On the Lie algebra with multiple brackets

Rafael S. González D'León

Department of Mathematics
University of Kentucky
Lexington, KY

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UNIVERSITY OF KENTUCKY

Lie bracket

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- $[x, y] + [y, x] = 0$ **(Antisymmetry)**
- $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$ **(Jacobi Identity)**

Free Lie algebra

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Examples of generators:

$$[1, 2]$$

$$[[3, 4], 3]$$

$$[[[3, 4], 3], [1, 2]]$$

$\mathcal{L}ie(n)$

$\mathcal{L}ie(n)$ is the component of the free lie algebra on $[n]$ generated by all the possible bracketings of $\{1, 2, \ldots, n\}$ containing each label exactly once (the multilinear component). Let's call these bracketings bracketed permutations.

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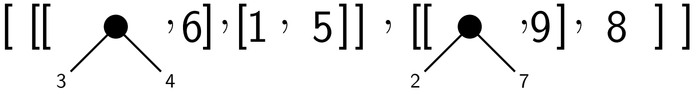
$\mathcal{L}ie(n)$ has the structure of an $\mathfrak{S}_n$-module.

There is another way to describe the generators

Generating set for $\mathcal{L}ie(n)$

$$[\,[[[3,4],6],[1,5]]\,,\,[[[2,7],9],8]\,]$$

Generating set for $\mathcal{L}ie(n)$



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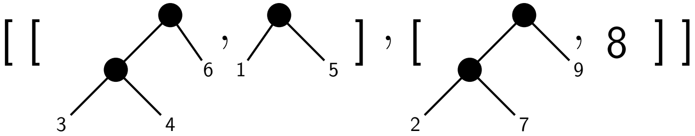

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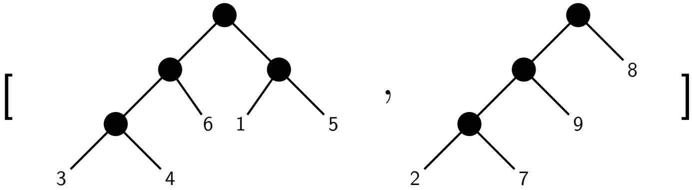

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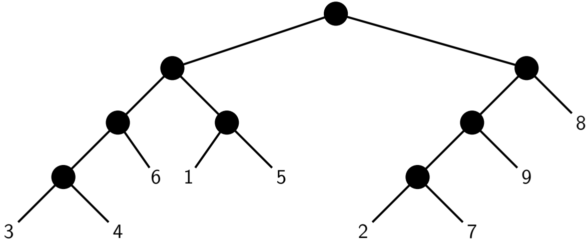
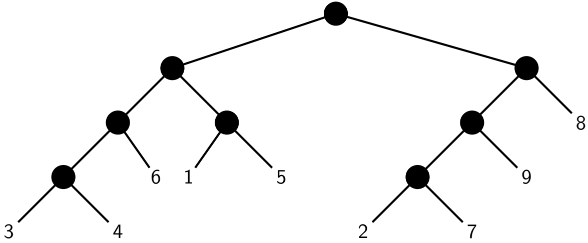


Generating set for $\mathcal{L}ie(n)$

A leaf-labeled binary tree



Let's turn the page temporarily to visit a combinatorial object.

Poset of partitions

A partition of $[n]$ is a collection of disjoint sets $\{B_1, B_2, \ldots, B_n\}$ such that their union $\bigcup_i B_i = [n]$.

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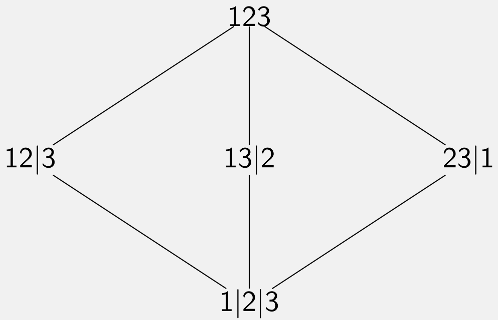
Let Π_n be the partially ordered set (poset) of partitions of $[n]$ with the order relation above.

Example (Π_3)



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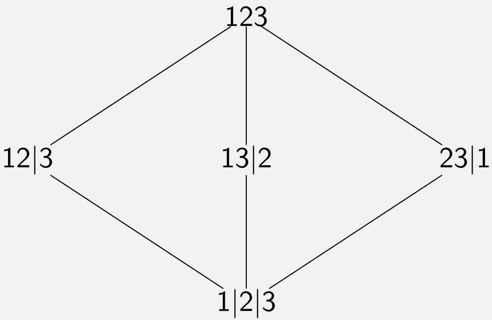

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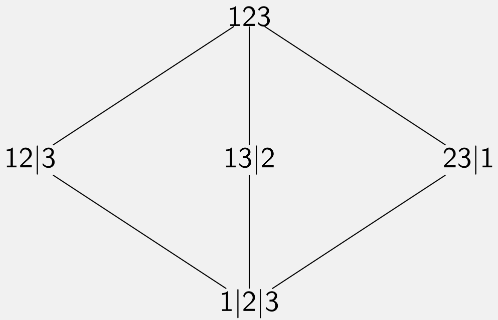

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Π_n has a top element, the block 123.

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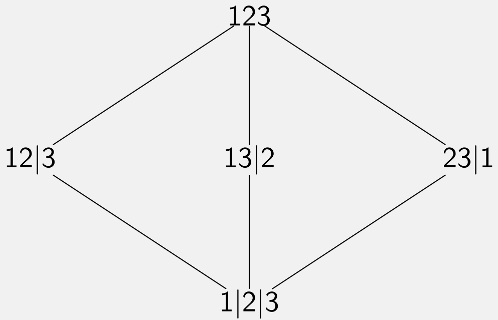

A chain is a totally ordered subset of P.

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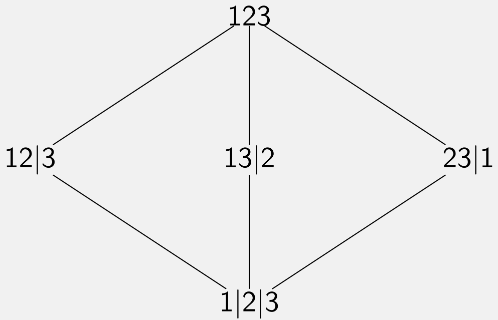

A chain is a totally ordered subset of P.

Example: in Π_3, $1|2|3 < 12|3$ is a chain as well as $1|2|3 < 123$.

Cohomology of a poset

Let P be a finite and bounded poset. We define (reduced) chain and cochain complexes

$$\cdots \underset{\delta_r}{\overset{\partial_{r+1}}{\rightleftarrows}} C_r(P) \underset{\delta_{r-1}}{\overset{\partial_r}{\rightleftarrows}} C_{r-1}(P) \underset{\delta_{r-2}}{\overset{\partial_{r-1}}{\rightleftarrows}} \cdots$$

where

$$C_r(P) = \mathbb{C}\{r\text{-chains in } P\}$$

and

$$\partial_r(\alpha_0 < \alpha_1 < \cdots < \alpha_r) = \sum_{i=0}^{r} (-1)^i (\alpha_0 < \cdots < \hat{\alpha}_i < \cdots < \alpha_r)$$

Cohomology of a poset

$\widetilde{H}^*(P)$ is the reduced cohomology of this complex.

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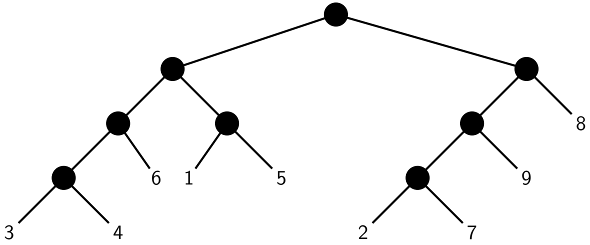
$\widetilde{H}^{top}(P) = \mathbb{C}\{\text{maximal chains}\}/\{\text{cohomology relations}\}$

Another set generated by leaf-labeled binary trees

Order the internal nodes of the binary tree in postorder (recursively left subtree $<$ right subtree $<$ root):



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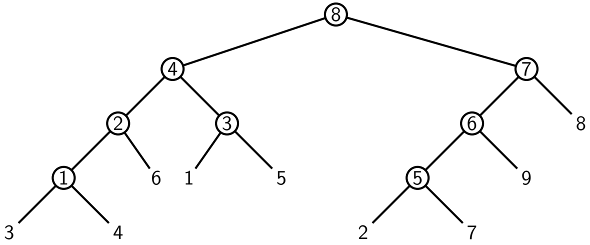

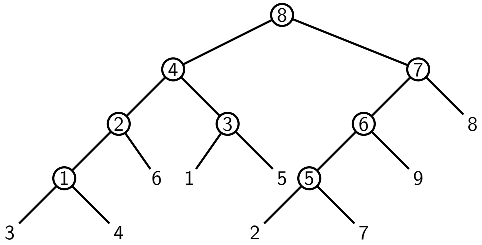
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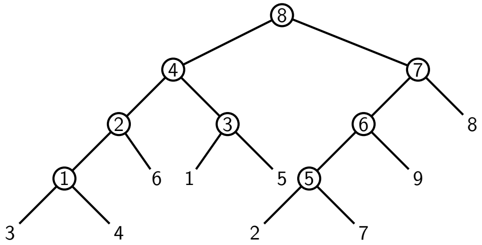
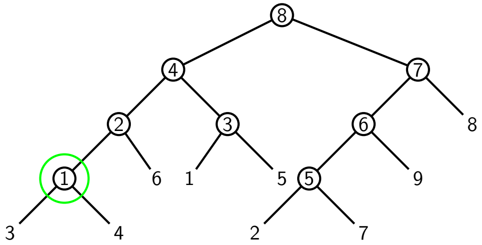


1|2|3|4|5|6|7|8|9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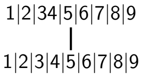
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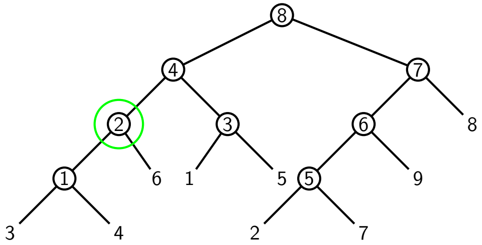
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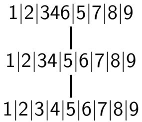
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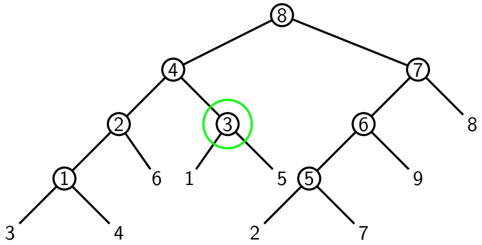
1|2|34|5|6|7|8|9
1|2|3|4|5|6|7|8|9

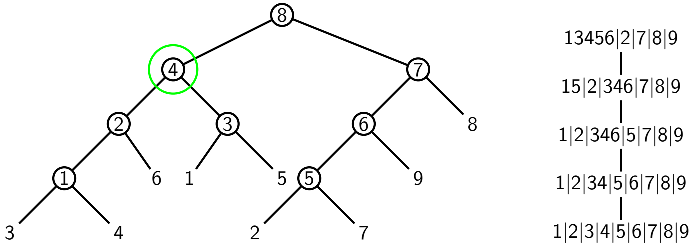


15|2|346|7|8|9

1|2|346|5|7|8|9

1|2|34|5|6|7|8|9

1|2|3|4|5|6|7|8|9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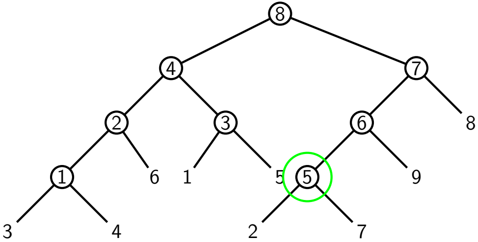
15|2|346|7|8|9
1|2|346|5|7|8|9
1|2|34|5|6|7|8|9
1|2|3|4|5|6|7|8|9



13456|27|8|9
13456|2|7|8|9
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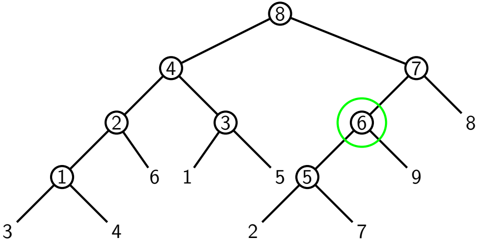


13456|279|8

13456|27|8|9

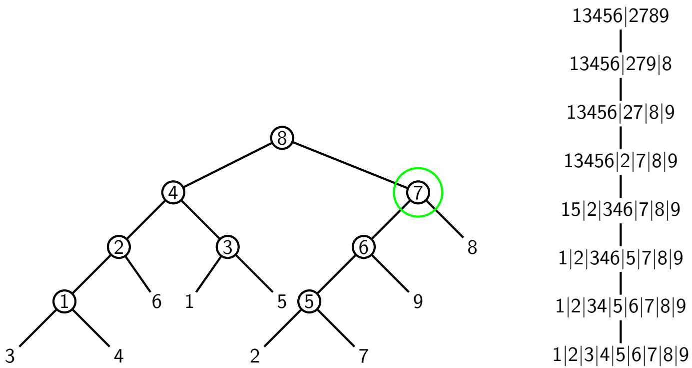
13456|2|7|8|9

15|2|346|7|8|9

1|2|346|5|7|8|9

1|2|34|5|6|7|8|9

1|2|3|4|5|6|7|8|9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13456|27|8|9
13456|2|7|8|9
15|2|346|7|8|9
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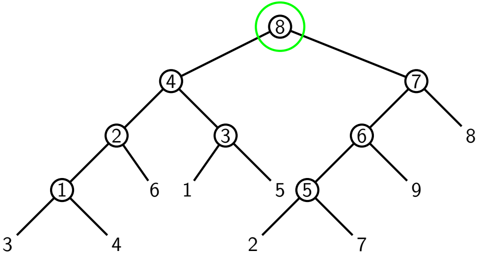
1|2|34|5|6|7|8|9
1|2|3|4|5|6|7|8|9



123456789

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13456|279|8

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15|2|346|7|8|9

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1|2|34|5|6|7|8|9

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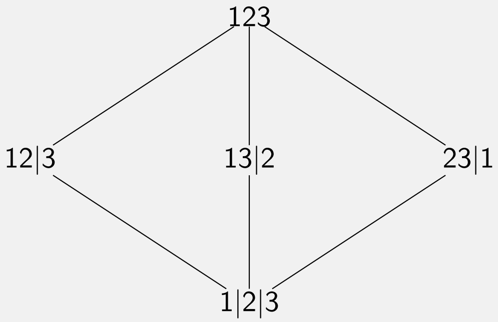
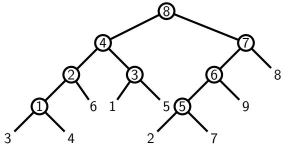
A maximal chain in the poset of partitions Π_n!

Example (Π_3)





123456789
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13456|279|8
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15|2|346|7|8|9
1|2|346|5|7|8|9
1|2|34|5|6|7|8|9
1|2|3|4|5|6|7|8|9

Remark

Not every maximal chain in Π_n is of this form (postorder is not enough!). But every maximal chain is cohomology equivalent to a chain of this form.

The isomorphism

Theorem (Joyal(1985), Barcelo (1988), Wachs (1998))

$$\mathcal{L}ie(n) \cong_{\mathfrak{S}_n} \widetilde{H}^{top}(\Pi_n \setminus \{\hat{0}, \hat{1}\}) \otimes \mathrm{sgn}_n$$

- This relation has a long history.

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  - Natural correspondence between generating sets - M. Wachs (1998).

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Moral:

We can study $\mathcal{L}ie(n)$ by applying poset topology techniques to Π_n.

A few results

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- Stanley-Bjorner ascent-free chains $\Rightarrow$ Lyndon basis
  Wachs ascent-free chains $\Rightarrow$ Comb basis.
- $\mu(\Pi_n) = (-1)^{n-1}(n-1)! \Longrightarrow \dim \mathcal{L}ie(n) = (n-1)!$

The story with two brackets.

$\mathcal{L}ie_2(n)$

Consider two Lie brackets:

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- $[x, y] = -[y, x]$ (Antisymmetry)
- $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$ (Jacobi Identity)
- $\langle x, y\rangle = -\langle y, x\rangle$ (Antisymmetry)
- $\langle x, \langle y, z\rangle\rangle + \langle z, \langle x, y\rangle\rangle + \langle y, \langle z, x\rangle\rangle = 0$ (Jacobi Identity)

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- $\langle x, \langle y, z\rangle\rangle + \langle z, \langle x, y\rangle\rangle + \langle y, \langle z, x\rangle\rangle = 0$ (Jacobi Identity)
- They are compatible if any linear combination of the two brackets is also a Lie bracket.

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- $\langle x, \langle y, z\rangle\rangle + \langle z, \langle x, y\rangle\rangle + \langle y, \langle z, x\rangle\rangle = 0$ (Jacobi Identity)
- $[x, \langle y, z\rangle] + [z, \langle x, y\rangle] + [y, \langle z, x\rangle] + \langle x, [y, z]\rangle$
  $+ \langle z, [x, y]\rangle + \langle y, [z, x]\rangle = 0$ (Mixed Jacobi Identity)

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- $[x, \langle y, z \rangle] + [z, \langle x, y \rangle] + [y, \langle z, x \rangle] + \langle x, [y, z] \rangle + \langle z, [x, y] \rangle + \langle y, [z, x] \rangle = 0$ (Mixed Jacobi Identity)

Denote by $\mathcal{L}ie_2(n)$ the multilinear component of the free doubly-bracketed Lie algebra on $[n]$.

$\mathcal{L}ie_2(n)$ is generated by bracketed permutations of the form:

$$\langle[\langle[3,4],6\rangle,[1,5]],\langle\langle[2,7],9\rangle,8\rangle\rangle$$

Denote by $\mathcal{L}ie_2(n,i)$ the component of $\mathcal{L}ie_2(n)$ generated by bracketed permutations with exactly i brackets of the first type.

Results on $\mathcal{L}ie_2(n)$ and $\mathcal{L}ie_2(n, i)$

Theorem (Dotsenko-Koroshkin (2007),Liu (2008))

$$\dim \mathcal{L}ie_2(n) = n^{n-1}$$

(The number of rooted trees on $[n]$*).*

Results on $\mathcal{L}ie_2(n)$ and $\mathcal{L}ie_2(n,i)$

Theorem (Dotsenko-Koroshkin (2007),Liu (2008))

$$\dim \mathcal{L}ie_2(n) = n^{n-1}$$

(The number of rooted trees on $[n]$).

Theorem (Liu (2008))

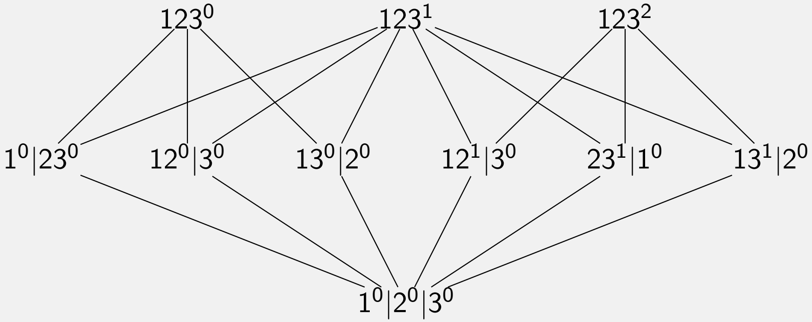
$$\dim \mathcal{L}ie_2(n,i) = |\mathcal{T}_{n,i}|$$

(the number of rooted trees on $[n]$ with i descents).

The poset of weighted partitions Π_n^w

- V. Dotsenko and A. Khoroshkin defined the poset of weighted partitions Π_n^w.

Example (Π_3^w)



It follows from a result of B. Vallette or from a proof using Wachs technique that

Theorem

$$\mathcal{L}ie(n,i) \cong_{\mathfrak{S}_n} H^{top}((\hat{0},[n]^i)) \otimes \mathrm{sgn}_n$$

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- Ascent-free chains $\Rightarrow$ bicolored Lyndon basis.

Question (Liu (2008))

Is it possible to define $\mathcal{L}ie_k(n)$ for any $k \geq 1$ so that it has nice dimension formulas like those for $\mathcal{L}ie(n)$ and $\mathcal{L}ie_2(n)$? What are the right combinatorial objects for $\mathcal{L}ie_k(n)$, if it can be defined?

Preliminary definitions

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We say that a set of Lie brackets on a vector space is compatible if any linear combination of them is a Lie bracket.

$\mathcal{L}ie(\mu)$

For a weak composition μ define $\mathcal{L}ie(\mu)$ to be the multilinear component of the free multibracketed Lie algebra on $[n]$ generated by bracketed permutations with μ_j brackets of type j for each j.

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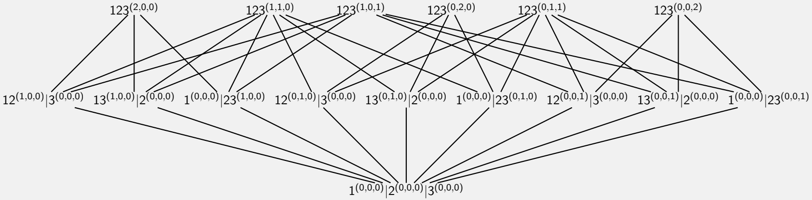
Example: $\mathcal{L}ie(0,2,0,1,2)$ is generated by bracketed permutations with two brackets of type 2, one bracket of type 4 and two brackets of type 5.

Is there a poset associated with $\mathcal{L}ie(\mu)$?

The poset of weighted partitions Π_n^k

Example ($k = n = 3$)



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- Recall that the maximal elements in Π_n^k are of the form $[n]^\mu$ where $\mu \in \mathrm{wcomp}_{n-1}$ with $\mathrm{supp}(\mu) \subseteq [k]$.

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Theorem (G (2013))

$$\mathcal{L}ie(\mu) \simeq_{\mathfrak{S}_n} \widetilde{H}^{top}((\hat{0}, [n]^\mu)) \otimes \mathrm{sgn}_n$$

What is dim $\mathcal{L}ie(\mu)$?.

Let $\mathbf{x} = (x_1, x_2, \dots)$ and $\mathbf{x}^\mu = \prod_{i \geq 1} x_i^{\mu_i}$.

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Normalized trees

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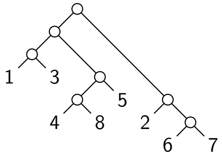

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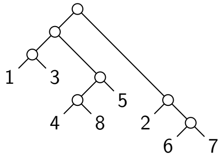

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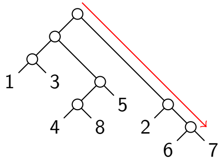

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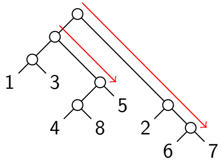

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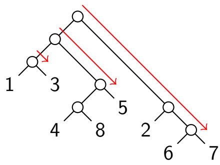

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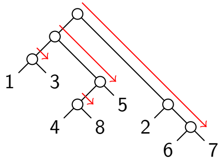

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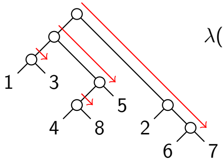

$$\lambda(T) = (3, 2, 1, 1)$$

Theorem (G (2013))

$$\sum_{\mu \in \mathrm{wcomp}_{n-1}} \dim \mathcal{L}ie(\mu)\, \mathbf{x}^{\mu} = \sum_{T \in Nor_n} e_{\lambda(T)}(\mathbf{x})$$

where e_λ is the elementary symmetric function indexed by λ.

Why is this?

The steps:

Ascent-free chains of the EL-labeling $\Leftrightarrow$ Multicolored Lyndon trees $\Leftrightarrow$ Multicolored Combs

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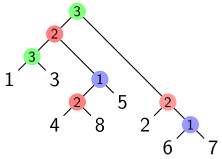
Multicolored combs look like:



Elementary symmetric functions

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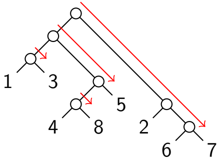
And for a number partition λ of n (i.e. a weak composition $(\lambda_1, \lambda_2, \cdots)$ of n with weakly decreasing values $\lambda_1 \geq \lambda_2 \geq \cdots$) define

$$e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots$$

What is the contribution of a normalized tree to the generating function?



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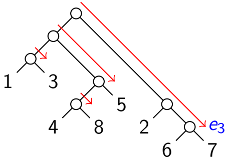

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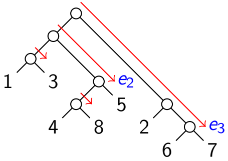

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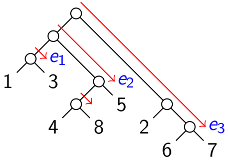

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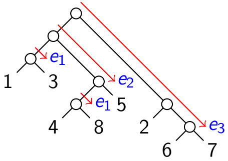

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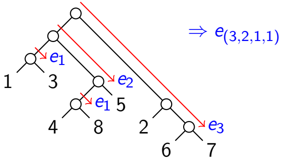

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