3 = 0$$

whose roots are:

$$y = \left[\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}\right]^{\frac{1}{3}}$$

Now let the discriminant Δ be the term inside the square root above, i.e.

$$\Delta = \left(\frac{q}{2}\right)^2 + \left(\frac{P}{3}\right)^3$$

then we will have two cases that will depend on whether the discriminant is positive or negative.

Case 1: $\Delta > 0$ Then we will have one real root and a complex conjugate pair

The first root is given by

$$z_1 = sign(h)\cdot\left(|h|\right)^{\frac{1}{3}} - \frac{A}{3}$$

where,

$$h = \frac{q}{2} + \sqrt{\Delta}$$

The other roots can then be obtained by using the values of the first root:

$$z_2 = \frac{-\left(A + z_1\right) + \sqrt{\left(A + z_1\right)^2 - 4\cdot\left[B + z_1\cdot\left(A + z_1\right)\right]}}{2}$$

$$z_3 = \frac{-\left(A + z_1\right) - \sqrt{\left(A + z_1\right)^2 - 4\cdot\left[B + z_1\cdot\left(A + z_1\right)\right]}}{2}$$

Case 2: $\Delta < 0$ There will be three real roots.

The first root will be obained as follows (whose proof is given below):

$$z_1 = \left[2\cdot\left(\sqrt{\frac{-p}{3}}\right)\right]\cdot cos\left(\frac{atan\left(\frac{\sqrt{-\Delta}}{\frac{q}{2}}\right)}{3}\right) - \frac{A}{3}$$

And the two remaining roots can be determined by the following equations:

$$z_2 = \frac{-\left(A + z_1\right) + \sqrt{\left(A + z_1\right)^2 - 4\cdot\left[B + z_1\cdot\left(A + z_1\right)\right]}}{2}$$

$$z_3 = \frac{-\left(A + z_1\right) - \sqrt{\left(A + z_1\right)^2 - 4\cdot\left[B + z_1\cdot\left(A + z_1\right)\right]}}{2}$$

Proof for formula to obtain the first root:

Let $h = \frac{q}{2} + i\sqrt{-\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}$

whose magnitude and angle are given by

$$|h| = \sqrt{\left(\frac{-p}{3}\right)^3}$$

$$\theta = arg(h) = atan\left(\frac{\sqrt{-\Delta}}{\frac{q}{2}}\right)$$

allowing one to evaluate the cube root of the polar representation:

$$y = h^{\frac{1}{3}} = \sqrt{\frac{-p}{3}}\cdot e^{i\cdot\frac{\theta}{3}}$$

from which we obtain

$$x = \sqrt{\frac{-p}{3}}\cdot e^{i\cdot\frac{\theta}{3}} + \frac{\frac{-p}{3}}{\sqrt{\frac{-p}{3}}\cdot e^{i\cdot\frac{\theta}{3}}}$$

$$x = \sqrt{\frac{-p}{3}} \cdot \left(e^{i \cdot \theta} + e^{-i \cdot \theta}\right) = 2 \cdot \sqrt{\frac{-p}{3}} \cdot \cos\left(\frac{\theta}{3}\right)$$

or

$$z_1 = x - \frac{A}{3} = 2 \cdot \sqrt{\frac{-p}{3}} \cdot \cos\left(\frac{\theta}{3}\right) - \frac{A}{3}$$