Linearization

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Main Result:

Given a nonlinear function $f(x_1, \ldots, x_n)$ and a specified point $\mathbf{p} = (x_{1o}, \ldots, x_{no})$, we want a linear approximation around the point $\mathbf{p}$ is given by

$$f(x_1, \ldots, x_n) \approx \alpha_0 + \alpha_1(x_1 - x_{1o}) + \cdots + \alpha_n(x_n - x_{no}) \tag{1}$$

Then,

$$\begin{aligned} \alpha_0 &= f(x_{1o}, \ldots, x_{no}) \\ \alpha_1 &= \left.\frac{\partial f}{\partial x_1}\right|_{(x_{1o},\ldots,x_{no})} \\ &\vdots \\ \alpha_n &= \left.\frac{\partial f}{\partial x_n}\right|_{(x_{1o},\ldots,x_{no})} \end{aligned} \tag{2}$$

Example:

Given $f(T, C) = C \cdot \exp(-E_r/T)$ where $E_r = 1000\ K$.

Then, for the specified point $(T_0, C_0) = (350K, 0.5\ g/liter)$,

$$\begin{aligned} \alpha_0 &= f(T_0, C_0) = C_0 \cdot \exp(-E_r/T_0) = 0.0287 \\ \alpha_1 &= \left.\frac{\partial f}{\partial T}\right|_{(T_0,C_0)} = \frac{C_0 E_r}{T_0^2} \exp\left(-\frac{E_r}{T_0}\right) = 2.34 \times 10^{-4} \\ \alpha_2 &= \left.\frac{\partial f}{\partial C}\right|_{(T_0,C_0)} = \exp\left(-\frac{E_r}{T_0}\right) = 0.0574 \end{aligned}$$

Thus,

$$\begin{gathered} f(T, C) \approx 0.0287 \frac{g}{liter} + \left(2.34 \times 10^{-4} \frac{g}{K \cdot liter}\right)(T - 350K) \\ + (0.0574)\left(C - 0.5 \frac{g}{liter}\right) \end{gathered}$$

Derivation:

The Taylor series expansion for a multivariable function $f(x_1, \dots, x_n)$ around a specified point $\mathbf{p} = (x_{10}, \dots, x_{n0})$ is an infinite series given by

$$f(x_1, \dots, x_n) = \alpha_0 + S_1(x_1, \dots, x_n) + S_2(x_1, \dots, x_n) + \cdots \tag{3}$$

where,

$$S_1(x_1, \dots, x_n) = \alpha_1(x_1 - x_{10}) + \cdots + \alpha_n(x_n - x_{n0})$$

$$\begin{aligned} S_2(x_1, \dots, x_n) = {} & \alpha_{11}(x_1 - x_{10})^2 + \alpha_{12}(x_1 - x_{10})(x_2 - x_{20}) + \cdots \\ & +\alpha_{1n}(x_1 - x_{10})(x_n - x_{n0}) \\ & +\alpha_{22}(x_2 - x_{20})^2 + \alpha_{23}(x_2 - x_{20})(x_3 - x_{30}) + \cdots \\ & +\alpha_{2n}(x_2 - x_{20})(x_n - x_{n0}) \\ & \vdots \\ & +\alpha_{nn}(x_n - x_{n0})^2 \end{aligned}$$

and terms for S_k will be a sum of terms of the form: $\beta(x_1 - x_{10})^{\ell_1} \cdots (x_n - x_{n0})^{\ell_n}$ where β is a coefficient, $0 \le \ell_i \le k$ and $\sum_{i=1}^{k} \ell_i = k$.

When we set $(x_1, \dots, x_n) = (x_{10}, \dots, x_{n0})$ in (3), we note that $S_1, S_2, \dots$ all drop to zero, leaving only α_0 on the right hand side of the equation. Thus,

$$f(x_{10}, \dots, x_{n0}) = \alpha_0$$

Next, take the partial derivative of f with respect to x_i, then

$$\frac{\partial f}{\partial x_i} = \alpha_i + \frac{\partial S_2}{\partial x_i} + \cdots + \frac{\partial S_k}{\partial x_i} + \cdots$$

With each term of S_k of the form: $\beta(x_1 - x_{10})^{\ell_1} \cdots (x_n - x_{n0})^{\ell_n}$, $\sum_i \ell_i = k$, the partial derivative of this term with respect to x_i will be zero if $\ell_i = 0$ or equal to

$$\beta \ell_i (x_1 - x_{10})^{\ell_1} \cdots (x_i - x_{i0})^{\ell_i - 1} \cdots (x_n - x_{n0})^{\ell_n}$$

if $\ell_i \ge 1$, which also becomes zero once we set $(x_1, \dots, x_n) = (x_{10}, \dots, x_{n0})$. Thus, after taking the partial derivative and the setting $(x_1, \dots, x_n) = (x_{10}, \dots, x_{n0})$, we find that

$$\left. \frac{\partial f}{\partial x_i} \right|_{x_{10}, \dots, x_{n0}} = \alpha_i$$

Although the other coefficients in the Taylor series can be found by taking higher order partial derivatives, we turn ourselves instead to the situation in which $(x_1, \dots, x_n)$ is close to point $(x_{10}, \dots, x_{n0})$, i.e. $|x_i - x_{i0}| < \delta$ where $\delta < 1$ is a small number. The linearization process then depends on that fact that $\cdots < \delta^3 < \delta^2 < \delta$, which then allows us to consider dropping the higher order terms in (3), i.e.

$$f(x_1, \dots, x_n) \approx \alpha_0 + \alpha_1(x_1 - x_{10}) + \cdots + \alpha_n(x_n - x_{n0})$$

where

$$\alpha_0 = f(x_{10}, \dots, x_{n0})$$

and

$$\alpha_i = \left.\frac{\partial f}{\partial x_i}\right|_{x_{10},\dots,x_{n0}}$$