Short Review of Basic Mathematics

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Chapter 1

Review of Functions

1.1 Mathematical Identities

1. $e^{-b} = \frac{1}{e^b}$

2. $e^{a+b} = e^a e^b$

3. $e^{\ln(x)} = x$

4. $\ln\left(e^x\right) = x$

5. $\ln(ab) = \ln(a) + \ln(b)$

6. $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$

7. $\ln\left(x^a\right) = a\ln(x)$

8. $\log(x) = \frac{\ln(x)}{\ln(10)} \approx \frac{\ln(x)}{2.303}$

9. $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$

10. $\sin(a+b) = \cos(a)\sin(b) + \sin(a)\cos(b)$

11. $\cos^2(x) + \sin^2(x) = 1$

12. $\cos^2(x) - \sin^2(x) = \cos(2x)$

13. $2\sin(x)\cos(x) = \sin(2x)$

14. $ax^2 + bx + c = 0 \quad \rightarrow \quad x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

15. $(a+b)(a-b) = a^2 - b^2$

16. $(a+b)^2 = a^2 + 2ab + b^2$

17. $(a+b)^n = a^n + na^{n-1}b + \ldots + \left(\dfrac{n!}{(n-k)!k!}\right) a^{n-k}b^k + \ldots + b^n$

. where $n! = (n)(n-1)\cdots(3)(2)(1)$ and $0! = 1$

Define the imaginary number: $i = \sqrt{-1}$

18. $(a+ib)(a-ib) = a^2 + b^2$

19. $e^{it} = \cos(t) + i\sin(t)$

20. $e^{-it} = \cos(t) - i\sin(t)$

21. $\cos(t) = \dfrac{e^{it} + e^{-it}}{2}$

22. $\sin(t) = \dfrac{e^{it} - e^{-it}}{2i}$

23. $\dfrac{d}{dt} t^a = a\ t^{a-1}$

24. $\dfrac{d}{dt} a^t = a^t \ln(a)$

25. $\dfrac{d}{dt} e^{at} = ae^{at}$

26. $$\frac{d}{dt}\ln(at) = \frac{1}{t}$$

27. $$\frac{d}{dt}\sin(t) = \cos(t)$$

28. $$\frac{d}{dt}\cos(t) = -\sin(t)$$

29. Derivative of Sums: $$\frac{d}{dt}\Big(f(t)+g(t)\Big) = \frac{df}{dt} + \frac{dg}{dt}$$

30. Derivative of Products: $$\frac{d}{dt}\Big(f(t)g(t)\Big) = \left(\frac{df}{dt}\right) g(t) + f(t)\left(\frac{dg}{dt}\right)$$

31. Derivative of Fractions: $$\frac{d}{dt}\left(\frac{f(t)}{g(t)}\right) = \frac{1}{f^2(t)}\left(f(t)\frac{dg}{dt} - g(t)\left(\frac{df}{dt}\right)\right)$$

32. Chain Rule: $$\frac{d}{dt}f\Big(u(t)\Big) = \frac{df}{du}\frac{du}{dt}$$

33. $$\int_a^b \Big(p\ f(t)\ +\ q\ g(t)\Big)\,dt = p\int_a^b f(t)dt + q\int_a^b g(t)dt$$

34. $$\int_a^b f(t)dt + \int_b^c f(t)dt = \int_a^c f(t)dt$$

35. Integration by parts: $$\int_{v(a)}^{v(b)} u\ dv = \Big(u(b)v(b) - u(a)v(a)\Big) - \int_{u(a)}^{u(b)} v\ du$$

1.2 Drills

1. Find $\dfrac{d}{dt}\left(2e^{1+3t} - t^2 + 10\right)$

2. Find $\dfrac{d}{dt}\left(\ln(at)\, e^{t+it^2}\right)$

3. Find the roots of : $3x^2 + 2x = 4$

4. Find the solution of : $2\log(x+3) = \log(x+15)$

5. Expand $(x+y)^4$ into a polynomial in x and y

6. Prove the following identity:

$$(a+ib)\,e^{p+iq} + (a-ib)\,e^{p-iq} = e^{p}\Big(2a\cos(q) - 2b\sin(q)\Big)$$

7. Find $\displaystyle\int_0^{10} \left(2 + t^2 e^{-2t}\right) dt$

8. Let r be a function of t. Show that the following is true:

$$\frac{d}{dt}\left(2\pi r^2 + 5r\right) = (4\pi r + 5)\,\frac{dr}{dt}$$

9. Find k and ϕ such that: $2\cos(3t) + 4\sin(3t) = k\sin(3t+\phi)$

10. Find $\displaystyle\int_0^2 t f(t) dt,$ where $f(t) = \begin{cases} t & \text{if } t \le 1 \\ 2-t & \text{if } t > 1 \end{cases}$

1.3 Answers to Drills

1. $6e^{1+3t} - 2t$

2. $\left(\dfrac{1}{t} + \ln(at)(1+2it)\right) e^{t+it^2}$

3. $x_1 = \dfrac{1+\sqrt{13}}{3}$, $x_2 = \dfrac{1-\sqrt{13}}{3}$

4. $x = 1$

5. $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

6.

$$\begin{aligned}
(a+ib)\,e^{p+iq} + (a-ib)\,e^{p-iq} &= e^{p}\Big((a+ib)(\cos(q) + i\sin(q)) \\
&\qquad (a-ib)(\cos(q) - i\sin(q))\Big) \\
&= e^{p}\Big(2a\cos(q) - 2b\sin(q)\Big)
\end{aligned}$$

7. $$\frac{81}{4} - \frac{221}{4}e^{-20}$$

8. Hint: use chain rule

9.

$$\begin{aligned} 2\cos(3t) + 4\sin(3t) &= k\sin(3t+\phi) \\ &= (k\sin(\phi))\cos(3t) + (k\cos(\phi))\sin(3t) \end{aligned}$$

Comparing coefficients on both sides

$$\begin{aligned} k\cos(\phi) &= 4 \\ k\sin(\phi) &= 2 \end{aligned} \quad \rightarrow \quad \begin{aligned} k &= \sqrt{20} \\ \phi &= \text{Atan}\left(\tfrac{1}{2}\right) \end{aligned}$$

10. $$\int_0^1 t^2 dt + \int_1^2 (2-t)t dt = 1$$

Chapter 2

Review of ODE Solutions

2.1 First Order Differential Equations

1. **Separable Type**

- **Form :** $\dfrac{dx}{dt} = f(t)g(x)$ subject to $x(0) = x_o$
- **Solution:** $\displaystyle\int_{x_o}^{x} \frac{1}{g(x)}dx = \int_0^t f(t)dt$
- **Example:**

$$\frac{dx}{dt} = 5\frac{x}{1+t} \qquad x(0) = 2$$

then $f(t) = \dfrac{5}{1+t}$ and $g(x) = x$,

$$\begin{aligned}
\int_{x_0}^{x} \frac{1}{x}dx &= \int_0^t \frac{5}{1+t}dt \\
\ln(x) - \ln(x_0) &= 5\Big(\ln(1+t) - \ln(1)\Big) \\
\ln\left(\frac{x}{x_0}\right) &= 5\ln(1+t) \\
\frac{x}{x_0} &= (1+t)^5 \\
x &= x_0(1+t)^5
\end{aligned}$$

2. **Linear First Order**

- **Form :** $\dfrac{dx}{dt} + p(t)x = q(t)$ subject to $x(0) = x_o$

- **Solution:** Let $\phi = \exp\left(\int p(t)dt\right)$ be an integrating factor, then

$$x = \frac{1}{\phi(t)}\left(x_0 + \int_0^t q(\tau)\phi(\tau)d\tau\right)$$

- **Example:**

$$\frac{dx}{dt} + 2tx = 4t \qquad x(0) = x_0$$

then $p(t) = 2t$ and $q(t) = 4t$,

$$\phi(t) = e^{\int 2tdt} = e^{t^2}$$

and

$$\begin{aligned} x &= e^{-t^2}\left(x_0 + \int_0^t 4\tau e^{\tau^2} d\tau\right) \\ &= e^{-t^2}\left(x_0 - 2 + 2e^{t^2}\right) \\ &= 2 + (x_0 - 2)\, e^{-t^2} \end{aligned}$$

2.2 High Order Linear ODEs with Constant Coefficients

- **Form :** $a_n \frac{d^n x}{dt^n} + \cdots + a_1 \frac{dx}{dt} + a_0 x = q(t)$

  subject to $x(0) = x_{o,0}, \ \ldots, \ \frac{d^{n-1}x}{dt^{n-1}}(0) = x_{o,n-1}$

- **Solution:** $x(t) = x_{\mathrm{c}}(t) + x_{\mathrm{p}}(t)$

  where

  - x_{c} is the *complementary solution* which solves

$$a_n \frac{d^n x_{\mathrm{c}}}{dt^n} + \cdots + a_1 \frac{dx_{\mathrm{c}}}{dt} + a_0 x_{\mathrm{c}} = 0$$

  - x_{p} is the *particular solution* which solves

$$a_n \frac{d^n x_{\mathrm{p}}}{dt^n} + \cdots + a_1 \frac{dx_{\mathrm{p}}}{dt} + a_0 x_{\mathrm{p}} = q(t)$$

- **Example: Second Order**

$$a_2 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = q(t)$$

Step 1: Find $x_{\rm c}$:

1. Get the **characteristic polynomial**,

$$a_2 s^2 + a_1 s + a_0 = 0$$

2. Solve for the roots, also known as **eigenvalues**,

$$r_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2 a_0}}{2a_2} \qquad r_2 = \frac{-a_1 - \sqrt{a_1^2 - 4a_1 a_0}}{2a_2}$$

3. Form the complementary solution to be

$$x_{\rm c} = C_1 y_1(t) + C_2 y_2(t)$$

where

$$\begin{aligned} y_1(t) &= e^{r_1 t} \\ y_2(t) &= \begin{cases} e^{r_2 t} & \text{if } r_1 \neq r_2 \\ t e^{r_1 t} & \text{if } r_1 = r_2 \end{cases} \end{aligned}$$

where C_1 and C_2 are constants, evaluated later to fit the initial conditions

Step 2: Find $x_{\rm p}$:

Method : Variation of Parameters

1. Evaluate the **Wronskian**: $W(t) = y_1 \frac{dy_2}{dt} - y_2 \frac{dy_1}{dt}$
2. Calculate u_1 and u_2:

$$u_1(t) = -\int \frac{y_2(t)q(t)}{a_2 W(t)} dt \qquad u_2(t) = \int \frac{y_1(t)q(t)}{a_2 W(t)} dt$$

3. Form the particular solution: $x_{\rm p}(t) = u_1(t) y_1(t) + u_2(t) y_2(t)$

Step 3: Use initial conditions to determine the values of C_1 and C_2.

- **Numerical Example:**

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} - 6x = e^{-t}\sin(2t) \qquad \text{subject to} \quad x(0) = 1, \ \frac{dx}{dt}(0) = 0$$

  – Step 1: complementary solution

$$s^2 + s - 6 = 0 \qquad \rightarrow r_1 = -3 \ ; \ r_2 = 2 \quad \rightarrow y_1 = e^{-3t} \ ; \ y_2 = e^{2t}$$

$$x_c = C_1 e^{-3t} + C_2 e^{2t}$$

  – Step 2: particular solution

$$\begin{aligned} W(t) &= 5e^{-t} \\ u_1(t) &= \frac{1}{20} e^{2t} \Big(\cos(2t) - \sin(2t)\Big) \\ u_2(t) &= \frac{1}{65} e^{-3t} \Big(-2\cos(2t) - 3\sin(2t)\Big) \\ \rightarrow x_p(t) &= \frac{1}{52} e^{-t} \Big(\cos(2t) - 5\sin(2t)\Big) \end{aligned}$$

  – Step 3: determine integration constant

$$\begin{aligned} x &= x_c + x_p \\ &= C_1 e^{-3t} + C_2 e^{2t} + \frac{1}{52} e^{-t} \Big(\cos(2t) - 5\sin(2t)\Big) \\ x(0) &= 0 = C_1 + C_2 + \frac{1}{52} \\ \frac{dx}{dt}(0) &= 0 = -3C_1 + 2C_2 - \frac{11}{52} \end{aligned}$$

$$\rightarrow C_1 = \frac{7}{20} \qquad C_2 = \frac{41}{65}$$

Solution: $$x(t) = \frac{7}{20} e^{-3t} + \frac{41}{65} e^{2t} + \frac{1}{52} e^{-t} \Big(\cos(2t) - 5\sin(2t)\Big)$$

2.3 Drills

Solve the following differential equations

1. $\dfrac{dx}{dt} = e^{-t}(x+2)$ subject to $x(0) = 1$

2. $\frac{dx}{dt} + (1+t)\,x = \sin(2t) \quad$ subject to $x(0) = 1$

3. $\frac{d}{dt}\left(e^{2t}x\right) = e^{-t} \quad$ subject to $x(0) = 2$

4. $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = 3 + e^{-t}\sin(t) \quad$ subject to $x(0) = 2$ and $\frac{dx}{dt}(0) = 0$

5. $\frac{d^2x}{dt^2} - e^{-2t} = -5\frac{dx}{dt} - 6x \quad$ subject to $x(0) = 0$ and $\frac{dx}{dt}(0) = 0$

2.4 Answers to Drills

1. $x(t) = \exp\left(1 + \ln(3) - e^{-t}\right) - 2$

2. $x(t) = \frac{1}{\phi(t)}\left(1 + \int_0^t \phi(\tau)\sin(2\tau)d\tau\right) \qquad$ where $\phi(t) = \exp\left(\frac{t^2+2t}{2}\right)$

3. $x(t) = \frac{3 - e^{-t}}{e^{2t}}$

4. $x(t) = \frac{3}{4} + \left(3t + \frac{7}{4}\right)e^{-2t} - \frac{1}{2}e^{-t}\cos(t)$

5. $x(t) = e^{-3t} + (t-1)\,e^{-2t}$

Chapter 3

Review of Matrix Operations

3.1 Basic Operations

1. Addition:

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix} = \begin{pmatrix} (a_{11}+b_{11}) & \cdots & (a_{1n}+b_{1n}) \\ \vdots & \ddots & \vdots \\ (a_{m1}+b_{m1}) & \cdots & (a_{mn}+b_{mn}) \end{pmatrix}$$

   Example: $$\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \end{pmatrix} + \begin{pmatrix} -1 & 2 & 2 \\ 2 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 1 \\ 4 & 3 & 3 \end{pmatrix}$$

2. Scalar Product:

$$\gamma \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} \gamma a_{11} & \cdots & \gamma a_{1n} \\ \vdots & \ddots & \vdots \\ \gamma a_{m1} & \cdots & \gamma a_{mn} \end{pmatrix}$$

   Example: $$3 \begin{pmatrix} 2 & 4 \\ 2 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 6 & 12 \\ 6 & 3 \\ 3 & -3 \end{pmatrix}$$

3. Matrix Product:

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{np} \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{1p} \\ \vdots & \ddots & \vdots \\ c_{m1} & \cdots & c_{mp} \end{pmatrix}$$

   where,

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \qquad i = 1, \ldots, m \quad j = 1, \ldots, p$$

Example:

$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & 2 & 3 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 2 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} (-1+2-1) & (2+2+0) \\ (-2+4+3) & (4+4+0) \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 5 & 8 \end{pmatrix}$$

4. **Transpose:**

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}^T = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}$$

Example: $$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$

3.2 Determinants

Notation: $\det(A) = |A|$

1. 2 × 2 Matrices:

$$\left| \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \right| = a_{11}a_{22} - a_{12}a_{21}$$

Example: $$\left| \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \right| = 1 \cdot 2 - 2 \cdot 3 = -4$$

2. $n \times n$ Matrices:

Method 1: row expansion

(a) Given matrix A, choose any row, say row i.

(b) Let $A_{ij\downarrow}$ be matrix A with row i and column j removed.

(c) The determinant can be determined recursively by

$$|A| = \sum_{j=1}^{n} (-1)^{i+j} \left| A_{ij\downarrow} \right|$$

Example: Expand along row 2,

$$\begin{aligned} \left| \begin{pmatrix} 1 & 2 & -1 \\ 3 & 2 & 0 \\ -1 & 1 & 1 \end{pmatrix} \right| &= 3 \cdot (-1)^{2+1} \left| \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \right| + 2 \cdot (-1)^{2+2} \left| \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right| \\ &\quad + 0 \cdot (-1)^{2+3} \left| \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} \right| \\ &= -9 \end{aligned}$$

Method 2: column expansion

(a) Given matrix A, choose any column, say column j.

(b) Let $A_{ij\downarrow}$ be matrix A with row i and column j removed.

(c) The determinant can be recursively determined by

$$|A| = \sum_{i=1}^{n} (-1)^{i+j} \left| A_{ij\downarrow} \right|$$

Example: Expand along column 2,

$$\begin{aligned} \left| \begin{pmatrix} 1 & 2 & -1 \\ 3 & 2 & 0 \\ -1 & 1 & 1 \end{pmatrix} \right| &= 2 \cdot (-1)^{1+2} \left| \begin{pmatrix} 3 & 0 \\ -1 & 1 \end{pmatrix} \right| + 2 \cdot (-1)^{2+2} \left| \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right| \\ &\quad + 1 \cdot (-1)^{3+2} \left| \begin{pmatrix} 1 & -1 \\ 3 & 0 \end{pmatrix} \right| \\ &= -9 \end{aligned}$$

3.3 Matrix Inverse

1. <u>2×2 Matrices:</u>

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} = \frac{1}{|A|} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

Example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}^{-1} = \frac{1}{1 \cdot 2 - 2 \cdot 3} \begin{pmatrix} 2 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -1/2 & 1/2 \\ 3/4 & -1/4 \end{pmatrix}$$

Checking: $\quad A^{-1}A = \begin{pmatrix} -1/2 & 1/2 \\ 3/4 & -1/4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

2. <u>$n \times n$ Matrices:</u>

(a) Given matrix A, let $A_{ij\downarrow}$ be matrix A with row i and column j removed.

(b) Build a new matrix B,

$$B = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix}$$

where $b_{ij} = (-1)^{i+j} \left| A_{ij\downarrow} \right|$.

(c) The inverse of A is given by

$$A^{-1} = \frac{1}{|A|} B^T$$

Example: Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 0 \\ 0 & 1 & -2 \end{pmatrix}$ then

$$b_{11} = \left|\begin{pmatrix} 2 & 0 \\ 1 & -2 \end{pmatrix}\right| \; ; \; b_{12} = -\left|\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}\right| \; ; \; b_{13} = \left|\begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}\right|$$

$$b_{21} = -\left|\begin{pmatrix} 2 & 3 \\ 1 & -2 \end{pmatrix}\right| \; ; \; b_{22} = \left|\begin{pmatrix} 1 & 3 \\ 0 & -2 \end{pmatrix}\right| \; ; \; b_{23} = -\left|\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\right|$$

$$b_{31} = \left|\begin{pmatrix} 2 & 3 \\ 2 & 0 \end{pmatrix}\right| \; ; \; b_{32} = -\left|\begin{pmatrix} 1 & 3 \\ 3 & 0 \end{pmatrix}\right| \; ; \; b_{33} = \left|\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\right|$$

$$B = \begin{pmatrix} -4 & 6 & 3 \\ 7 & -2 & -1 \\ -6 & 9 & -4 \end{pmatrix}$$

Expanding along row 3,

$$|A| = 0 + 1 \cdot (-1)^{3+2} \cdot \left|\begin{pmatrix} 1 & 3 \\ 3 & 0 \end{pmatrix}\right| - 2 \cdot (-1)^{3+3} \left|\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\right| = 17$$

Thus, the inverse is given by

$$A^{-1} = \frac{1}{17} \begin{pmatrix} -4 & 6 & 3 \\ 7 & -2 & -1 \\ -6 & 9 & -4 \end{pmatrix}^T = \begin{pmatrix} -4/17 & 7/17 & -6/17 \\ 6/17 & -2/17 & 9/17 \\ 3/17 & -1/17 & -4/17 \end{pmatrix}$$

Checking:

$$A^{-1}A = \begin{pmatrix} -4/17 & 7/17 & -6/17 \\ 6/17 & -2/17 & 9/17 \\ 3/17 & -1/17 & -4/17 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 0 \\ 0 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

3.4 Drills

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 & 1 \\ 3 & 2 & 0 \\ 1 & 2 & 2 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$ and $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

and let s be a scalar variable.

Evaluate the following:

1. $AA^T - B$

2. ACA^T

3. B^{-1}

4. $\left(sI_2 - C\right)^{-1}$

5. BC

3.5 Answer to Drills

1. $AA^T = \begin{pmatrix} 5 & 11 & 3 \\ 11 & 25 & 7 \\ 3 & 7 & 2 \end{pmatrix} \quad \rightarrow \quad AA^T - B = \begin{pmatrix} 4 & 12 & 2 \\ 8 & 23 & 7 \\ 2 & 5 & 0 \end{pmatrix}$

2. $ACA^T = \begin{pmatrix} 18 & 44 & 13 \\ 40 & 98 & 29 \\ 11 & 27 & 8 \end{pmatrix}$

3. $B^{-1} = \begin{pmatrix} 4/14 & 4/14 & -2/14 \\ -6/14 & 1/14 & 3/14 \\ 4/14 & -3/14 & 5/14 \end{pmatrix}$

4. $(sI_2 - C)^{-1} = \frac{1}{s^2 - 4s + 1} \begin{pmatrix} s-2 & 1 \\ 3 & s-2 \end{pmatrix}$

5. Not applicable! Sizes of B and C are not conformable for the product BC.