Using Spreadsheets to Simulate Process Dynamics - A Short Tutorial

I. Euler Method for Solving Initial Value Problems

Given: a differential equation

$$\frac{dx}{dt} = f(t, x, u)$$

the function $u(t)$, and initial condition $x(0)$.

Required: a series of number-pairs,

$$(t_0, x_0), (t_1, x_1), (t_2, x_2), \ldots, (t_N, x_N)$$

such that for $0 \le k \le N$

$$x_k = x(t = t_k)$$

approximate the solution of the given differential equation.

Method:

1. Let $t_0, t_1, \ldots, t_N$ be evenly spaced, i.e. $t_k = k\Delta t$.
2. If Δt is sufficiently small, the derivative dx/dt could be approximated by the finite difference

$$\frac{dx}{dt} \approx \frac{\Delta x}{\Delta t}$$

where $\Delta x = x_{k+1} - x_k$.

3. Now substitute these approximations to the original differential equation to obtain

$$\frac{dx}{dt} \approx \frac{\Delta x}{\Delta t} = f(t_k, x_k, u_k)$$

$$x_{k+1} - x_k = \Delta t\, f(t_k, x_k, u_k)$$

$$x_{k+1} = x_k + \Delta t\, f(t_k, x_k, u_k)$$

   where we have set the evaluation of the function f at $t = t_k$, and so $u_k = u(t = t_k)$.

   The last equation above is known as the recursion equation, also known as the finite difference approximation. All terms on the left-hand side of the recursion equation are known (current values used). Evaluating the left-hand side yields (predicts) the next value of x in the series.

4. Start the simulation by using the initial conditions, i.e. $k = 0$,

$$x_1 = x_0 + \Delta t\, f(0, x_0, u_0)$$

5. Continue until we get x_N, i.e.

$$x_2 = x_1 + \Delta t\, f(\Delta t, x_1, u_1)$$

$$x_3 = x_1 + \Delta t\, f(2\Delta t, x_2, u_2)$$

$$\vdots$$

$$x_N = x_{N-1} + \Delta t\, f((N-1)\Delta t, x_{N-1}, u_{N-1})$$

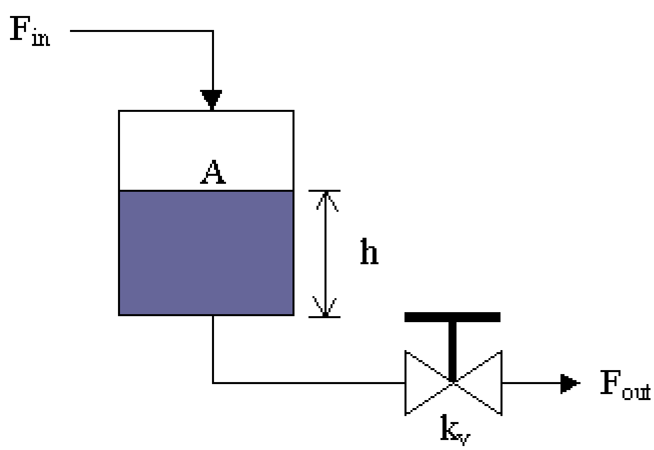
   (you can then plot x_k vs. t_k)

II. Spreadsheet Implementation

For discussion purposes, suppose we have the process model for liquid level change in a cylindrical tank.

$$\frac{dh}{dt} = \frac{F_{in} - F_{out}}{A}$$

where F_{in} is the volumetric flow rate into the tank, $F_{out} = k_v \sqrt{h}$, is the volumetric flow rate out of the tank (behaving according to Torricelli's law) with k_v as valve coefficient, and A is the cross-sectional area.



The main purpose of a simulation is to observe changes in the behavior when certain parameters of the system are tweaked.

For our scenario, let us choose the following settings:

$$F_{in} = 1 \text{ ft}^3/\text{sec}$$

$$k_v = 1.25 \text{ ft}^{2.5}$$

$$A = 0.75 \text{ ft}^2$$

and we will treat k_v and A as parameters. For our initial condition, let $h(0) = 1$ ft.

To compare our case with the discussion of Euler's method above, we have $x = h, u = F_{in}$, and

$$f(\,t, x, u\,) = f(\,t, h, F_{in}\,) = \frac{\left(\,F_{in} - k_v\sqrt{h}\right)}{A}$$

so the recursion equation is given by

$$h_{k+1} \;=\; h_k \;+\; \Delta t\,\frac{\left(\,F_{in,k} - k_v\,\sqrt{h_k}\right)}{A}$$

Now, let us implement this in a spreadsheet (figures below were generated using Microsoft Excel):

1. First, lay out the constants (say $\Delta t = 0.1$) and parameters, and fill-in the time column. Also, you can fill-in F_{in} values and the h_0 value.

	A	B	C	D	E	F
1						
2	**Simulation Example for Tank Level Dynamics**					
3						
4			**Constants:**			
5						
6				**delta_t =**	0.1	
7						
8			**Parameters:**			
9						
10				*Case 1*		
11						
12				**A =**	0.75	
13				**kv =**	1.25	
14						
15	**Iter. No.(k)**	**Time**		**Fin**	**h**	
16	0	0		1	1	
17	1	0.1		1		
18	2	0.2	=A18*E6	1		
19	3	0.3		1		
20	4	0.4		1		
21	5	0.5		1		
22	6	0.6		1		

Continue to say t=20 (k=200)

2. Plug in the recursion formula for h_1 and copy the formula to the cells below.

	B	C	D	E
5				
6			**delta_t =**	0.1
7				
8		**Parameters:**		
9				
10			*Case 1*	
11				
12			**A =**	0.75
13			**kv =**	1.25
14				
15	**Time**		**Fin**	**h**
16	0		1	1
17	0.1		1	0.966667
18	0.2		1	0.936135
19	0.3		1	0.908211
20	0.4		1	0.882711
21	0.5		1	0.859457
22	0.6		1	0.838278

=E16+E6*(D16-E$13*SQRT(E16))/E$12

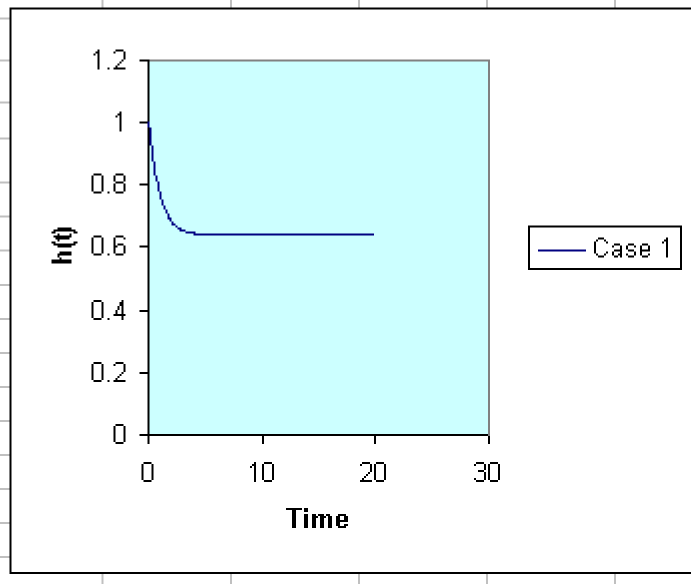
Take note of the absolute and relative cell addressing. We chose to use the address E$12 for A and address E$13 for k_v because we are planning to copy the block from cell D10 to E216 to a location to the left, to begin another case study.

3. Plot h vs time.

Case 1

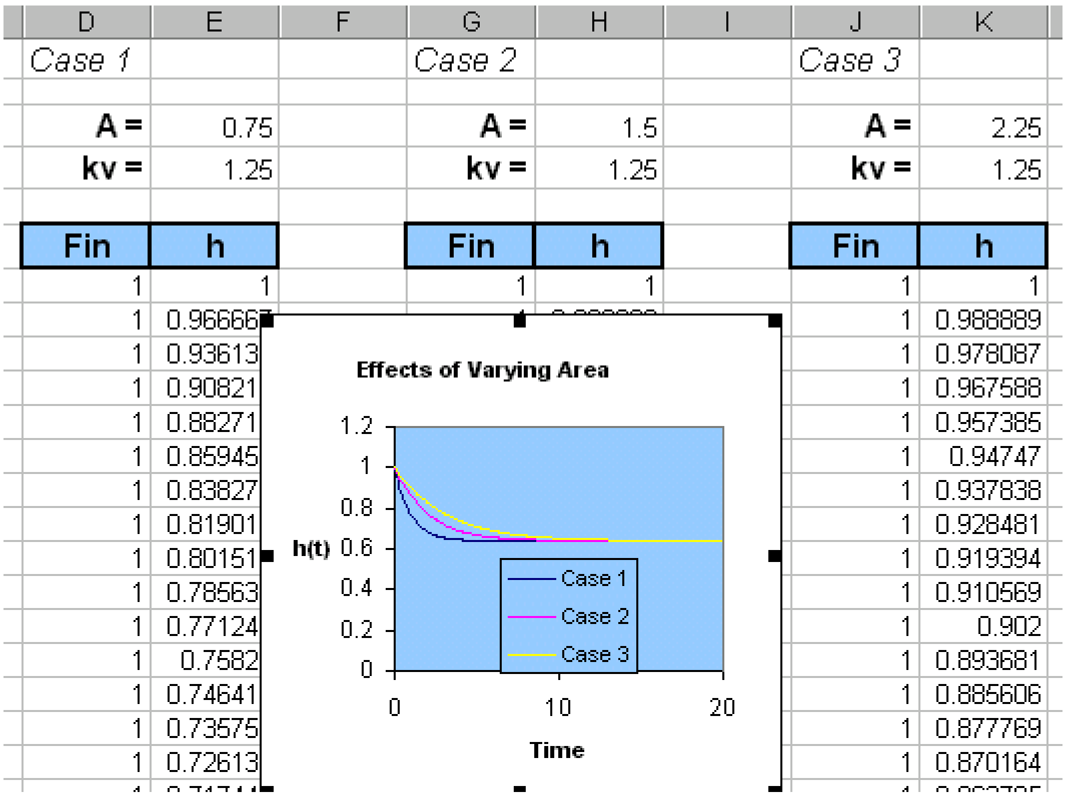
A =	0.75
kv =	1.25

Fin	h
1	1
1	0.966667
1	0.936135
1	0.908211
1	0.882711
1	0.859457
1	0.838278
1	0.819016
1	0.801517
1	0.785638
1	0.771244
1	0.75821



You could actually change the value of A in cell E12 and then observe how the response will change accordingly.

4. As mentioned earlier, to investigate how the response will behave to a set of parameter changes, we can collect several cases by copying the block from cell D10 to E216 to another location and then change one of the parameters, say *A* below



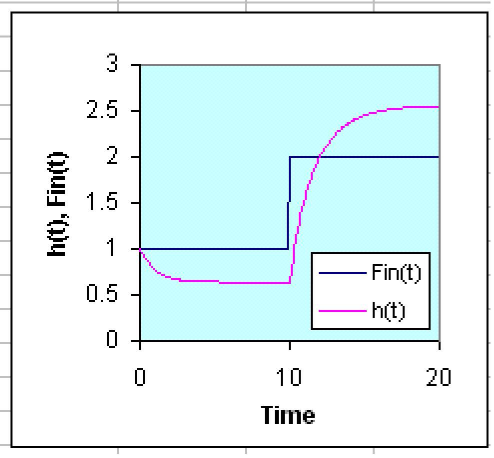
D	E	F	G	H	I	J	K
Case 1			*Case 2*			*Case 3*	
A =	0.75		**A =**	1.5		**A =**	2.25
kv =	1.25		**kv =**	1.25		**kv =**	1.25
Fin	**h**		**Fin**	**h**		**Fin**	**h**
1	1		1	1		1	1
1	0.96666					1	0.988889
1	0.93613					1	0.978087
1	0.90821					1	0.967588
1	0.88271					1	0.957385
1	0.85945					1	0.94747
1	0.83827					1	0.937838
1	0.81901					1	0.928481
1	0.80151					1	0.919394
1	0.78563					1	0.910569
1	0.77124					1	0.902
1	0.7582					1	0.893681
1	0.74641					1	0.885606
1	0.73575					1	0.877769
1	0.72613					1	0.870164

5. As another example, suppose instead of $F_{in}(t)=1.0$, we decide to investigate a case where

$$F_{in} = 1 \text{ when } t < 10, \text{ and } F_{in} = 2 \text{ when } t \geq 10$$

	B	M	N
9			
10		*Case 4*	
11			
12		**A =**	0.75
13		**kv =**	1.25
14			
15	**Time**	**Fin**	**h**
112	9.6	1	0.64001[illegible]
113	9.7	1	0.6[illegible]001
114	9.8	1	0.640009
115	9.9	1	0.640008
116	10	2	0.640007
117	10.1	2	0.77334
118	10.2	2	0.89344
119	10.3	2	1.00257
120	10.4	2	1.102356
121	10.5	2	1.194034
122	10.6	2	1.278581
123	10.7	2	1.35679
124	10.8	2	1.429322
125	10.9	2	1.496731

=IF($B114<10,1,2)



III. Higher Order Differential Equations

We will limit the discussion to second order, but the pattern should hold for orders greater than 2.

$$\frac{d^2x}{dt^2} = f\left(t, x, \frac{dx}{dt}, u\right)$$

We first need to introduce new variables to denote the derivatives. Let $v = dx/dt$, then we can reduce the original second order equation to a set of 2 first order equations given by

$$\frac{dx}{dt} = v$$

$$\frac{dv}{dt} = f(t, x, v, u)$$

Following the same approach as before of approximating derivatives by finite differences, we get two recursion equations

$$x_{k+1} = x_k + \Delta t \; v_k$$

$$v_{k+1} = v_k + \Delta t \; f(t_k, x_k, v_k, u_k)$$

The simulation will be initialized by conditions $x_0 = x(0)$ and $v_0 = dx/dt(0)$.

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