Derivation of Taylor Series Expansion

Objective:

Given $f(x)$, we want a power series expansion of this function with respect to a chosen point x_o, as follows:

$$f(x) = a_0 + a_1(x - x_o) + a_2(x - x_o)^2 + a_3(x - x_o)^3 + \cdots \qquad (1)$$

(Find the values of $a_o, a_1, \ldots$ such that the equation holds)

Method:

General idea: process both sides of the equation and choose values of x so that only one unknown appears each time.

To obtain a_0: Set $x = x_o$ in equation (1):

$$f(x_o) = a_0$$

To obtain a_1: First take the derivative of equation (1)

$$\frac{df(x)}{dx} = a_1 + 2a_2(x - x_o) + 3a_3(x - x_o)^2 + \cdots \qquad (2)$$

Next, set $x = x_o$.

$$\left.\frac{df(x)}{dx}\right|_{x=x_o} = a_1$$

To obtain a_2: First take the derivative of equation (2)

$$\frac{d^2 f(x)}{dx^2} = 2a_2 + (3 \cdot 2)a_3(x - x_o) + (4 \cdot 3)a_4(x - x_o)^2 + \cdots \qquad (3)$$

Next, set $x = x_o$,

$$\left.\frac{d^2 f(x)}{dx^2}\right|_{x=x_o} = 2a_2$$

To generalize:

$$a_n = \left(\frac{1}{n!}\right) \left.\frac{d^n f(x)}{dx^n}\right|_{x=x_o}$$

Summary:

The Taylor series expansion of $f(x)$ with respect to x_o is given by:

$$f(x) = f(x_o) + \left(\left.\frac{df}{dx}\right|_{x=x_o}\right)(x-x_o) + \left(\frac{1}{2}\left.\frac{d^2 f}{dx^2}\right|_{x=x_o}\right)(x-x_o)^2 + \left(\frac{1}{3!}\left.\frac{d^3 f}{dx^3}\right|_{x=x_o}\right)(x-x_o)^3 + \cdots$$

Generalization to multivariable function:

$$\begin{aligned} f(x,y,z) = {} & a_{000} + a_{100}(x-x_o) + a_{010}(y-y_o) + a_{001}(z-z_o) \\ & + a_{200}(x-x_o)^2 + a_{110}(x-x_o)(y-y_o) + a_{101}(x-x_o)(z-z_o) \\ & + a_{020}(y-y_o)^2 + a_{011}(y-y_o)(z-z_o) \\ & + a_{002}(z-z_o)^2 \\ & + \cdots + a_{\alpha\beta\gamma}(x-x_o)^{\alpha}(y-y_o)^{\beta}(z-z_o)^{\gamma} + \cdots \end{aligned} \tag{5}$$

Using similar methods as described in the previous section,

$$a_{\alpha\beta\gamma} = \left(\frac{1}{\alpha!\beta!\gamma!}\right) \left.\frac{\partial^{(\alpha+\beta+\gamma)} f}{\partial x^{\alpha}\partial y^{\beta}\partial z^{\gamma}}\right|_{x=x_o,\,y=y_o,\,z=z_o}$$