# Covering Groups with Proper Subgroups

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# Outline

# Finite and Minimal Covers

## Definition

A *finite cover* of a group $G$ is a collection $\mathcal{C} = \{H_1, ..., H_n\}$ of proper subgroups of $G$ such that $G = \bigcup_{i=1}^{n} H_i$. Such a cover $\mathcal{C}$ is called *minimal* if $|\mathcal{C}| \leq |\mathcal{D}|$ for every finite cover $\mathcal{D}$ of $G$.

# Covers cont'd

Not every group admits a finite cover by proper subgroups (e.g. cyclic groups). However,

**Fact**

Any group with a finite noncyclic homomorphic image is a union of finitely many proper subgroups.

Throughtout the remainder of this talk we will onkly be concerned with finite noncyclic groups.

# Covering Numbers

**Definition**

Let $G$ be a group with a finite noncyclic homomorphic image. The *covering number*, $\sigma(G)$, of $G$ is the size of a minimal cover of $G$, i.e. $\sigma(G) = \min\{|\mathcal{C}| : \mathcal{C} \text{ is a finite cover of } G\}$.

# Primary Elements

## Definition

Let $g \in G$. We say that $g$ is a *principal element* of $G$ if the cyclic subgroup $< g >$ generated by $g$ is maximal among cyclic subgroups of $G$.

Note that a collection $\{H_1, ..., H_n\}$ of proper subgroups of $G$ is a cover of $G$ if and only if $\bigcup_{i=1}^{n} H_i$ contains all of the principal elements of $G$.

# Maximal Subgroups

Suppose that $\mathcal{C}$ is a cover of a finite group $G$ by proper subgroups. Replacing each member of $\mathcal{C}$ by maximal subgroup of $G$ containing it, we obtain a cover $\mathcal{C}'$ of $G$ by maximal subgroups and $|\mathcal{C}'| \leq |\mathcal{C}|$. Consequently, when computing the covering number of a finite group, it suffices to consider covers by maximal subgroups.

# Some Known Results

- M.J. Tomkinson: If $G$ is a finite solvable group and $p^\alpha$ is the order of the smallest chief factor of $G$ with more than one complement then $\sigma(G) = p^\alpha + 1$.
- R.A. Bryce, V. Fedri, and L. Serena: If $G \cong PSL(2,q), PGL(2,q)$ or $GL(2,q)$ and $q \neq 2,5,7,9$, then $\sigma(G) = \frac{1}{2}q(q+1)$ if $q$ is even and $\sigma(G) = \frac{1}{2}q(q+1)+1$ if $q$ is odd.
- A. Maróti: $\sigma(\mathbb{S}_n) = 2^{n-1}$ if $n$ is odd and $n \neq 9$, and $\sigma(\mathbb{S}_n) \leq 2^{n-1}$ if $n$ is even.

# What is Known about $\sigma(\mathbb{A}_n)$

- A. Maróti: $\sigma(\mathbb{A}_n) \geq 2^{n-2}$ with equaility if and only if $n \equiv 2 \pmod 4$.
- J.H.E Cohn: $\sigma(\mathbb{A}_5) = 10$.
- R.A. Bryce et al.: $\mathbb{A}_6 \cong PSL(2,9) \Rightarrow \sigma(\mathbb{A}_6) = 16$.
- L-C Kappe and J. Redden: $\sigma(\mathbb{A}_7) = 31, \sigma(\mathbb{A}_8) = 71$, and $127 \leq \sigma(\mathbb{A}_9) \leq 157$.
- R.F. Morse: $141 \leq \sigma(\mathbb{A}_9)$.

# The Mathieu Groups and Their Covering Numbers

The Mathieu groups $M_{11}, M_{12}, M_{22}, M_{23}$, and $M_{24}$ were the first sporadic simple groups to be discovered. Each is a multiply transitive group and each can be realized as the automorphism group of a Steiner system.

- P. E. Holmes: $\sigma(M_{11}) = 23$, $\sigma(M_{22}) = 771$, and $\sigma(M_{23}) = 41079$.
- L-C Kappe, D. Nikolova-Popova and E. Swartz: $\sigma(M_{12}) = 208$.

# Outline

# Maximal Subgroups of $\mathbb{A}_9$

We begin with the conjugacy classes of maximal subgroups of $\mathbb{A}_9$ (from the Atlas of Finite Groups):

| Class | Isomorphism Type | Number |
|---|---|---|
| $\mathcal{M}_1$ | $\mathbb{A}_8$ | 9 |
| $\mathcal{M}_2$ | $\mathbb{S}_7$ | 36 |
| $\mathcal{M}_3$ | $(\mathbb{A}_6 \times \mathbb{Z}_3) : \mathbb{Z}_2$ | 84 |
| $\mathcal{M}_4$ | $L_2(8) : \mathbb{Z}_3$ | 120 |
| $\mathcal{M}_5$ | $L_2(8) : \mathbb{Z}_3$ | 120 |
| $\mathcal{M}_6$ | $(\mathbb{A}_5 \times \mathbb{A}_4) : \mathbb{Z}_2$ | 126 |
| $\mathcal{M}_7$ | $\mathbb{Z}_3^3 : \mathbb{S}_4$ | 280 |
| $\mathcal{M}_8$ | $\mathbb{Z}_3^2 : 2\mathbb{A}_4$ | 840 |

# Principal Elements of $\mathbb{A}_9$

We also determine the principal elements of $\mathbb{A}_9$:

| Cycle Type | Order | Number |
|---|---|---|
| $4^2 \cdot 1^1$ | 4 | 11340 |
| $6^1 \cdot 2^1 \cdot 1^1$ | 6 | 30240 |
| $7^1 \cdot 1^2$ | 7 | 25920 |
| $9^1$ | 9 | 40320 |
| $5^1 \cdot 2^2$ | 10 | 9072 |
| $4^1 \cdot 3^1 \cdot 2^1$ | 12 | 15120 |
| $5^1 \cdot 3^1 \cdot 1^1$ | 15 | 24192 |

# An Upper Bound for $\sigma(\mathbb{A}_9)$

- The subgroups from $\mathcal{M}_1$ and $\mathcal{M}_2$ cover all principal elements except those of order 9.
- It turns out that the elements of order 9 can be covered with 56 subgroups from each of classes $\mathcal{M}_4$ and $\mathcal{M}_5$.
- An upper bound for the covering number of $\mathbb{A}_9$ is $9 + 36 + 112 = 157$.

# Is This Cover Minimal?

## Theorem

*The covering number of $\mathbb{A}_9$ is 157.*

## Sketch of the Proof.

1. Construct the $40902 \times 1615$ incidence matrix between the cyclic subgroups generated by the principal elements and the maximal subgroups of $\mathbb{A}_9$.
2. Use integer linear programming to compute the minimal number of subgroups sufficient to cover the principal elements.

□

# Outline

# Maximal Subgroups of $\mathbb{A}_{11}$

We begin the same way as before, with the conjugacy classes of maximal subgroups of $\mathbb{A}_{11}$:

| Class | Isomorphism Type | Number |
|---|---|---|
| $\mathcal{M}_1$ | $\mathbb{A}_{10}$ | 11 |
| $\mathcal{M}_2$ | $\mathbb{S}_9$ | 55 |
| $\mathcal{M}_3$ | $(\mathbb{A}_8 \times \mathbb{Z}_3) : \mathbb{Z}_2$ | 165 |
| $\mathcal{M}_4$ | $(\mathbb{A}_7 \times \mathbb{A}_4) : \mathbb{Z}_2$ | 330 |
| $\mathcal{M}_5$ | $(\mathbb{A}_6 \times \mathbb{A}_5) : \mathbb{Z}_2$ | 462 |
| $\mathcal{M}_6$ | $M_{11}$ | 2520 |
| $\mathcal{M}_7$ | $M_{11}$ | 2520 |

# Principal Elements of $A_{11}$

| Cycle Type | Order | Number |
|---|---|---|
| $5^2 \cdot 1^1$ | 5 | 798336 |
| $6^1 \cdot 3^1 \cdot 2^1$ | 6 | 1108800 |
| $6^1 \cdot 2^1 \cdot 1^3$ | 6 | 554400 |
| $8^1 \cdot 2^1 \cdot 1^1$ | 8 | 2494800 |
| $9^1 \cdot 1^2$ | 9 | 2217600 |
| $11^1$ | 11 | 3628800 |
| $6^1 \cdot 4^1 \cdot 1^1$ | 12 | 1663200 |
| $4^2 \cdot 3^1$ | 12 | 415800 |
| $4^1 \cdot 3^1 \cdot 2^1 \cdot 1^2$ | 12 | 831600 |
| $7^1 \cdot 2^2$ | 14 | 712800 |
| $5^1 \cdot 3^2$ | 15 | 443520 |
| $5^1 \cdot 3^1 \cdot 1^3$ | 15 | 443520 |
| $5^1 \cdot 4^1 \cdot 2^1$ | 20 | 997920 |
| $7^1 \cdot 3^1 \cdot 1^1$ | 21 | 1900800 |

# Handling Subgroups of Order 11

In this case we are able to determine the covering number without resorting to linear programming. The first step is proving the following:

## Proposition

The 2520 subgroups form class $\mathcal{M}_6$ (or $\mathcal{M}_7$) are sufficient to cover the cyclic subgroups of $\mathbb{A}_{11}$ of order 11. Moreover, these cyclic subgroups cannot be covered with fewer than 2520 maximal subgroups of $\mathbb{A}_{11}$.

An immediate consequence is that the covering number of $\mathbb{A}_{11}$ is at least 2520.

# An Upper Bound for the Covering Number

Each principal element $\sigma$ not of order 11 satifies at least one of the following:

- $\sigma$ fixes a point.
- $\sigma$ fixes a 2-subset of $\{1, 2, ..., 11\}$.
- $\sigma$ fixes a 3-subset of $\{1, 2, ..., 11\}$.

Consequently we can cover $\mathbb{A}_{11}$ by $11 + 55 + 165 + 2520 = 2751$ maximal subgroups.

# Establishing the Lower Bound

We claim that this cover is minimal. The idea of the proof is as follows: Suppose $\mathcal{C}$ is a cover of $\mathbb{A}_{11}$ by maximal subgroups, and let $x_i = |\mathcal{M}_i \cap \mathcal{C}|, i = 1, \ldots, 5$. Then,

1. $x_3 + x_4 \geq 165$
2. $x_1 < 11 \Rightarrow x_3 + x_4 + x_5 \geq 330$, and
3. $x_2 < 55 \Rightarrow x_2 + x_3 + x_4 + x_5 \geq 221$.

# The lower Bound cont'd

- The basic idea is that we overestimate the number of elements of certain primary types that get covered by a collection of subgroups to obtain an inequality involving the $x_i$.
- The trick is to get estimates that are accurate enough to be useful.

# The lower Bound cont'd

For example, we can look at the elements of type $4^2 \cdot 3^1$ which appear only in the maximal subgroups from classes $\mathcal{M}_3$ and $\mathcal{M}_4$. Each subgroup from class $\mathcal{M}_3$ or $\mathcal{M}_4$ contains exactly 2520 of these elements, and there are a total of 415800 of them in $\mathbb{A}_{11}$. Then we must have $2520(x_3 + x_4) \geq 415800$, and so $x_3 + x_4 \geq 415800/2520 = 165$.

# Main Result cont'd

Having established these claims, one has that if $\mathcal{C}$ is a minimal cover of $\mathbb{A}_{11}$, then

- $x_1 = 11$
- $x_2 = 55$
- $x_3 + x_4 \geq 165$ and
- $|\mathcal{C} \cap (\mathcal{M}_6 \cup \mathcal{M}_7)| \geq 2520$.

Consequently $|\mathcal{C}| \geq 11 + 55 + 165 + 2520 = 2751$, thereby establishing

**Theorem**

*The covering number of $\mathbb{A}_{11}$ is 2751.*

# Outline

# The Maximal Subgroups of $M_{24}$

| Class | Isomorphism Type | Number |
|---|---|---|
| $\mathcal{M}_1$ | $M_{23}$ | 24 |
| $\mathcal{M}_2$ | $M_{22} : \mathbb{Z}_2$ | 276 |
| $\mathcal{M}_3$ | $\mathbb{Z}_2^4 : \mathbb{A}_8$ | 759 |
| $\mathcal{M}_4$ | $M_{12} : \mathbb{Z}_2$ | 1288 |
| $\mathcal{M}_5$ | $\mathbb{Z}_2^6 : \mathbb{Z}_3.\mathbb{S}_6$ | 1771 |
| $\mathcal{M}_6$ | $L_3(4) : \mathbb{S}_3$ | 2024 |
| $\mathcal{M}_7$ | $\mathbb{Z}_2^6 : (L_3(2) \setminus \mathbb{S}_3)$ | 3795 |
| $\mathcal{M}_8$ | $L_2(23)$ | 40320 |
| $\mathcal{M}_9$ | $L_2(7)$ | 1457280 |

# Principal Elements of $M_{24}$

| Cycle Type | Order | Number |
|---|---|---|
| $8^2 \cdot 4^1 \cdot 2^1 \cdot 1^2$ | 8 | 15301440 |
| $10^2 \cdot 2^2$ | 10 | 12241152 |
| $11^2 \cdot 1^2$ | 11 | 22256640 |
| $12^1 \cdot 6^1 \cdot 4^1 \cdot 2^1$ | 12 | 20401920 |
| $12^2$ | 12 | 20401920 |
| $14^1 \cdot 7^1 \cdot 2^1 \cdot 1^1 a$ | 14 | 17487360 |
| $14^1 \cdot 7^1 \cdot 2^1 \cdot 1^1 b$ | 14 | 17487360 |
| $15^1 \cdot 5^1 \cdot 3^1 \cdot 1^1 a$ | 15 | 16321536 |
| $15^1 \cdot 5^1 \cdot 3^1 \cdot 1^1 b$ | 15 | 16321536 |
| $21^1 \cdot 3^1 a$ | 21 | 11658240 |
| $21^1 \cdot 3^1 b$ | 21 | 11658240 |
| $23^1 \cdot 1^1 a$ | 23 | 10644480 |
| $23^1 \cdot 1^1 b$ | 23 | 10644480 |

# An Upper Bound for $\sigma(M_{24})$

We note that $\mathcal{M}_1 \cup \mathcal{M}_4 \cup \mathcal{M}_6$ is a cover of $M_{24}$ by 3336 subgroups, and therefore we have $\sigma(M_{24}) \leq 3336$.

# Establishing the Lower Bound

Suppose that $\mathcal{C}$ is a cover of $M_{24}$ by maximal subgroups and let $x_i = |\mathcal{C} \cap \mathcal{M}_i|$ for $i = 1, \ldots, 9$. As we did for $\mathbb{A}_{11}$, we derive a system of linear inequalities in the $x_i$.

# Establishing the Lower Bound

For example, consider the elements with cycle type $23^1 \cdot 1^1$ which appear in the subgroups from classes $\mathcal{M}_1$ and $\mathcal{M}_8$ only. There are 40320 principal cyclic subgroups of order 23 in each subgroup from class $\mathcal{M}_1$, and 24 in each subgroup from class $\mathcal{M}_8$. Consequently,

$$40320x_1 + 24x_8 \geq 967680.$$

Simplifying, we have

$$1680x_1 + x_8 \geq 40320.$$

# Establishing the Lower Bound

Proceeding accordingly with the elements of types $21^1 \cdot 3^1$, $12^2$, $10^2 \cdot 2^2$, and $12^1 \cdot 6^1 \cdot 4^1 \cdot 2^1$ we derive the following system of linear inequalities:

- $1680x_1 + x_8 \geq 40320$
- $15x_6 + 8x_7 \geq 30360$
- $3960x_4 + 2880x_5 + 1344x_7 + 253x_8 \geq 5100480$
- $308x_2 + 99x_4 + 24x_5 \geq 42504$
- $385x_2 + 140x_3 + 165x_4 + 120x_5 + 28x_7 \geq 106260.$

# Establishing the Lower Bound

These inequalities, along with the conditions $0 \leq x_i \leq |\mathcal{M}_i|$ must be satisfied for any cover $\mathcal{C}$ of $\mathcal{M}_{24}$. We find the minimum value of $x_1 + x_2 + ... + x_9$ subject to these constraints using linear programming, which is indeed 3336, thereby establishing

**Theorem**

The covering number of $M_{24}$ is 3336.

Thank you for listening!