1. The following matrix G is a generator matrix of a binary $[5,3]$ code C:

$$G = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

(i) (7 points) List all information sets.

Answer:

$$\{1,2,3\},\ \{1,2,5\},\ \{1,3,5\},\ \{2,3,4\},\ \{2,3,5\},\ \{2,4,5\},\ \{3,4,5\}$$

(ii) (6 points) Apply row operations on G to find a standard generator matrix G' of C.

Answer:

$$G' = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

(iii) (6 points) Find a parity check matrix H of C.

Answer:

$$H = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix}.$$

2. The following matrix H is the parity check matrix of a binary code:

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

(a) (5 points) Determine the minimum distance d of the code. Explain your answer.

Answer: $d = 2$

H has no all-zero columns, and columns 2 and 4 are linearly dependent.

(b) (16 points) List a set of coset leaders and their syndromes.

Answer:

Leader	Syndrome
00000	000
00001	100
00010 (or 01000)	001
00100	011
10000	111
00011 (or 01001)	101
00110 (or 01100)	010
10010 (or 11000)	110

(c) (5 points) Use syndrome decoding to decode the vector $y = (1, 0, 1, 0, 0)$.

Answer: $(1, 0, 1, 0, 1)$.

3. (6 points) Is there a binary linear $[12, 6, 5]$ code? Explain your answer.

Answer: No.

The code parameters violate the sphere-packing bound:

$$1 + 12 + \binom{12}{2} = 1 + 12 + 66 = 79 > 2^{12-6} = 2^6 = 64.$$

4. (6 points) Is there a binary $[10, 6, 4]$ code? Explain your answer.

Answer: No.

A punctured code wold have parameters $[9, 6, d*]$ with $d* = 3$ or $d* = 4$. However the parameters $[9, 6, 3]$ and $[9, 6, 4]$ violate the sphere-packing bound:

$$1 + 9 > 2^{9-6} = 2^3 = 8.$$

5. (8 points) Write down the standard generator matrix of a binary $[7, 6, 2]$ code.

Answer:

$$\begin{pmatrix} 1000001 \\ 0100001 \\ 0010001 \\ 0001001 \\ 0000101 \\ 0000011 \end{pmatrix}.$$

6. (6 points) Give the parity check matrix of a binary $[6, 3, 3]$ code.

Answer:

$$\begin{pmatrix} 000111 \\ 011001 \\ 101010 \end{pmatrix}.$$

7. (a) (10 points) Give the parity check matrix H of the binary Hamming code of length 15.

Answer:

$$H = \begin{pmatrix} 000000011111111 \\ 000111100001111 \\ 011001100110011 \\ 101010101010101 \end{pmatrix}.$$

(b) (8 points) Using the parity check matrix H from part (a), find the syndrome of $y = (111110000000000)$.

Answer: $(0, 0, 0, 1)^T$.

(c) (5 points) Decode the vector $y = (111110000000000)$

Answer: (011110000000000).

The syndrome of y is equal to the first column of H. Thus, we correct the first component of y from 1 to 0.

8. (6 points) Give the parity check matrix H of the Hamming code of length 6 over $GF(5)$. Order the columns of H lexicographically, and choose the columns so that the first nonzero entry in each column is equal to 1.

Answer:

$$H = \begin{pmatrix} 011111 \\ 101234 \end{pmatrix}.$$