Successive Substitution

(by Dr. Tomas Co 5/7/2008)

Definition:

A numerical method for solving a nonlinear equation for the unknown.

Main Idea:

1. Rewrite a nonlinear function into a form given by

$$x = f(x) \tag{1}$$

2. Starting with an initial guess, x_0, evaluate $f(x_0)$ to yield x_1. Continue the iteration

$$x_{k+1} = f(x_k) \qquad k = 1,2,\ldots \tag{2}$$

until the result no longer changes to within a specified tolerance, i.e. after m iterations where

$$|x_{m+1} - x_m| \leq \epsilon \tag{3}$$

Example:

Find x that solves the following equation

$$x^3 + 2x + 2 = 10e^{-2x^2} \tag{4}$$

Rearranging equation (4) to the following form,

$$x = f(x) = \sqrt{-\frac{1}{2}\ln\left(\frac{x^3 + 2x + 2}{10}\right)} \tag{5}$$

Then the spreadsheet can be implemented as given in Figure 1.

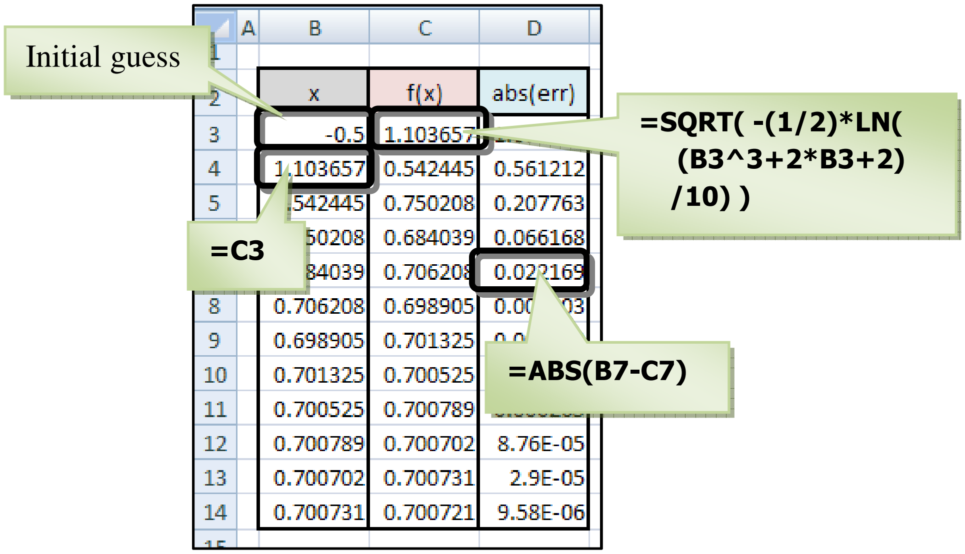

Figure 1. Solution using successive substitution.

Remarks:

1. The convergence is highly dependent on how ones defines $f(x)$. For instance, if we rearranged equation (4) to be

$$x = f(x) = \frac{1}{2}\left(10e^{-2x^2} - (x^3 + 2)\right) \tag{6}$$

then the method will diverge.

2. Let x^* be the solution and x_0 be the initial condition. One sufficient condition for convergence is that the slope of $f(x)$ is between 1 and -1 as shown in Figure 2 and 3.

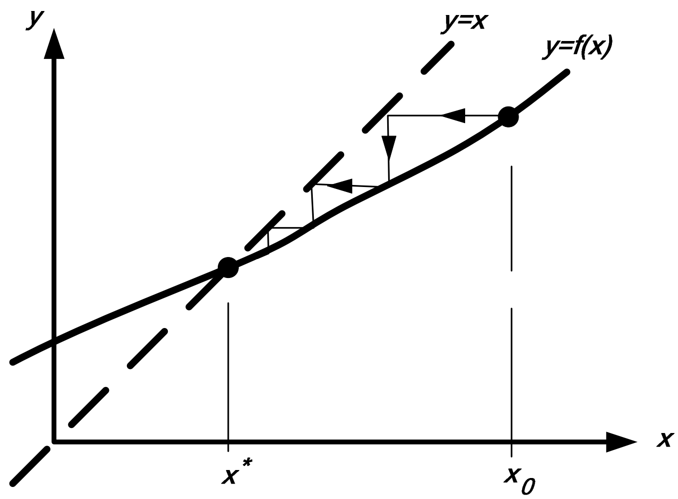

Figure 2. Slope of $f(x)$ in the range $x^* \pm x_0$ is between 0 and 1.

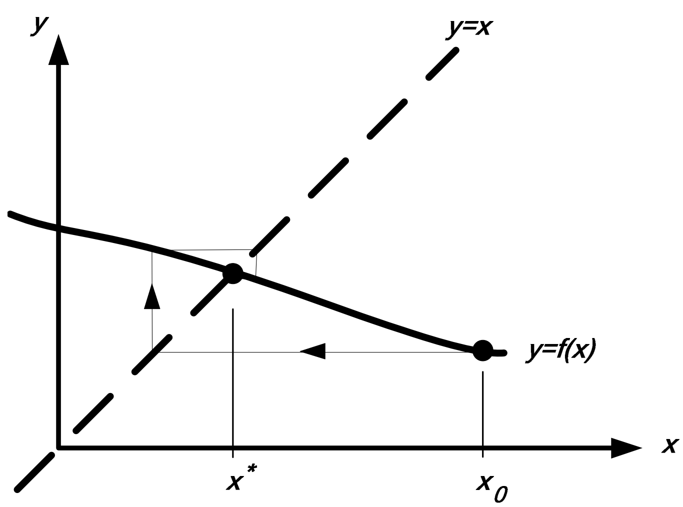

Figure 3. Slope of $f(x)$ in the range $x^* \pm x_0$ is between -1 and 0.