Cubic Interpolation in MathCad
(Dr. Tom Co 10/11/2008)

Introduction.

Standard physical data are usually given in the form of tables, e.g. steam tables. Alternatively, the data can come as graphs and, in some cases, as empirical formulas. These are considered to be reliable data. While graphical forms are more compact, the tabular forms minimize imprecision or errors that could arise from reading of the graphs. However, tabulated data can not cover a continuum of values. This means that interpolation is needed to obtain values that lie between tabulated values.

Linear Interpolation

The simplest interpolation approach is linear interpolation. It evaluates the desired value to lie in a line connecting two consecutive points of a table.

Consider the values given in Table 1. Suppose we want to determine the value of y corresponding to $x = 2.25$. Focusing on the entries $(x, y) = (2.0, 0.368)$ and $(x, y) = (2.5, 0.105)$,

Table 1. *x*-*y* data.

x	*y*
0.0	0.368
0.5	0.779
1.0	1.00
1.5	0.779
2.0	0.368
2.5	0.105
3.0	0.018

we can use the property of similar triangles as shown in Figure 1 to obtain

$$\frac{y - 0.105}{0.368 - 0.105} = \frac{2.25 - 2.5}{2 - 2.5} \quad \rightarrow \quad y = 0.105 + (0.368 - 0.105)\left(\frac{2.25 - 2.5}{2 - 2.5}\right)$$

$$y = 0.236$$

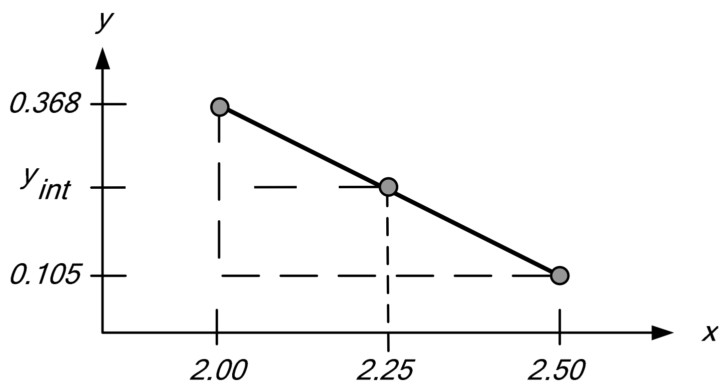

Figure 1. Linear interpolation example.

Linear interpolation, however, can lead to inaccuracies when the slope from one data segment is significantly different from the neighboring segments. Another disadvantage is the loss of smoothness at the tabulated points as shown in Figure 2.

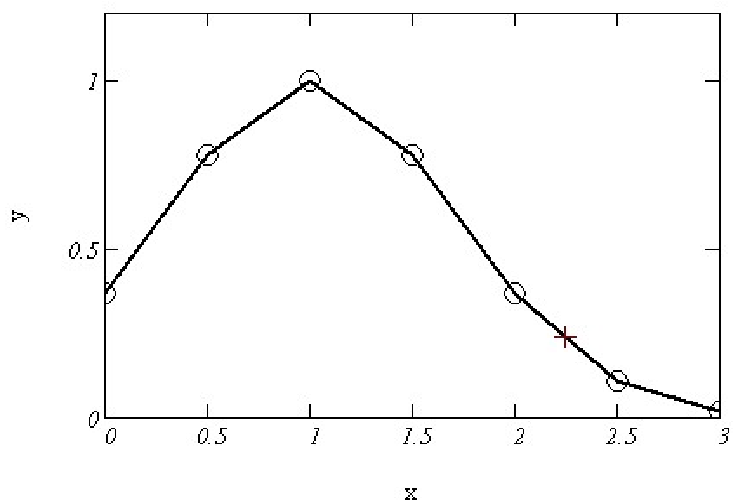

Figure 2. Linear interpolation lines. The circles (o) are the data points given in Table 1, while the cross (+) symbol represents the interpolated point for $x = 2.25$.

Cubic Interpolation

Another approach is to use a cubic polynomial to evaluate interpolated values. Details of this approach can be found in Appendix 1 and 2. This method obtains a piecewise continuous function that has continuous first and second order derivatives. Figure 3 shows how cubic interpolation is applied on the data given in Table 2.

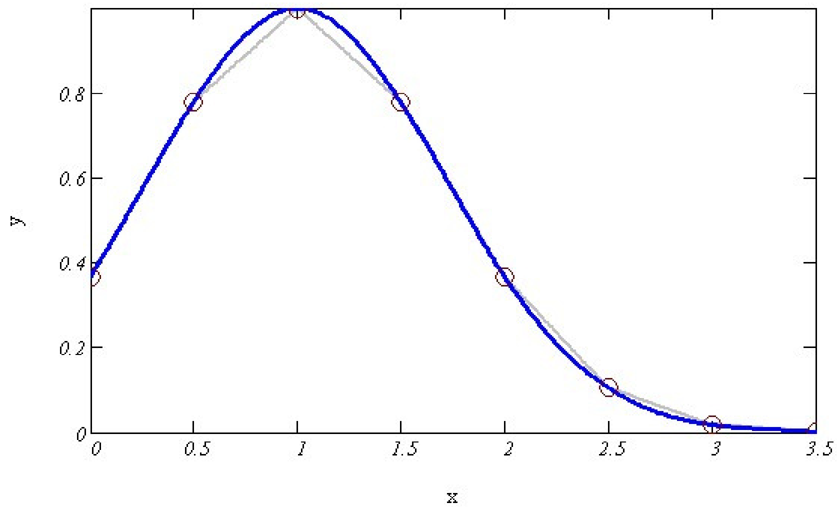

Figure 3. Cubic interpolation overlayed with linear interpolated data.

MathCad Procedures

Let the independent variable data be given by $xdata_i$, $i = 0,1 \ldots, N$ and the dependent variable by given by $ydata_i$, $i = 0,1, \ldots, N$.

A. **Linear Interpolation**

$$y(x) := \textbf{linterp}(\,xdata, ydata, x)$$

B. **Cubic Interpolation**

- First, obtain the vector of second-order derivatives

$$\begin{aligned} & vs := \textbf{lspline}\ (\,xdata, ydata\,) \\ \text{or}\quad & vs := \textbf{pspline}\ (\,xdata, ydata\,) \\ \text{or}\quad & vs := \textbf{cspline}\ (\,xdata, ydata\,) \end{aligned}$$

- Next, obtain the interpolation

$$y(x) := \textbf{interp}\ (\,vs, xdata, ydata, x\,)$$

C. **2D Cubic Interpolation**

Assume that the data is given in a data table by vectors $xdata_i$, $ydata_j$ and and matrix $zdata_{i,j}$, $i, j = 0,1, \ldots, N$. Let $Mxy := \textbf{augment}(xdata, ydata)$, then

$$\begin{aligned} & vs := \textbf{cspline}\ (\,Mxy, zdata) \\ & z(x, y) := \textbf{interp}\ \left(\,vs, Mxy, zdata, \begin{pmatrix} x \\ y \end{pmatrix}\right) \end{aligned}$$

Appendix 1. Cubic Interpolation

Consider two consecutive points a and b whose values are given by (x_a, y_a) and (x_b, y_b), respectively, and whose second derivatives are given by v_a and v_b, respectively. Then to determine the value corresponding to $x \in [x_a, x_b]$, first obtain the polynomial

$$p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$$

to satisfy the following four conditions for $\alpha_1, \alpha_2, \alpha_3, \alpha_4$:

$$\begin{aligned} y_a &= \alpha_0 + \alpha_1 x_a + \alpha_2 x_a^2 + \alpha_3 x_a^3 \\ y_a &= \alpha_0 + \alpha_1 x_b + \alpha_2 x_b^2 + \alpha_3 x_b^3 \\ v_a &= 2\alpha_2 + 6\alpha_3 x_a \\ v_b &= 2\alpha_2 + 6\alpha_3 x_b \end{aligned}$$

Thereafter, evaluate $y = p(x)$.

Appendix 2. Obtaining Second Derivatives Needed for Cubic Interpolation

To determine the second derivatives needed for cubic interpolation, we assume cubic polynomials for $x \in [x_k, x_{k+1}]$ given by

$$p_k(x) = A_{k,0} + A_{k,1} x + A_{k,2} x^2 + A_{k,3} x^3$$

subject to additional constraints that the piecewise-continuous function passing through the given points will have continuous derivatives and continuous second order derivatives throughout the domain including the "knot" points, i.e. at the points where two polynomial meet.

First consider three consecutive points a, b and c described by $(x_a, y_a), (x_b, y_b)$ and (x_c, y_c), respectively. Let $p_L(x)$ and $p_R(x)$ be the polynomials for $x \in [x_a, x_b]$ and $x \in [x_b, x_c]$, respectively,

$$p_L(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$$

$$p_R(x) = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3$$

Let v_a, v_b and v_c be the second order derivatives at a, b and c, respectively. Then

$$v_a = 2\alpha_2 + 6\alpha_3 x_a \qquad v_b = 2\alpha_2 + 6\alpha_3 x_b \qquad v_b = 2\beta_2 + 6\beta_3 x_b \qquad v_c = 2\beta_2 + 6\beta_3 x_a$$

From which we obtain

$$\alpha_3 = \frac{1}{6}(v_a - v_b)(x_a - x_b) \qquad \alpha_2 = \frac{1}{2}(v_b - 6\alpha_3 x_b)$$

$$\beta_3 = \frac{1}{6}(v_b - v_c)(x_b - x_c) \qquad \beta_2 = \frac{1}{2}(v_b - 6\beta_3 x_b)$$

At the knot points, $y_a = p_L(x_a), y_b = p_L(x_b), y_b = p_R(x_b)$ and $y_c = p_R(x_c)$, which results in

$$\alpha_1 = \frac{y_b - y_a}{x_b - x_a} - \alpha_2(x_b + x_a) - \alpha_3(x_b^2 + x_a x_b + x_a^2)$$

$$\beta_1 = \frac{y_c - y_b}{x_c - x_b} - \beta_2(x_c + x_b) - \beta_3(x_c^2 + x_b x_c + x_b^2)$$

By forcing continuity in the first order derivative at $x = x_b$, we have

$$\alpha_1 + 2\alpha_2 x_b + 3\alpha_3 x_b^2 = \beta_1 + 2\beta_2 x + 3\beta_3 x_b^2$$

After substituting the values found earlier for the α_i's and the β_i's into the last equation, we end up with

$$\frac{x_b - x_a}{6} v_a + \frac{x_c - x_a}{3} v_b + \frac{x_c - x_b}{6} v_c = \left(\frac{y_c - y_b}{x_c - x_b}\right) - \left(\frac{y_b - y_a}{x_b - x_a}\right)$$

This means that for a data table consisting of $(N + 1)$ points: $(x_0, y_0), \ldots, (x_N, y_N)$, we have $(N - 1)$ equations for $k = 1,2, \ldots, N - 1$, given by

$$\frac{x_k - x_{k-1}}{6} v_{k-1} + \frac{x_{k+1} - x_{k-1}}{3} v_k + \frac{x_{k+1} - x_k}{6} v_{k+1} = \left(\frac{y_{k+1} - y_k}{x_{k+1} - x_k}\right) - \left(\frac{y_k - y_{k-1}}{x_k - x_{k-1}}\right)$$

However, there are $(N + 1)$ unknowns, which are the second order derivatives $v_0, v_1, \ldots, v_N$. Thus, we still need to specify two more conditions. There are three different ways that MathCad allows for these two conditions:

a) **lspline( )** , which specifies $v_0 = 0$ and $v_N = 0$.
b) **pspline( )**, which specifies $v_0 = v_1$ and $v_N = v_{N-1}$.
c) **cspline( )**, which specifies

$$\frac{v_0 - v_1}{x_0 - x_1} = \frac{v_2 - v_1}{x_2 - x_1} \qquad \text{and} \qquad \frac{v_{N-2} - v_{N-1}}{x_{N-2} - x_{N-1}} = \frac{v_N - v_{N-1}}{x_N - x_{N-1}}$$