LECTURE 21. Method of Characteristics and Similarity Transformation Methods

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1. **A Quasilinear First Order PDE.**

- Standard form:

$$\xi(x,y,u)\frac{\partial u}{\partial x}+\eta(x,y,u)\frac{\partial u}{\partial x}=\zeta(x,y,u) \tag{1}$$

- Cauchy boundary condition:

$$u=u_o(x_o(a),y_o(a))$$

2. **Method of Characteristics**

- Main Idea:

  The desired solution is to take various paths of u along parameter/path s that starts at points in the curve defined in the Cauchy boundary condition, i.e., at $(x_o(a),y_o(a))$.

  These paths of the solution u are along the "characteristic curves" whose projection on the x-y plane are the "characteristic base curves". The complete solution then become the surface resulting from "covering/combining" the various characteristic curves together. (see Figure 1).

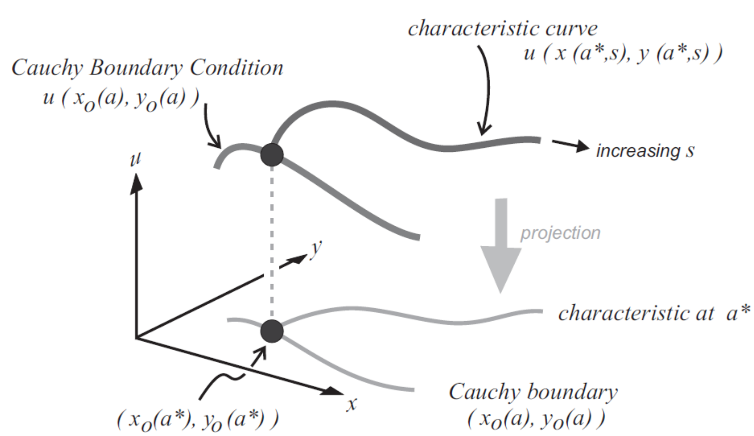

Figure 1. The solution path along the characteristic curve.

- So, start with taking the differential $u(x,y)$,

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

Next, dividing by ds, and rearranging,

$$\left(\frac{dx}{ds}\right)\frac{\partial u}{\partial x} + \left(\frac{dy}{ds}\right)\frac{\partial u}{\partial y} = \left(\frac{du}{ds}\right)$$

Comparing this with the PDE in (1), i.e.,

$$\xi(x, y, u)\frac{\partial u}{\partial x} + \eta(x, y, u)\frac{\partial u}{\partial x} = \zeta(x, y, u)$$

We obtain the characteristic equations:

$$\begin{aligned} \frac{dx}{ds} &= \xi(x, y, u) \\ \frac{dy}{ds} &= \eta(x, y, u) \\ \frac{dx}{ds} &= \zeta(x, y, u) \end{aligned} \tag{2}$$

Subject to the initial conditions:

$$@\, s = 0: \quad x = x_o(a), \;\; y = y_o(a), \;\; u = u_o(x_o(a), y_o(a)) \tag{3}$$

Example:

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 2x^2 - 6y^2 \qquad \text{with B.C.}: u(x = 1, y) = 3y^2 + 2y - 5$$

Then the characteristic equations are:

$$\begin{aligned} \frac{dx}{ds} &= x \\ \frac{dy}{ds} &= -y \\ \frac{du}{ds} &= 2x^2 - 6y^2 \end{aligned}$$

Boundary Conditions: $x_o = 1, \qquad y_o = a, \qquad u_o = 3a^2 + 2a - 5$

Solving,

$$\begin{aligned} x &= x_o e^s = e^s \\ y &= y_o e^{-s} = a e^{-s} \\ u &= u_o + \int_0^s (2e^{2s} - 6a^2 e^{-2s})\, ds \end{aligned}$$

$$= e^{2s} + 3a^2e^{-2s} + 2a - 6$$

In this case, we could put u back in terms of x and y, with $e^{2s} = x^2$, $(ae^{-s})^2 = y^2,\ a = ye^s = xy,$

$$\rightarrow \quad u = x^2 + 3y^2 + 2xy - 6$$

Remarks:

i) The process can be easily generalized to n independent variables. This will mean solving $n + 1$ first order ODEs.

ii) However, the solution surface will have a discontinuity when two characteristics intersect each other. This point is known as a "shock". In these cases, the solution will need additional conditions to proceed.

3. Alternate forms and General Solutions

- If the ODEs in (2) were divided by each other to remove the dependency on s, we end up with the alternate form of the characteristic equations.

$$\frac{dx}{dy} = \frac{\xi}{\eta} \quad \rightarrow \quad \frac{dx}{\xi} = \frac{dy}{\eta} \quad and \quad \frac{du}{dy} = \frac{\zeta}{\eta} \rightarrow \frac{dy}{\eta} = \frac{du}{\zeta}$$

$$or \qquad \frac{dx}{\xi} = \frac{dy}{\eta} = \frac{du}{\zeta} \tag{4}$$

Note: if any of the denominator functions in (4) is zero, we remove that ratio from the equation and set the corresponding variable as constant, e.g. if $\eta = 0$ then $y =$ constant.

Equation (4) is used when one is interested in the general solutions instead. The equations in (4) will generate two solutions of the form,

$$\psi_1(x, y, u) = \text{constant} \quad ; \quad \psi_2(x, y, u) = \text{constant}$$

Thus, one form of a general solution is to set $\psi_2 = f(\psi_1)$ and if possible, rearrange to obtain u explicitly.

Example:

Using the previous example

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 2x^2 - 6y^2$$

We have

$$\frac{dx}{x} = \frac{dy}{-y} = \frac{du}{2x^2 - 6y^2}$$

Solving the first pair,

$$\ln(x) = \ln\left(\frac{1}{y}\right) + \ln(\psi_1) \rightarrow \quad \psi_1 = xy$$

For the second pair, we could choose $\frac{dx}{x} = \frac{du}{2x^2-6y^2}$

$$du = \left(2x - \frac{6y^2}{x}\right) dx$$

$$u = \int \left(2x - \frac{6\psi_1^2}{x^3}\right) dx$$

$$= x^2 + \frac{3\psi_1^2}{x^2} + \psi_2 = x^2 + 3y^2 + \psi_2$$

Since the general solution is given by,

$$\psi_2 = f(\psi_1)$$

$$u = x^2 + 3y^2 + f(xy) \tag{5}$$

To show that this satisfies the given PDE, let $z = xy$,

$$\frac{\partial u}{\partial x} = 2x + y\frac{df}{dz} \quad and \quad \frac{\partial u}{\partial y} = 6y + x\frac{df}{dz}$$

Then,

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 2x^2 + xy\frac{df}{dz} - 6y^2 - xy\frac{df}{dz} = 2x^2 - 6y^2$$

which checks out to be the given PDE.

Having the general equation such as (5), one can also use it find a specific solution to satisfy a given boundary condition. Recall the condition:

$$u(1, y) = 3y^2 + 2y - 5$$

Then, the functionality of f can be found to be,

$$3y^2 + 2y - 5 = 1 + 3y^2 + f(y) \rightarrow \quad f(y) = 2y - 6$$

Which then yields the same specific solution as in the previous example that satisfy the BC and PDE, i.e.,

$$u = x^2 + 3y^2 + 2xy - 6$$

4. Application to the wave equation

- Consider the linear wave equation (with unconstrained range for x):

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0 \tag{6}$$

  Subject to $u(x,0) = f(x)$ and $\partial u/\partial t(x,0) = g(x)$ with constant c.

- PDE (6) is a hyperbolic type, so we expect a wave-type of solution. Further the operator is also "reducible", i.e., the differential operator can be factored to two commutative first order PDE operators.

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) = \left(\frac{\partial}{\partial x} - \frac{1}{c}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t}\right) = \left(\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} - \frac{1}{c}\frac{\partial}{\partial t}\right)$$

  This means we have two complementary solutions:

  i) For u_1,

$$\left(\frac{\partial}{\partial x} - \frac{1}{c}\frac{\partial}{\partial t}\right)u_1 = 0$$

$$\rightarrow \frac{dx}{1} = \frac{dt}{-1/c} \rightarrow x = -ct + \psi_1 \rightarrow \psi_1 = x + ct$$

$$u_1 = \psi_2 = \psi(x + ct)$$

  i) For u_1,

$$\left(\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t}\right)u_1 = 0$$

$$\rightarrow \frac{dx}{1} = \frac{dt}{1/c} \rightarrow x = ct + \varphi_1 \rightarrow \varphi_1 = x - ct$$

$$u_2 = \varphi_2 = \varphi(x - ct)$$

  Then the complete solution is the superposition of u_1 and u_2,

$$u = \psi(x + ct) + \varphi(x - ct)$$

  Next, to fit with the boundary condition: $u(x,0) = f(x)$,

$$f(x) = \psi(x) + \varphi(x)$$

  While for the initial condition: $\partial u/\partial t(x,0) = g(x)$,

$$g(x) = c\frac{d\psi}{dx} - c\frac{d\varphi}{dx} \tag{7}$$

  Taking derivative of $f(x)$ with respect to x,

$$\frac{df}{dx} = \frac{d\psi}{dx} + \frac{d\varphi}{dx} \tag{8}$$

Solving (7) and (8) simultaneously for $d\psi/dx$ and $d\varphi/dx$,

$$\frac{d\psi}{dx} = \frac{1}{2}\left(\frac{df}{dx} + \frac{g}{c}\right) \;\; ; \;\; \frac{d\varphi}{dx} = \frac{1}{2}\left(\frac{df}{dx} - \frac{g}{c}\right)$$

And so,

$$\psi(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(\tau)d\tau + \kappa_1$$

$$\varphi(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(\tau)d\tau + \kappa_2$$

Note that $\kappa_1 + \kappa_2 = 0$ since $f(0) = \psi(0) + \varphi(0)$. Thus, we have the final solution to be,

$$u(x,t) = \frac{1}{2}\big(f(x+ct) + f(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)\, d\tau \tag{9}$$

Equation (9) is known as the d'Alembert's solution to the wave equation.

5. Similarity Transformation Method

- When a second order PDE admits a similarity transformation, a combination of variables exist that will reduce the PDE to a related ODE.
- **Theorem:**
  Let a second order PDE admit a similarity transformation $\tilde{x} = \lambda^{-\alpha}x$, $\tilde{y} = \lambda^{-\beta}y$ and $\tilde{u} = \lambda^{-\gamma}u$, i.e., for a PDE, $F\big(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}\big) = 0$, if

  $$F\big(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}\big) = F\big(\tilde{x}, \tilde{y}, \tilde{u}, \tilde{u}_{\tilde{x}}, \tilde{u}_{\tilde{y}}, \tilde{u}_{\tilde{x}\tilde{x}}, \tilde{u}_{\tilde{x}\tilde{y}}, \tilde{u}_{\tilde{y}\tilde{y}}\big)$$

  Then the PDE can be reduced to an ODE with

  $$\zeta = \frac{y^\alpha}{x^\beta} \text{ as the independent variable, } \;\; \psi = \frac{u^\alpha}{x^\gamma} \text{ as the independent variable}$$

Example: Consider the diffusion equation given by

$$\frac{\partial u}{\partial t} = \mu^2 \frac{\partial^2 u}{\partial x^2} \tag{10}$$

Subject to the condition: $u(x,0) = u_{init}$ and $u(0,t) = u_o$.

Using the transformations: $\tilde{t} = \lambda^\alpha t$, $\tilde{x} = \lambda^{-\beta}x$ and $\tilde{u} = \lambda^{-\gamma}u$, equation (10) becomes

$$\lambda^{\alpha-\gamma}\frac{\partial u}{\partial t} = \mu^2 \lambda^{2\beta-\gamma}\frac{\partial^2 u}{\partial x^2}$$

Thus, for similarity, we could set $\alpha = 1, \beta = 1/2$ and $\gamma = 0$. This means we can keep u as the dependent variable and we have only one independent variable $\zeta = x/\sqrt{t}$, or $u = u(\zeta)$.

Next, we use this combination of variables to reduce the PDE to an ODE:

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \left(\frac{du}{d\zeta}\right)\left(\frac{\partial \zeta}{\partial t}\right) \\
&= -\frac{1}{2}\left(\frac{du}{d\zeta}\right)\left(\frac{x}{t\sqrt{t}}\right) \\
&= -\frac{1}{2}\left(\frac{du}{d\zeta}\right)\left(\frac{\zeta}{t}\right) \\
\frac{\partial u}{\partial x} &= \left(\frac{du}{d\zeta}\right)\left(\frac{\partial \zeta}{\partial x}\right) \\
&= \left(\frac{du}{d\zeta}\right)\left(\frac{1}{\sqrt{t}}\right) \\
\frac{\partial^2 u}{\partial x^2} &= \left(\frac{d^2 u}{d\zeta^2}\right)\left(\frac{1}{t}\right)
\end{aligned}$$

Substituting these into the PDE,

$$\frac{d^2 u}{d\zeta^2} = -\frac{\zeta}{2\mu^2}\left(\frac{du}{d\zeta}\right)$$

This is a second order ODE with missing dependent variable, so let $p = du/d\zeta$,

$$\frac{dp}{d\zeta} = -\frac{\zeta}{2\mu^2}p \qquad \rightarrow \qquad \ln(p) = -\frac{\zeta^2}{4\mu^2} + \ln(C_1) \qquad \rightarrow \qquad p = C_1 \exp\left(-\left(\frac{\zeta}{2\mu}\right)^2\right)$$

$$\frac{du}{d\zeta} = C_1 \exp\left(-\left(\frac{\zeta}{2\mu}\right)^2\right)$$

$$\rightarrow \quad u(\zeta) - u(0) = C_1 \int_0^\zeta \exp\left(-\left(\frac{\zeta}{2\mu}\right)^2\right) d\zeta$$

At this point we note that the integral can be evaluated using the "error function" defined as,

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-q^2}\, dq$$

With the following additional properties: $\mathrm{erf}(0) = 0$ and $\mathrm{erf}(\infty) = 1$.

Thus, the solution is given by,

$$u(\zeta) - u(0) = A\, \mathrm{erf}\left(\frac{\zeta}{2\mu}\right) \quad \left(\text{where} \quad A = C_1 \mu\sqrt{\pi}\,\right)$$

Since the initial condition at $t = 0$ is at $\zeta = \infty$, and $x = 0$ occur with $\zeta = 0$,

$$A = u_{init} - u_o$$

Finally, we have the solution to the diffusion equation in terms of x and t as

$$\frac{u - u_o}{u_i - u_o} = \text{erf}\left(\frac{1}{2\mu} \cdot \frac{x}{\sqrt{t}}\right) \tag{11}$$

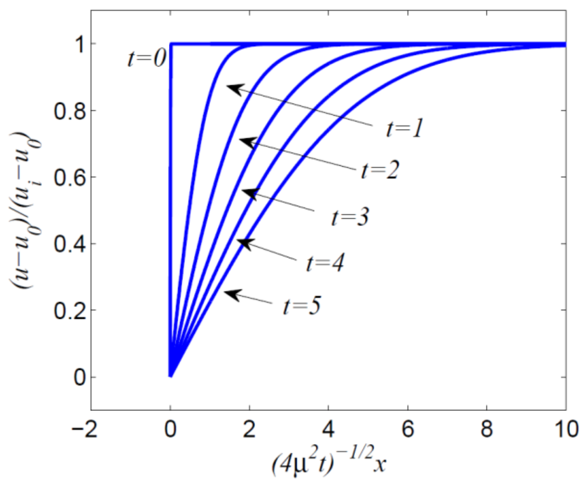

Figure 2. Diffusion profile as t increases.