Handicap incomplete tournaments of odd order

Dalibor Froncek
University of Minnesota Duluth

or…

Dalibor Froncek
University of Minnesota Duluth

or…

Short trip to the 19th century

and back

Dalibor Froncek
University of Minnesota Duluth

Incomplete round robin tournaments

Incomplete round robin tournaments

What if we have 8 teams but not enough time to play a complete tournament?

Say we can play only 4–6 games per team.

Incomplete round robin tournaments

What if we have 8 teams but not enough time to play a complete tournament?

Say we can play only 4–6 games per team.

What games we choose, if the teams are ranked according to their strength?

We have several options:

Incomplete round robin tournaments

What if we have 8 teams but not enough time to play a complete tournament?

Say we can play only 4–6 games per team.

What games we choose, if the teams are ranked according to their strength?

We have several options:

- All teams have the same strength of schedule

Incomplete round robin tournaments

What if we have 8 teams but not enough time to play a complete tournament?

Say we can play only 4–6 games per team.

What games we choose, if the teams are ranked according to their strength?

We have several options:

- All teams have the same strength of schedule
- The tournaments mimics the complete tournament (strongest team has easiest schedule)

Incomplete round robin tournaments

What if we have 8 teams but not enough time to play a complete tournament?

Say we can play only 4–6 games per team.

What games we choose, if the teams are ranked according to their strength?

We have several options:

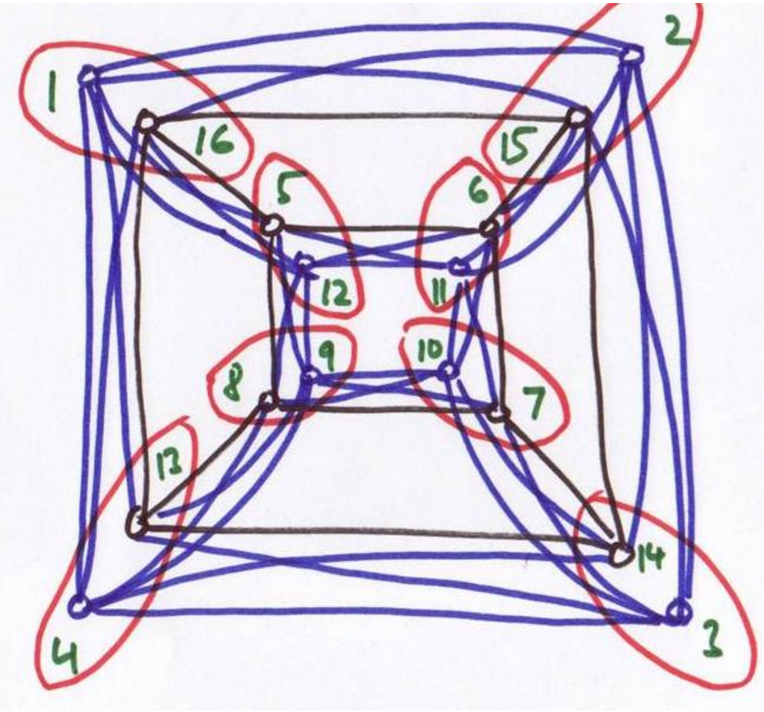
- All teams have the same strength of schedule
- The tournaments mimics the complete tournament (strongest team has easiest schedule)
- All teams have the same chance of winning (weakest team has easiest schedule)

Distance magic labeling (equal strenth)

Distance magic vertex labeling of a graph G with n vertices:

A bijection μ from the vertex set of G to $\{1, 2,\ldots, n\}$ such that sum $w(x)$ of labels of the neighbors of each vertex (called the *weight* of x) is equal to the same constant m.

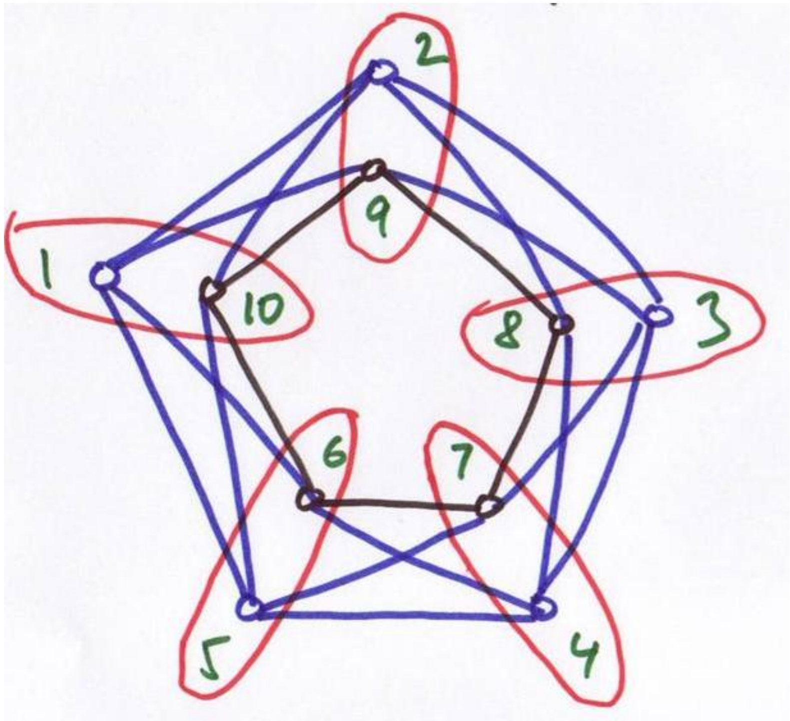
Distance magic labeling (equal strenth)



Distance magic vertex labeling of a graph G with n vertices:

A bijection μ from the vertex set of G to $\{1, 2,\ldots, n\}$ such that sum $w(x)$ of labels of the neighbors of each vertex (called the *weight* of x) is equal to the same constant m.

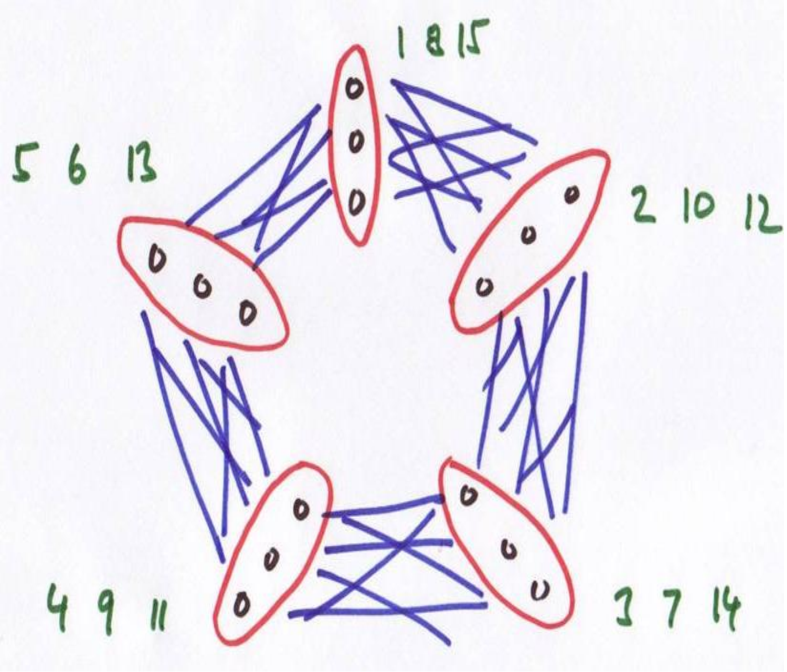
Distance magic labeling (equal strenth)



Distance magic vertex labeling of a graph *G* with *n* vertices:

A bijection μ from the vertex set of *G* to {1, 2,…, *n*} such that sum $w(x)$ of labels of the neighbors of each vertex (called the *weight* of *x*) is equal to the same constant m.

Distance magic labeling (equal strenth)



Distance magic vertex labeling of a graph *G* with *n* vertices:

A bijection μ from the vertex set of *G* to $\{1, 2, \ldots, n\}$ such that sum $w(x)$ of labels of the neighbors of each vertex (called the *weight* of *x*) is equal to the same constant *m*.

Theorem 1: There is no r-regular DM graph for r odd.

Theorem 2: For n even there is an r-regular DM graph with n vertices if and only if $r \equiv 0 \pmod 4$ or $n \equiv 0 \pmod 4$.

Theorem 3: For n odd there is an r-regular DM graph with n vertices if $q > 1$ is odd, $s > 0$, $r = 2^s q$, and $q|n$.

Theorem 4: For n odd there is an r-regular DM graph with n vertices if $q>1$ is odd, $s>0$, $r = 2^s q$, $r \leq (2n - 4)/7$.

Kotzig 5×9 array

1	2	3	4	5	6	7	8	9
9	7	5	3	1	8	6	4	2
5	6	7	8	9	1	2	3	4
1	2	3	4	5	6	7	8	9
9	8	7	6	5	4	3	2	1
25	**25**	**25**	**25**	**25**	**25**	**25**	**25**	**25**

Distance magic labeling in K_{45}

36 + 1	36 + 2	36 + 3	36 + 4	36 + 5	36 + 6	36 + 7	36 + 8	36 + 9
27 + 9	27 + 7	27 + 5	27 + 3	27 + 1	27 + 8	27 + 6	27 + 4	27 + 2
18 + 5	18 + 6	18 + 7	18 + 8	18 + 9	18 + 1	18 + 2	18 + 3	18 + 4
9 + 1	9 + 2	9 + 3	9 + 4	9 + 5	9 + 6	9 + 7	9 + 8	9 + 9
0 + 9	0 + 8	0 + 7	0+ 6	0 + 5	0 + 4	0 + 3	0 + 2	0 + 1
90 + 25	**90 + 25**	**90 + 25**	**90 + 25**	**90 + 25**	**90 + 25**	**90 + 25**	**90 + 25**	**90 + 25**

Tournament comparison

Team\Opps ranking	Complete RR	Incomplete RR FAIR	Incomplete RR EQUAL STRENGTH	Incomplete RR HANDICAP
1	35	$35 - m$	m	$k+1$
2	34	$34 - m$	m	$k+2$
3	33	$33 - m$	m	$k+3$
4	32	$32 - m$	m	$k+4$
5	31	$31 - m$	m	$k+5$
6	30	$30 - m$	m	$k+6$
7	29	$29 - m$	m	$k+7$
8	28	$28 - m$	m	$k+8$

Handicap tournaments

We want to find an *r*-factor such that the sum of rankings of the neighbors of team ***i*** (i.e., the games which **will** be played) will be equal to *k*+*i* for same constant *k*.

Handicap tournaments

We want to find an *r*-factor such that the sum of rankings of the neighbors of team ***i*** (i.e., the games which **will** be played) will be equal to $k+i$ for same constant k.

Team\Opps ranking	Incomplete RR HANDICAP
1	***k*+1**
2	***k*+2**
3	***k*+3**
4	***k*+4**
5	***k*+5**
6	***k*+6**
7	***k*+7**
8	***k*+8**

Which games to play?

We want to find an r-factor such that the sum of rankings of the neighbors of team ***i*** (i.e., the games which **will** be played) will be equal to $k+i$ for same constant k.

Distance-antimagic vertex labeling of a graph G:

A bijection μ from the vertex set of G to $\{1, 2, \ldots, n\}$ such that weights of all vertices form the set $\{k+1, k+2, \ldots, k+n\}$ for same constant k.

Which games to play?

So in fact a fair incomplete round robin tournament is a distance-antimagic graph

Distance-antimagic vertex labeling of a graph *G:*

A bijection μ from the vertex set of G to $\{1, 2,\ldots, n\}$ such that weights of all vertices form the set

$\{k+1, k+2,\ldots, k+n\}$

for same constant k.

Which games to play?

We want to find an r-factor such that the sum of rankings of the neighbors of team ***i*** (i.e., the games which will be played) will be equal to $k+i$ for same constant k.

Handicap *distance-antimagic vertex labeling* of a graph G:

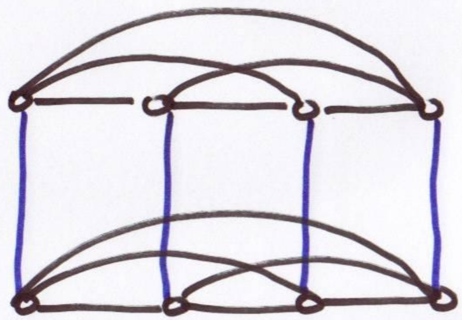
A bijection μ from the vertex set of G to $\{1, 2, \ldots, n\}$ such that weight of vertex ***i*** is equal to $k+i$ for same constant k.

Which games to play?

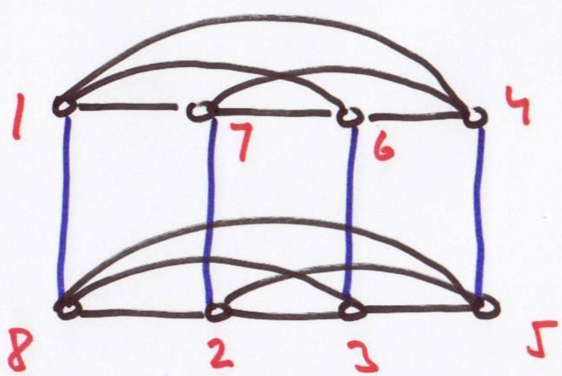
We want to find an r-factor such that the sum of rankings of the neighbors of team ***i*** (i.e., the games which will be played) will be equal to $k+i$ for same constant k.

We want to find an r-factor F which has an *handicap distance-antimagic vertex labeling.*

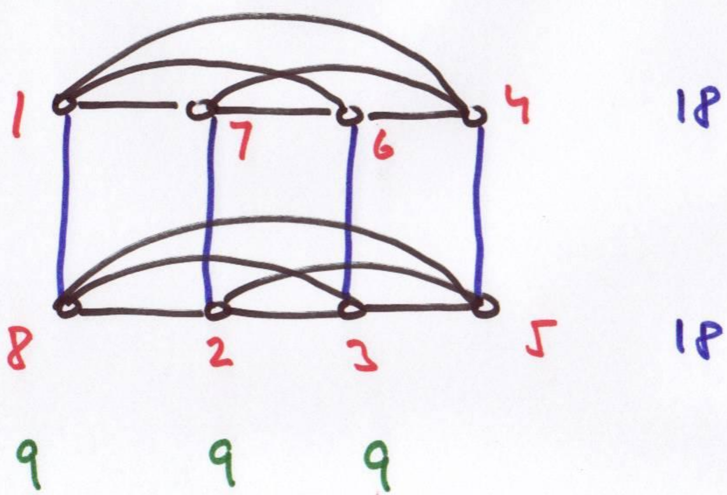
$K_2 \times K_4$



$K_2 \times K_4$



$K_2 \times K_4$



Remove an r-factor F with

a distance-antimagic vertex labeling with diff $= 2$

1	7	6	4	**18**
8	2	3	5	**18**
9	**9**	**9**	**9**	

$K_2 \quad K_4$

Remove an r-factor F with

a distance-antimagic vertex labeling with diff $= 2$

1	7	6	4	**18**
8	2	3	5	**18**
9	**9**	**9**	**9**	

Team 1 misses opponents with total rankings

18 + 9

Remove an r-factor F with

a distance-antimagic vertex labeling with diff = 2

1	7	6	4	**18**
8	2	3	5	**18**
9	**9**	**9**	**9**	

Team 1 misses opponents with total rankings

$$18 + 9 - 1 - 1 = 25$$

Remove an *r*-factor *F* with

a distance-antimagic vertex labeling with diff = 2

1	7	6	4	**18**
8	2	3	5	**18**
9	**9**	**9**	**9**	

Team 1 misses opponents with total rankings

$$18 + 9 - 1 - 1 = 25$$

therefore plays opponents with total rankings

$$35 - 25 = 10$$

Remove an r-factor F with

a distance-antimagic vertex labeling with diff = 2

1	7	6	4	**18**
8	**2**	3	5	**18**
9	**9**	**9**	**9**	

Team 2 misses opponents with total rankings

$$18 + 9 - 2 - 2 = 23$$

Remove an r-factor F with *a distance-antimagic vertex labeling with diff = 2*

1	7	6	4	**18**
8	**2**	3	5	**18**
9	**9**	**9**	**9**	

Team 2 misses opponents with total rankings

$$18 + 9 - 2 - 2 = 23$$

therefore plays opponents with total rankings

$$34 - 23 = 11$$

Remove an r-factor F with

a distance-antimagic vertex labeling with diff = 2

1	7	6	4	**18**
8	2	**3**	5	**18**
9	**9**	**9**	**9**	

Team 3 misses opponents with total rankings

18 + 9 – 3 – 3 = 21

Remove an *r*-factor *F* with *a distance-antimagic vertex labeling with diff* = 2

1	7	6	4	**18**
8	2	**3**	5	**18**
9	**9**	**9**	**9**	

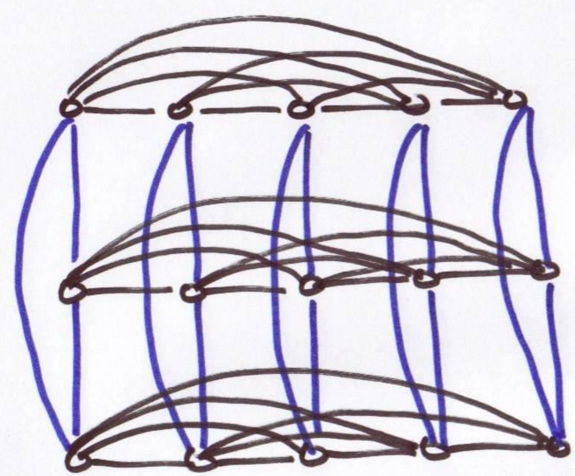
Team 3 misses opponents with total rankings

$$18 + 9 - 3 - 3 = 21$$

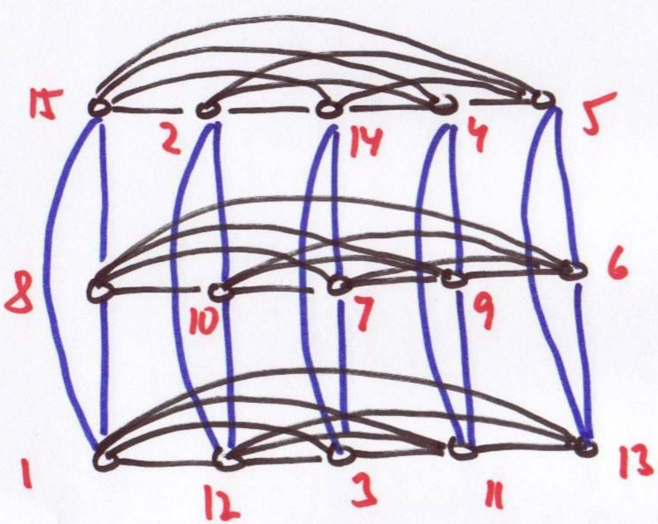
therefore plays opponents with total rankings

$$33 - 21 = 12$$

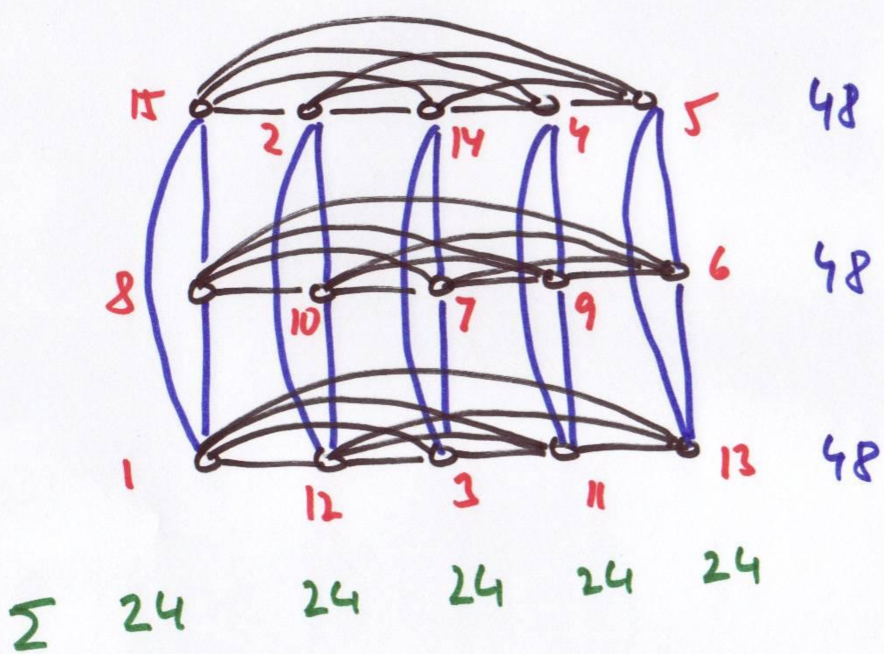
$K_3 \times K_5$



$K_3 \times K_5$



$K_3 \times K_5$



Remove an r-factor F with *a distance-antimagic vertex labeling with diff = 2*

15	2	14	4	5	**40**
8	10	7	9	6	**40**
1	12	3	11	13	**40**
24	**24**	**24**	**24**	**24**	

$K_3 \quad K_5$

Definition: A *magic rectangle* MR(a,b) is an $a \times b$ array with $a,b > 1$ in which the first ab positive integers are placed so that the sum over each column of MR(a,b) is $\sigma(a,b) = a(ab + 1)/2$ and the sum over each row is $\tau(a,b) = b(ab + 1)/2$.

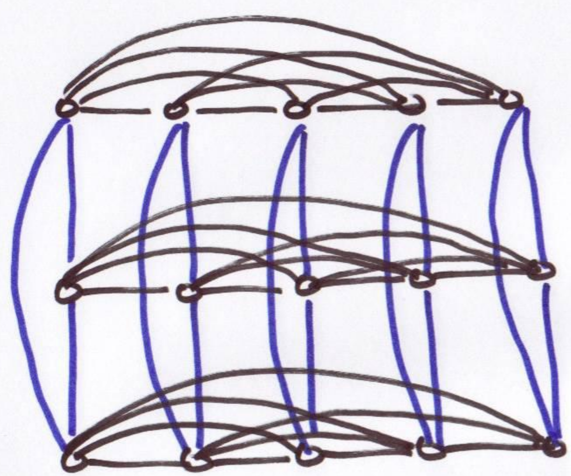
Definition: A *magic rectangle* MR(a,b) is an $a \times b$ array with $a,b > 1$ in which the first ab positive integers are placed so that the sum over each column of MR(a,b) is $\sigma(a,b) = a(ab + 1)/2$ and the sum over each row is $\tau(a,b) = b(ab + 1)/2$.

52	61	4	13	20	29	36	45
14	3	62	51	46	35	30	19
53	60	5	12	21	28	37	44
11	6	59	54	43	38	27	22
55	58	7	10	23	26	39	42
9	8	57	56	41	40	25	24
50	63	2	15	18	31	34	47
16	1	64	49	48	33	32	17

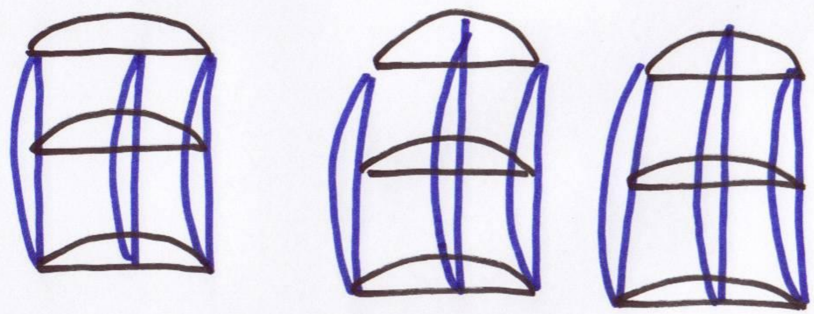
Definition: A *magic rectangle* MR(a,b) is an $a \times b$ array with $a,b > 1$ in which the first ab positive integers are placed so that the sum over each column of MR(a,b) is $\sigma(a,b) = a(ab + 1)/2$ and the sum over each row is $\tau(a,b) = b(ab + 1)/2$.

Theorem: (Harmuth, 1861) There is a magic rectangle MR(a,b) if and only if $a \equiv b \pmod 2$ except when $a = b = 2$.

$K_3 \times K_5$



$3 K_2 \times K_3$



Remove an *r*-factor *F* with *a distance-antimagic vertex labeling with diff = 2*

1	27	14	**42**	10	9	23	**42**	19	18	5
15	2	25	**42**	24	11	7	**42**	6	20	16
26	13	3	**42**	8	22	12	**42**	17	4	21
42	**42**	**42**		**42**	**42**	**42**		**42**	**42**	**42**

$3K_3 \quad K_3$

Remove an *r*-factor *F* with

a distance-antimagic vertex labeling with diff = 2

1	15	14	4	**34**	5	11	10	8
16	2	3	13	**34**	12	6	7	9
17	**17**	**17**	**17**		**17**	**17**	**17**	**17**

$$2\,K_2 \quad K_4$$

Definition: A *magic rectangle set* MRS($a,b;c$) is a collection of c arrays $a \times b$ with $a,b > 1$ in which the first abc positive integers are placed so that the sum over each column of every MR(a,b) is $\sigma(a,b) = ac(abc + 1)/2$ and the sum over each row is $\tau(a,b) = bc(abc + 1)/2$.

Definition: A *magic rectangle set* MRS($a,b;c$) is a collection of c arrays $a \times b$ with $a,b > 1$ in which the first abc positive integers are placed so that the sum over each column of every MR(a,b) is $\sigma(a,b) = ac(abc + 1)/2$ and the sum over each row is $\tau(a,b) = bc(abc + 1)/2$.

1	27	14	**42**	10	9	23	**42**	19	18	5
15	2	25	**42**	24	11	7	**42**	6	20	16
26	13	3	**42**	8	22	12	**42**	17	4	21
42	**42**	**42**		**42**	**42**	**42**		**42**	**42**	**42**

Definition: A *magic rectangle set* MRS($a,b;c$) is a collection of c arrays $a \times b$ with $a,b > 1$ in which the first abc positive integers are placed so that the sum over each column of every MR(a,b) is $\sigma(a,b) = ac(abc + 1)/2$ and the sum over each row is $\tau(a,b) = bc(abc + 1)/2$.

Theorem: There no magic rectangle set MRS($a,b;c$) for a odd if b or c is even.

Definition: A *magic rectangle set* MRS($a,b;c$) is a collection of c arrays $a \times b$ with $a,b > 1$ in which the first abc positive integers are placed so that the sum over each column of every MR(a,b) is $\sigma(a,b) = ac(abc + 1)/2$ and the sum over each row is $\tau(a,b) = bc(abc + 1)/2$.

Theorem: There no magic rectangle set MRS($a,b;c$) for a odd if b or c is even.

Theorem: There is a magic rectangle set MRS($a,b;c$) whenever $a \equiv b \equiv 0 \pmod 2$, $2 \leq a$, $4 \leq b$.

Definition: A *magic rectangle set* MRS($a,b;c$) is a collection of c arrays $a \times b$ with $a,b > 1$ in which the first abc positive integers are placed so that the sum over each column of every MR(a,b) is $\sigma(a,b) = ac(abc + 1)/2$ and the sum over each row is $\tau(a,b) = bc(abc + 1)/2$.

Theorem: There no magic rectangle set MRS($a,b;c$) for a odd if b or c is even.

Theorem: There is a magic rectangle set MRS($a,b;c$) whenever $a \equiv b \equiv 0 \pmod 2$, $2 \le a$, $4 \le b$.

Theorem: There is a magic rectangle set MRS($a,b;c$) whenever $a \equiv b \equiv c \equiv 1 \pmod 2$.

Definition: A *magic rectangle set* MRS($a,b;c$) is a collection of c arrays $a \times b$ with $a,b > 1$ in which the first abc positive integers are placed so that the sum over each column of every MR(a,b) is $\sigma(a,b) = ac(abc + 1)/2$ and the sum over each row is $\tau(a,b) = bc(abc + 1)/2$.

Theorem: There no magic rectangle set MRS($a,b;c$) for a odd if b or c is even.

Theorem: There is a magic rectangle set MRS($a,b;c$) whenever $a \equiv b \equiv 0 \pmod 2$, $2 \leq a$, $4 \leq b$.

Theorem: There is a magic rectangle set MRS($a,b;c$) whenever $a \equiv b \equiv c \equiv 1 \pmod 2$.

Proof:

Definition: A *magic rectangle set* MRS($a,b;c$) is a collection of c arrays $a \times b$ with $a,b > 1$ in which the first abc positive integers are placed so that the sum over each column of every MR(a,b) is $\sigma(a,b) = ac(abc + 1)/2$ and the sum over each row is $\tau(a,b) = bc(abc + 1)/2$.

Theorem: There no magic rectangle set MRS($a,b;c$) for a odd if b or c is even.

Theorem: There is a magic rectangle set MRS($a,b;c$) whenever $a \equiv b \equiv 0 \pmod 2$, $2 \le a$, $4 \le b$.

Theorem: There is a magic rectangle set MRS($a,b;c$) whenever $a \equiv b \equiv c \equiv 1 \pmod 2$. ***Really?***

Proof:

Definition: A *magic rectangle set* MRS($a,b;c$) is a collection of c arrays $a \times b$ with $a,b > 1$ in which the first abc positive integers are placed so that the sum over each column of every MR(a,b) is $\sigma(a,b) = ac(abc + 1)/2$ and the sum over each row is $\tau(a,b) = bc(abc + 1)/2$.

Theorem: There no magic rectangle set MRS($a,b;c$) for a odd if b or c is even.

Theorem: There is a magic rectangle set MRS($a,b;c$) whenever $a \equiv b \equiv 0 \pmod 2$, $2 \leq a$, $4 \leq b$.

Theorem: There is a magic rectangle set MRS($a,b;c$) whenever $a \equiv b \equiv c \equiv 1 \pmod 2$. ***Really?***

Proof: The margin of this napkin is to small…

Theorem: There no magic rectangle set MRS($a,b;c$) for a odd if b or c is even.

Theorem: There is a magic rectangle set MRS($a,b;c$) whenever $a \equiv b \equiv 0 \pmod 2$, $2 \leq a$, $4 \leq b$.

Theorem: There is a magic rectangle set MRS($a,b;c$) when $a \equiv b \equiv c \equiv 1 \pmod 2$, $c|a$ or $c|b$, and $a,b > 1$.

Theorem: There is a magic rectangle set MRS($a,b;c$) when $a \equiv b \equiv c \equiv 1 \pmod 2$, $a \leq b$, $d|c$ and $d \leq a$.

Example — MRS(7,11;15)

Take MR(7,11)

76	56	42	55	14	0	63	49	35	21	7	418
5	8	33	22	29	36	43	50	57	64	71	418
2	9	16	32	30	37	44	53	58	65	72	418
38	45	52	59	66	73	3	10	17	24	31	418
74	60	46	23	18	4	67	51	39	25	11	418
1	19	15	47	61	75	12	26	40	54	68	418
70	69	62	28	48	41	34	27	20	13	6	418
266	266	266	266	266	266	266	266	266	266	266	

Example — MRS(7,11;15)

Take MR(7,11)

76	56	42	55	14	0	63	49	35	21	7	418
5	8	33	22	29	36	43	50	57	64	71	418
2	9	16	32	30	37	44	53	58	65	72	418
38	45	52	59	66	73	3	10	17	24	31	418
74	60	46	23	18	4	67	51	39	25	11	418
1	19	15	47	61	75	12	26	40	54	68	418
70	69	62	28	48	41	34	27	20	13	6	418
266	266	266	266	266	266	266	266	266	266	266	

Multiply every element by 15

1140	840	630	825	210	0	945	735	525	315	105	6270
75	120	495	330	435	540	645	750	855	960	1065	6270
30	135	240	480	450	555	660	795	870	975	1080	6270
570	675	780	885	990	1095	45	150	255	360	465	6270
1110	900	690	345	270	60	1005	765	585	375	165	6270
15	285	225	705	915	1125	180	390	600	810	1020	6270
1050	1035	930	420	720	615	510	405	300	195	90	6270
3990	3990	3990	3990	3990	3990	3990	3990	3990	3990	3990	

Example — MRS(7,11;15)

Take KA(3,5)

0	1	2	3	4
3	4	0	1	2
3	1	4	2	0
6	6	6	6	6

Example — MRS(7,11;15)

Take KA(3,5)

0	1	2	3	4
3	4	0	1	2
3	1	4	2	0
6	6	6	6	6

Lift it

10	11	12	13	14	+10
8	9	5	6	7	+5
3	1	4	2	0	+0
21	21	21	21	21	

Example — MRS(7,11;15)

Take KA(3,5)

0	1	2	3	4
3	4	0	1	2
3	1	4	2	0
6	6	6	6	6

Lift it

10	11	12	13	14	+10
8	9	5	6	7	+5
3	1	4	2	0	+0
21	21	21	21	21	

Example — MRS(7,11;15)

Take KA(3,5)

0	1	2	3	4
3	4	0	1	2
3	1	4	2	0
6	6	6	6	6

Lift it

10	11	12	13	14	+10
8	9	5	6	7	+5
3	1	4	2	0	+0
21	21	21	21	21	

Pick a column to construct LS(3)

10	8	3									
8	3	10									
3	10	8									

Example — MRS(7,11;15)

Take KA(3,5)

0	1	2	3	4
3	4	0	1	2
3	1	4	2	0
6	6	6	6	6

Lift it

10	11	12	13	14	+10
8	9	5	6	7	+5
3	1	4	2	0	+0
21	21	21	21	21	

Pick a column to construct LS(3)

10	8	3	11	3	11	3	11	3	11	3	77
8	3	10	4	10	4	10	4	10	4	10	77
3	10	8	6	8	6	8	6	8	6	8	77

Expand rows

Example — MRS(7,11;15)

Take KA(3,5)

0	1	2	3	4
3	4	0	1	2
3	1	4	2	0
6	6	6	6	6

Lift it

10	11	12	13	14	+10
8	9	5	6	7	+5
3	1	4	2	0	+0
21	21	21	21	21	

Pick a column to construct LS(3)

10	8	3	11	3	11	3	11	3	11	3	77
8	3	10	4	10	4	10	4	10	4	10	77
3	10	8	6	8	6	8	6	8	6	8	77
11	4	6									
3	10	8									
11	4	6									
3	10	8									
49	49	49									

Expand rows and columns

Example — MRS(7,11;15)

Take KA(3,5)

0	1	2	3	4
3	4	0	1	2
3	1	4	2	0
6	6	6	6	6

Lift it

10	11	12	13	14	+10
8	9	5	6	7	+5
3	1	4	2	0	+0
21	21	21	21	21	

Pick a column to construct LS(3)

10	8	3	11	3	11	3	11	3	11	3	77
8	3	10	4	10	4	10	4	10	4	10	77
3	10	8	6	8	6	8	6	8	6	8	77
11	4	6	8	6	8	6	8	6	8	6	77
3	10	8	6	8	6	8	6	8	6	8	77
11	4	6	8	6	8	6	8	6	8	6	77
3	10	8	6	8	6	8	6	8	6	8	77
49	49	49	49	49	49	49	49	49	49	49	

Expand rows and columns

Fill the rest

Example — MRS(7,11;15)

10	8	3	11	3	11	3	11	3	11	3	77
8	3	10	4	10	4	10	4	10	4	10	77
3	10	8	6	8	6	8	6	8	6	8	77
11	4	6	8	6	8	6	8	6	8	6	77
3	10	8	6	8	6	8	6	8	6	8	77
11	4	6	8	6	8	6	8	6	8	6	77
3	10	8	6	8	6	8	6	8	6	8	77
49	49	49	49	49	49	49	49	49	49	49	

8	3	10									
3	10	8									
10	8	3									

Rotate the Latin Square

Example — MRS(7,11;15)

10	8	3	11	3	11	3	11	3	11	3	77
8	3	10	4	10	4	10	4	10	4	10	77
3	10	8	6	8	6	8	6	8	6	8	77
11	4	6	8	6	8	6	8	6	8	6	77
3	10	8	6	8	6	8	6	8	6	8	77
11	4	6	8	6	8	6	8	6	8	6	77
3	10	8	6	8	6	8	6	8	6	8	77
49	49	49	49	49	49	49	49	49	49	49	

8	3	10	4	10	4	10	4	10	4	10	77
3	10	8	6	8	6	8	6	8	6	8	77
10	8	3	11	3	11	3	11	3	11	3	77
4	6	11	3	11	3	11	3	11	3	11	77
10	8	3	11	3	11	3	11	3	11	3	77
4	6	11	3	11	3	11	3	11	3	11	77
10	8	3	11	3	11	3	11	3	11	3	77
49	49	49	49	49	49	49	49	49	49	49	

Rotate the Latin Square

Fill the rest as before

Example — MRS(7,11;15)

10	8	3	11	3	11	3	11	3	11	3	77
8	3	10	4	10	4	10	4	10	4	10	77
3	10	8	6	8	6	8	6	8	6	8	77
11	4	6	8	6	8	6	8	6	8	6	77
3	10	8	6	8	6	8	6	8	6	8	77
11	4	6	8	6	8	6	8	6	8	6	77
3	10	8	6	8	6	8	6	8	6	8	77
49	49	49	49	49	49	49	49	49	49	49	

8	3	10	4	10	4	10	4	10	4	10	77
3	10	8	6	8	6	8	6	8	6	8	77
10	8	3	11	3	11	3	11	3	11	3	77
4	6	11	3	11	3	11	3	11	3	11	77
10	8	3	11	3	11	3	11	3	11	3	77
4	6	11	3	11	3	11	3	11	3	11	77
10	8	3	11	3	11	3	11	3	11	3	77
49	49	49	49	49	49	49	49	49	49	49	

3	10	8	6	8	6	8	6	8	6	8	77
10	8	3	11	3	11	3	11	3	11	3	77
8	3	10	4	10	4	10	4	10	4	10	77
6	11	4	10	4	10	4	10	4	10	4	77
8	3	10	4	10	4	10	4	10	4	10	77
6	11	4	10	4	10	4	10	4	10	4	77
8	3	10	4	10	4	10	4	10	4	10	77
49	49	49	49	49	49	49	49	49	49	49	

Example — MRS(7,11;15)

Take KA(3,5)

0	1	2	3	4
3	4	0	1	2
3	1	4	2	0
6	6	6	6	6

Lift it

10	11	12	13	14	+10
8	9	5	6	7	+5
3	1	4	2	0	+0
21	21	21	21	21	

Pick a column to construct LS(3)

Example — MRS(7,11;15)

Take KA(3,5)

0	1	2	3	4
3	4	0	1	2
3	1	4	2	0
6	6	6	6	6

Lift it

10	11	12	13	14	+10
8	9	5	6	7	+5
3	1	4	2	0	+0
21	21	21	21	21	

Pick a column to construct LS(3)

14	7	0									
7	0	14									
0	14	7									

Example — MRS(7,11;15)

Take KA(3,5)

0	1	2	3	4
3	4	0	1	2
3	1	4	2	0
6	6	6	6	6

Lift it

10	11	12	13	14	+10
8	9	5	6	7	+5
3	1	4	2	0	+0
21	21	21	21	21	

Pick a column to construct LS(3)

14	7	0	14	0	14	0	14	0	14	0	77
7	0	14	0	14	0	14	0	14	0	14	77
0	14	7	7	7	7	7	7	7	7	7	77
14	0	7	7	7	7	7	7	7	7	7	77
0	14	7	7	7	7	7	7	7	7	7	77
14	0	7	7	7	7	7	7	7	7	7	77
0	14	7	7	7	7	7	7	7	7	7	77
49	49	49	49	49	49	49	49	49	49	49	

Fill the rest

Example — MRS(7,11;15)

Take KA(3,5)

0	1	2	3	4
3	4	0	1	2
3	1	4	2	0
6	6	6	6	6

Lift it

10	11	12	13	14	+10
8	9	5	6	7	+5
3	1	4	2	0	+0
21	21	21	21	21	

Pick a column to construct LS(3)

14	7	0	14	0	14	0	14	0	14	0	77
7	0	14	0	14	0	14	0	14	0	14	77
0	14	7	7	7	7	7	7	7	7	7	77
14	0	7	7	7	7	7	7	7	7	7	77
0	14	7	7	7	7	7	7	7	7	7	77
14	0	7	7	7	7	7	7	7	7	7	77
0	14	7	7	7	7	7	7	7	7	7	77
49	49	49	49	49	49	49	49	49	49	49	

Fill the rest

Repeat...

Example — MRS(7,11;15)

1140	840	630	825	210	0	945	735	525	315	105	6270
75	120	495	330	435	540	645	750	855	960	1065	6270
30	135	240	480	450	555	660	795	870	975	1080	6270
570	675	780	885	990	1095	45	150	255	360	465	6270
1110	900	690	345	270	60	1005	765	585	375	165	6270
15	285	225	705	915	1125	180	390	600	810	1020	6270
1050	1035	930	420	720	615	510	405	300	195	90	6270
3990	3990	3990	3990	3990	3990	3990	3990	3990	3990	3990	

Example — MRS(7,11;15)

1140	840	630	825	210	0	945	735	525	315	105	6270
75	120	495	330	435	540	645	750	855	960	1065	6270
30	135	240	480	450	555	660	795	870	975	1080	6270
570	675	780	885	990	1095	45	150	255	360	465	6270
1110	900	690	345	270	60	1005	765	585	375	165	6270
15	285	225	705	915	1125	180	390	600	810	1020	6270
1050	1035	930	420	720	615	510	405	300	195	90	6270
3990	3990	3990	3990	3990	3990	3990	3990	3990	3990	3990	

8	3	10	4	10	4	10	4	10	4	10	77
3	10	8	6	8	6	8	6	8	6	8	77
10	8	3	11	3	11	3	11	3	11	3	77
4	6	11	3	11	3	11	3	11	3	11	77
10	8	3	11	3	11	3	11	3	11	3	77
4	6	11	3	11	3	11	3	11	3	11	77
10	8	3	11	3	11	3	11	3	11	3	77
49	49	49	49	49	49	49	49	49	49	49	

Example — MRS(7,11;15)

1140	840	630	825	210	0	945	735	525	315	105	6270
75	120	495	330	435	540	645	750	855	960	1065	6270
30	135	240	480	450	555	660	795	870	975	1080	6270
570	675	780	885	990	1095	45	150	255	360	465	6270
1110	900	690	345	270	60	1005	765	585	375	165	6270
15	285	225	705	915	1125	180	390	600	810	1020	6270
1050	1035	930	420	720	615	510	405	300	195	90	6270
3990	3990	3990	3990	3990	3990	3990	3990	3990	3990	3990	

8	3	10	4	10	4	10	4	10	4	10	77
3	10	8	6	8	6	8	6	8	6	8	77
10	8	3	11	3	11	3	11	3	11	3	77
4	6	11	3	11	3	11	3	11	3	11	77
10	8	3	11	3	11	3	11	3	11	3	77
4	6	11	3	11	3	11	3	11	3	11	77
10	8	3	11	3	11	3	11	3	11	3	77
49	49	49	49	49	49	49	49	49	49	49	

1148	843	640	829	220	4	955	739	535	319	115	6347
78	130	503	336	443	546	653	756	863	966	1073	6347
40	143	243	491	453	566	663	806	873	986	1083	6347
574	681	791	888	1001	1098	56	153	266	363	476	6347
1120	908	693	356	273	71	1008	776	588	386	168	6347
19	291	236	708	926	1128	191	393	611	813	1031	6347
1060	1043	933	431	723	626	513	416	303	206	93	6347
4039	4039	4039	4039	4039	4039	4039	4039	4039	4039	4039	

Theorem: There no MRS($a,b;c$) for a odd if b or c is even.

Theorem: There is a magic rectangle set MRS($a,b;c$) whenever $a \equiv b \equiv 0 \pmod 2$, $2 \leq a$, $4 \leq b$, for any c.

Theorem: There is a magic rectangle set MRS($a,b;c$) when $a \equiv b \equiv c \equiv 1 \pmod 2$, $3 \leq a \leq b$, $d|c$ and $d \leq a$.

Corollary: There is a magic rectangle set MRS($a,b;c$) when $a \equiv b \equiv c \equiv 1 \pmod 2$ and $3 \leq c \leq a \leq b$.

Theorem: There no MRS($a,b;c$) for a odd if b or c is even.

Theorem: There is a magic rectangle set MRS($a,b;c$) whenever $a \equiv b \equiv 0 \pmod 2$, $2 \leq a$, $4 \leq b$, for any c.

Theorem: There is a magic rectangle set MRS($a,b;c$) when $a \equiv b \equiv c \equiv 1 \pmod 2$, $3 \leq a \leq b$, $d|c$ and $d \leq a$.

Corollary: There is a magic rectangle set MRS($a,b;c$) when $a \equiv b \equiv c \equiv 1 \pmod 2$ and $3 \leq c \leq a \leq b$.

Theorem: There is a magic rectangle set MRS($a,b;c$) when $a \equiv b \equiv c \equiv 1 \pmod 2$, $3 \leq a \leq b$, and $a \leq c$.

Example — MRS(3,5;7)

Example — MRS(3,5;7)

Take MR(3,5)

15	2	14	4	5	40
8	10	7	9	6	40
1	12	3	11	13	40
24	24	24	24	24	

Example — MRS(3,5;7)

Take MR(3,5)

15	2	14	4	5	40
8	10	7	9	6	40
1	12	3	11	13	40
24	24	24	24	24	

Multiply by 7 (base array)

105	14	98	28	35	280
56	70	49	63	42	280
7	84	21	77	91	280
168	168	168	168	168	

Example — MRS(3,5;7)

Take MR(3,5)

15	2	14	4	5	40
8	10	7	9	6	40
1	12	3	11	13	40
24	24	24	24	24	

Multiply by 7 (base array)

105	14	98	28	35	280
56	70	49	63	42	280
7	84	21	77	91	280
168	168	168	168	168	

Construct Kotzig Array KA(3,7)

1	2	3	4	5	6	7
7	5	3	1	6	4	2
4	5	6	7	1	2	3
12	12	12	12	12	12	12

Example — MRS(3,5;7)

Take MR(3,5)

15	2	14	4	5	40
8	10	7	9	6	40
1	12	3	11	13	40
24	24	24	24	24	

Multiply by 7 (base array)

105	14	98	28	35	280
56	70	49	63	42	280
7	84	21	77	91	280
168	168	168	168	168	

Construct Kotzig Array KA(3,7)

1	2	3	4	5	6	7
7	5	3	1	6	4	2
4	5	6	7	1	2	3
12	12	12	12	12	12	12

Construct LS(3)

1	4	7
7	1	4
4	7	1
12	12	12

Example — MRS(3,5;7)

Take MR(3,5)

15	2	14	4	5	40
8	10	7	9	6	40
1	12	3	11	13	40
24	24	24	24	24	

Multiply by 7 (base array)

105	14	98	28	35	280
56	70	49	63	42	280
7	84	21	77	91	280
168	168	168	168	168	

Construct Kotzig Array KA(3,7)

1	2	3	4	5	6	7
7	5	3	1	6	4	2
4	5	6	7	1	2	3
12	12	12	12	12	12	12

Construct LS(3) and complete 3x7 residual array

1	4	7	1	7	20
7	1	4	7	1	20
4	7	1	4	4	20
12	12	12	12	12	

Example — MRS(3,5;7)

Take MR(3,5) — Multiply by 7 (base array)

15	2	14	4	5	40
8	10	7	9	6	40
1	12	3	11	13	40
24	24	24	24	24	

105	14	98	28	35	280
56	70	49	63	42	280
7	84	21	77	91	280
168	168	168	168	168	

Construct Kotzig Array KA(3,7)

1	2	3	4	5	6	7
7	5	3	1	6	4	2
4	5	6	7	1	2	3
12	12	12	12	12	12	12

Construct LS(3) and complete 3x7 residual array

1	4	7	1	7	20
7	1	4	7	1	20
4	7	1	4	4	20
12	12	12	12	12	

Add base and residual arrays to obtain 1st rectangle

106	18	105	29	42	300
63	71	53	70	43	300
11	91	22	81	95	300
180	180	180	180	180	

Example — MRS(3,5;7)

Take MR(3,5)

15	2	14	4	5	40
8	10	7	9	6	40
1	12	3	11	13	40
24	24	24	24	24	

Multiply by 7 (base array)

105	14	98	28	35	280
56	70	49	63	42	280
7	84	21	77	91	280
168	168	168	168	168	

Construct Kotzig Array KA(3,7)

1	2	3	4	5	6	7
7	5	3	1	6	4	2
4	5	6	7	1	2	3
12	12	12	12	12	12	12

Construct LS(3) and complete 3x7 residual array

2	5	5	2	6	20
5	2	5	5	3	20
5	5	2	5	3	20
12	12	12	12	12	

Repeat for

another column

Add base and residual arrays to obtain 1st rectangle

107	19	103	30	41	300
61	72	54	68	45	300
12	89	23	82	94	300
180	180	180	180	180	

Example — MRS(3,5;7)

Take MR(3,5) | Multiply by 7 (base array)

15	2	14	4	5	40
8	10	7	9	6	40
1	12	3	11	13	40
24	24	24	24	24	

105	14	98	28	35	280
56	70	49	63	42	280
7	84	21	77	91	280
168	168	168	168	168	

Construct Kotzig Array KA(3,7)

1	2	3	4	5	6	7
7	5	3	1	6	4	2
4	5	6	7	1	2	3
12	12	12	12	12	12	12

Construct LS(3) and complete 3x7 residual array

3	6	3	3	5	20
3	3	6	3	5	20
6	3	3	6	2	20
12	12	12	12	12	

...and again

Add base and residual arrays to obtain 1st rectangle

108	20	101	31	40	300
59	73	55	66	47	300
13	87	24	83	93	300
180	180	180	180	180	

Example — MRS(3,5;7)

Take MR(3,5)

15	2	14	4	5	40
8	10	7	9	6	40
1	12	3	11	13	40
24	24	24	24	24	

Multiply by 7 (base array)

105	14	98	28	35	280
56	70	49	63	42	280
7	84	21	77	91	280
168	168	168	168	168	

Construct Kotzig Array KA(3,7)

1	2	3	4	5	6	7
7	5	3	1	6	4	2
4	5	6	7	1	2	3
12	12	12	12	12	12	12

Construct LS(3) and complete 3x7 residual array

7	3	2	7	1	20
2	7	3	2	6	20
3	2	7	3	5	20
12	12	12	12	12	

...until you are done!

Add base and residual arrays to obtain 1st rectangle

112	17	100	35	36	300
58	77	52	65	48	300
10	86	28	80	96	300
180	180	180	180	180	

Theorem: There no MRS($a,b;c$) for a odd if b or c is even.

Theorem: There is a magic rectangle set MRS($a,b;c$) whenever $a \equiv b \equiv 0 \pmod 2$, $2 \le a$, $4 \le b$, for any c.

Theorem: There is a magic rectangle set MRS($a,b;c$) when $a \equiv b \equiv c \equiv 1 \pmod 2$, $3 \le a \le b$, $d|c$ and $d \le a$.

Corollary: There is a magic rectangle set MRS($a,b;c$) when $a \equiv b \equiv c \equiv 1 \pmod 2$ and $3 \le c \le a \le b$.

Theorem: There is a magic rectangle set MRS($a,b;c$) when $a \equiv b \equiv c \equiv 1 \pmod 2$, $3 \le a \le b$, and $a \le c$.

Theorem: There no MRS($a,b;c$) for a odd if b or c is even.

Theorem: There is a magic rectangle set MRS($a,b;c$) whenever $a \equiv b \equiv 0 \pmod 2$, $2 \leq a$, $4 \leq b$, for any c.

Theorem: There is a magic rectangle set MRS($a,b;c$) when $a \equiv b \equiv c \equiv 1 \pmod 2$, $3 \leq a \leq b$, $d|c$ and $d \leq a$.

Corollary: There is a magic rectangle set MRS($a,b;c$) when $a \equiv b \equiv c \equiv 1 \pmod 2$ and $3 \leq c \leq a \leq b$.

Theorem: There is a magic rectangle set MRS($a,b;c$) when $a \equiv b \equiv c \equiv 1 \pmod 2$, $3 \leq a \leq b$, and $a \leq c$.

Theorem: There is a magic rectangle set MRS($a,b;c$) when $a \equiv b \equiv c \equiv 1 \pmod 2$ and $3 \leq a \leq b$.

Theorem: There no MRS($a,b;c$) for a odd if b or c is even.

Theorem: There is a magic rectangle set MRS($a,b;c$) whenever $a \equiv b \equiv 0 \pmod 2$, $2 \leq a$, $4 \leq b$, for any c.

Theorem: There is a magic rectangle set MRS($a,b;c$) when $a \equiv b \equiv c \equiv 1 \pmod 2$ and $3 \leq a \leq b$.

Search for the spectrum…

Theorem: (P. Kovar, T. Kovarova)
A regular HIT(n,r) does not exist when n and r are both even, or $r = 1, 2, (n-1), (n-2t)$,
or $r \equiv 1 \pmod 4$, and $n \equiv 2 \pmod 4$.

Search for the spectrum…

Theorem: (P. Kovar, T. Kovarova)
For r even and n odd, HIT(n,r) exists for all feasible values of n and r whenever $3 \leq r \leq n - 11$.

Search for the spectrum…

Theorem: (P. Kovar, T. Kovarova)
For r even and n odd, HIT(n,r) exists for all feasible values of n and r whenever $3 \leq r \leq n - 11$.

So we started looking at n even.

Search for the spectrum…

Theorem: (P. Kovar, T. Kovarova)
For r even and n odd, HIT(n,r) exists for all feasible values of n and r whenever $3 \leq r \leq n - 11$.

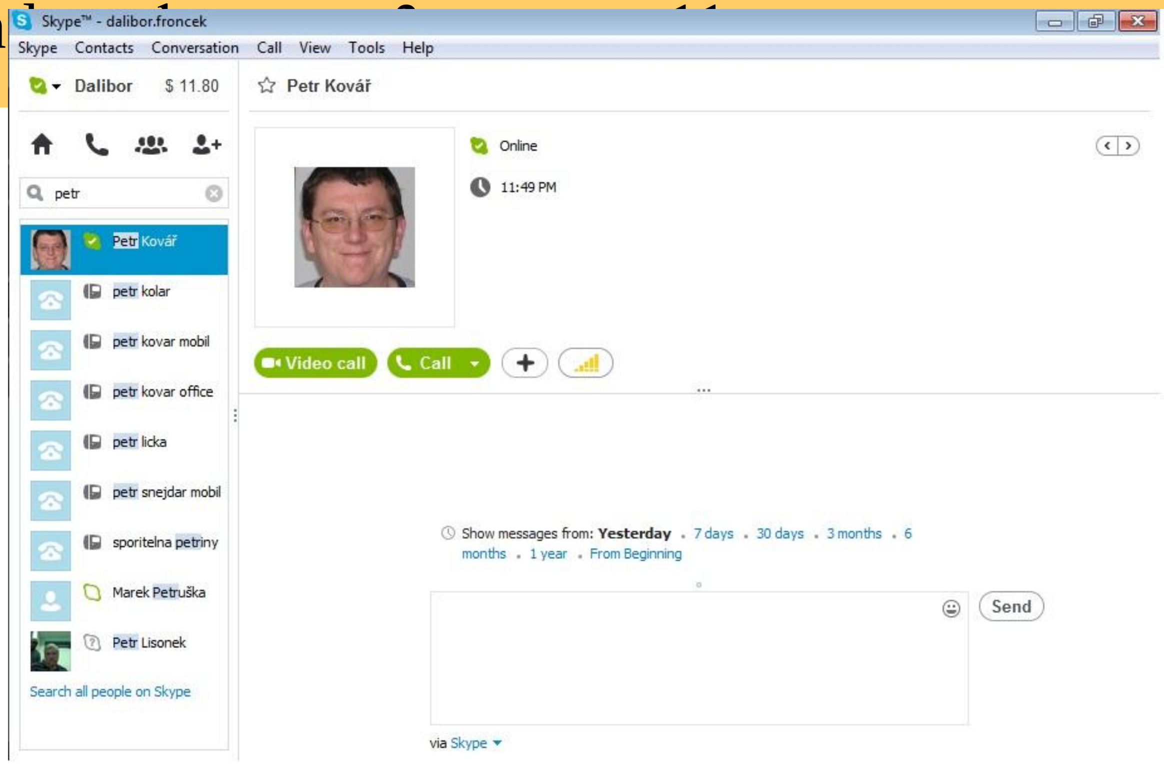
So we started looking at n even.

Theorem: (DF, A. Shepanik)
For $n \equiv 0 \pmod 8$, HIT(n,r) exists whenever $3 \leq r \leq n - 5$.
For $n \equiv 4 \pmod 8$, HIT(n,r) exists whenever $7 \leq r \leq n - 5$.

Search for the spectrum…

Theorem: (P. Kovar, T. Kovarova)

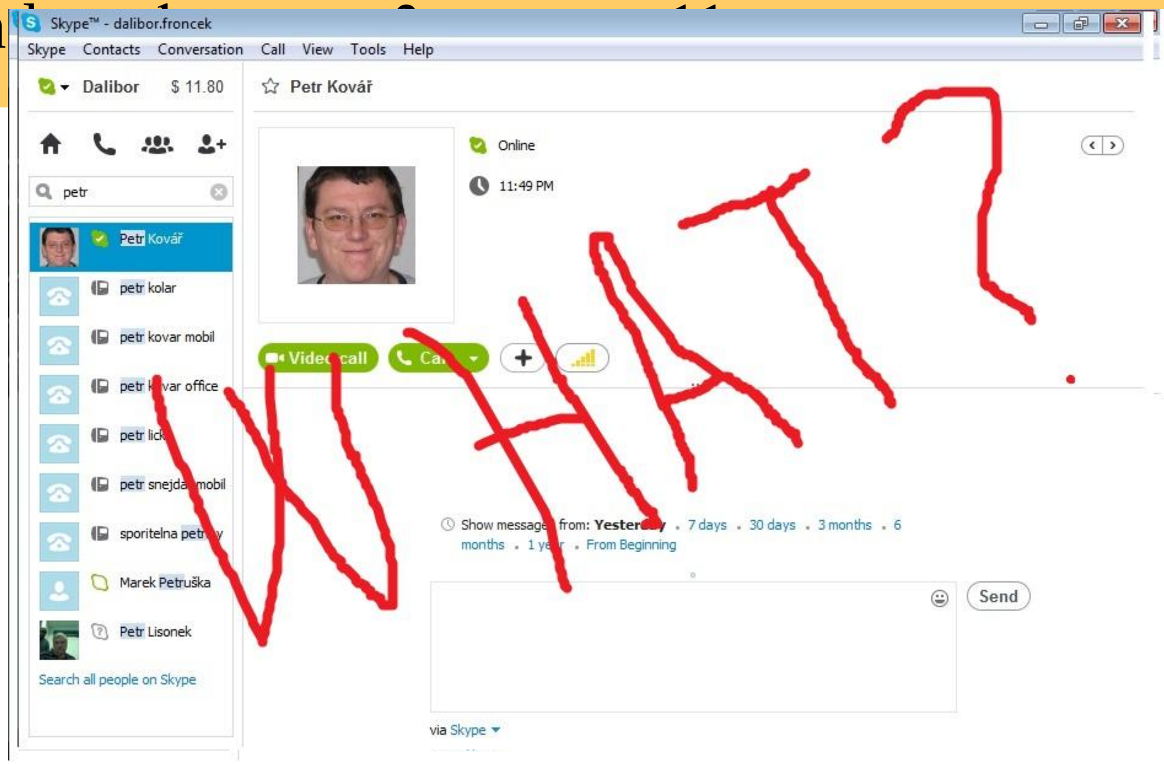
For r even and n odd, HIT(n,r) exists for all feasible values of n an



Search for the spectrum…

Theorem: (P. Kovar, T. Kovarova)

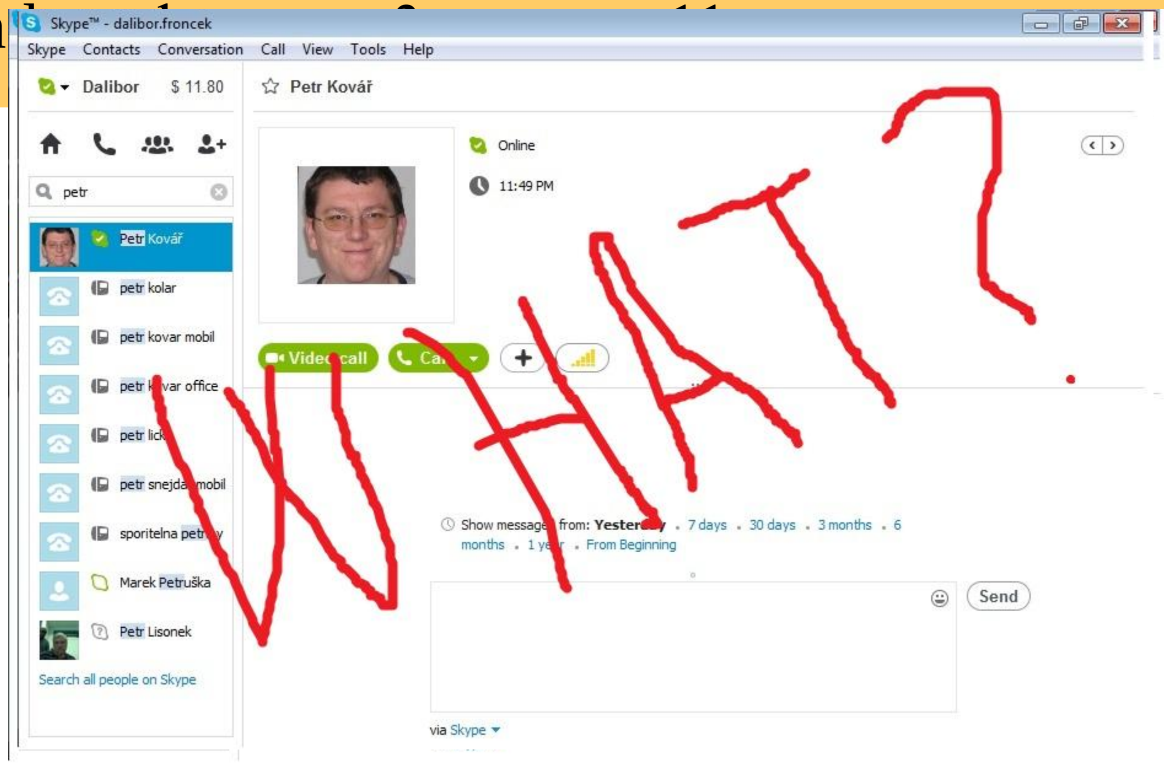
For r even and n odd, HIT(n,r) exists for all feasible values of n an



Search for the spectrum…

Theorem: (P. Kovar, T. Kovarova)

For r even and n **even**, HIT(n,r) exists for all feasible values of n an

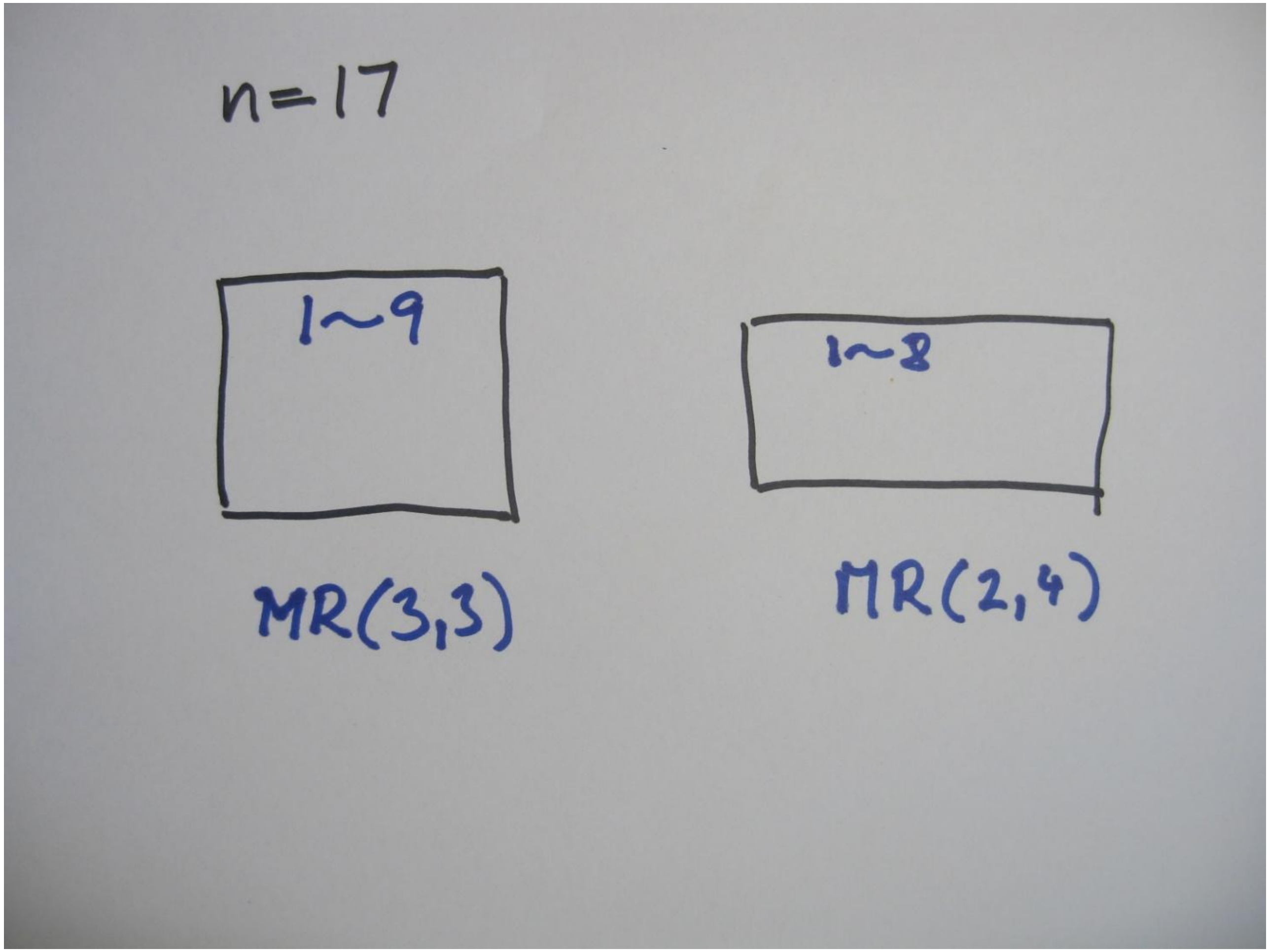


Search for the spectrum…

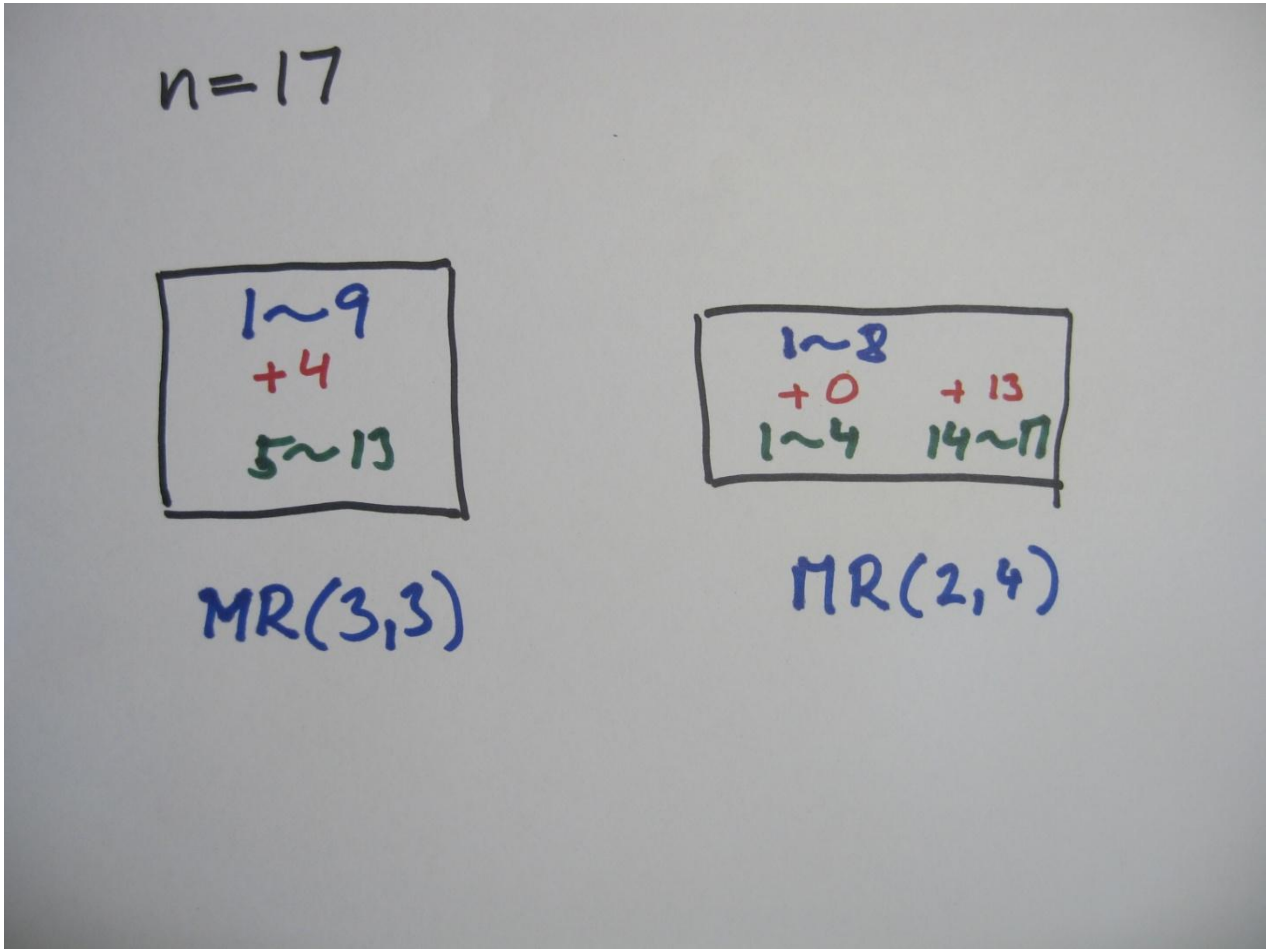
Theorem: (DF, P. Kovar, T. Kovarova, A. Shepanik)
For r even and n **even**, HIT(n,r) exists if and only if
$n \equiv 0 \pmod 4$ and $3 \le r \le n - 5$ or
$n \equiv 2 \pmod 4$ and $3 \le r \le n - 7$.

Search for the spectrum… for *n* odd

Search for the spectrum… for *n* odd



Search for the spectrum… for *n* odd

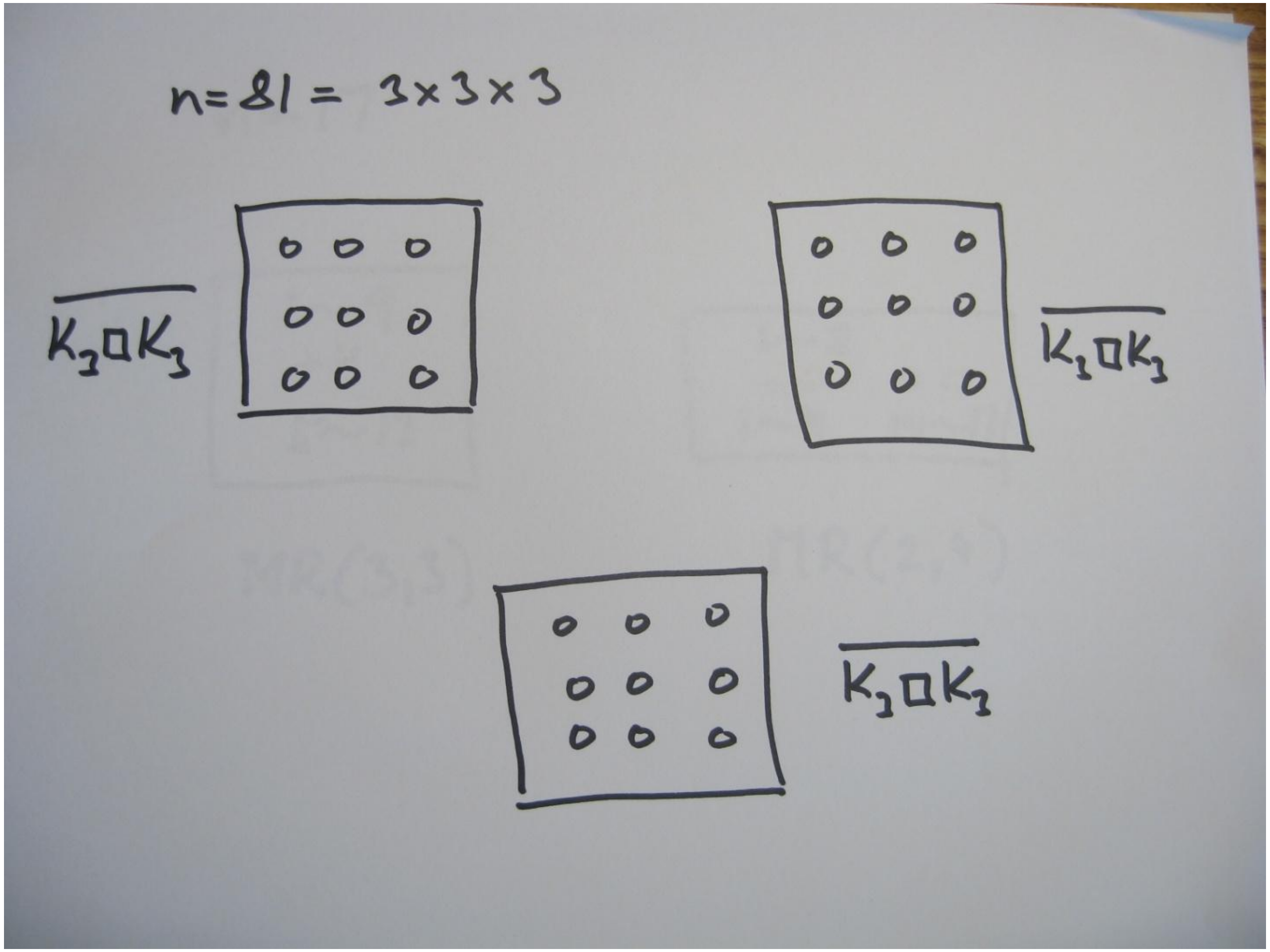


Search for the spectrum… for n odd

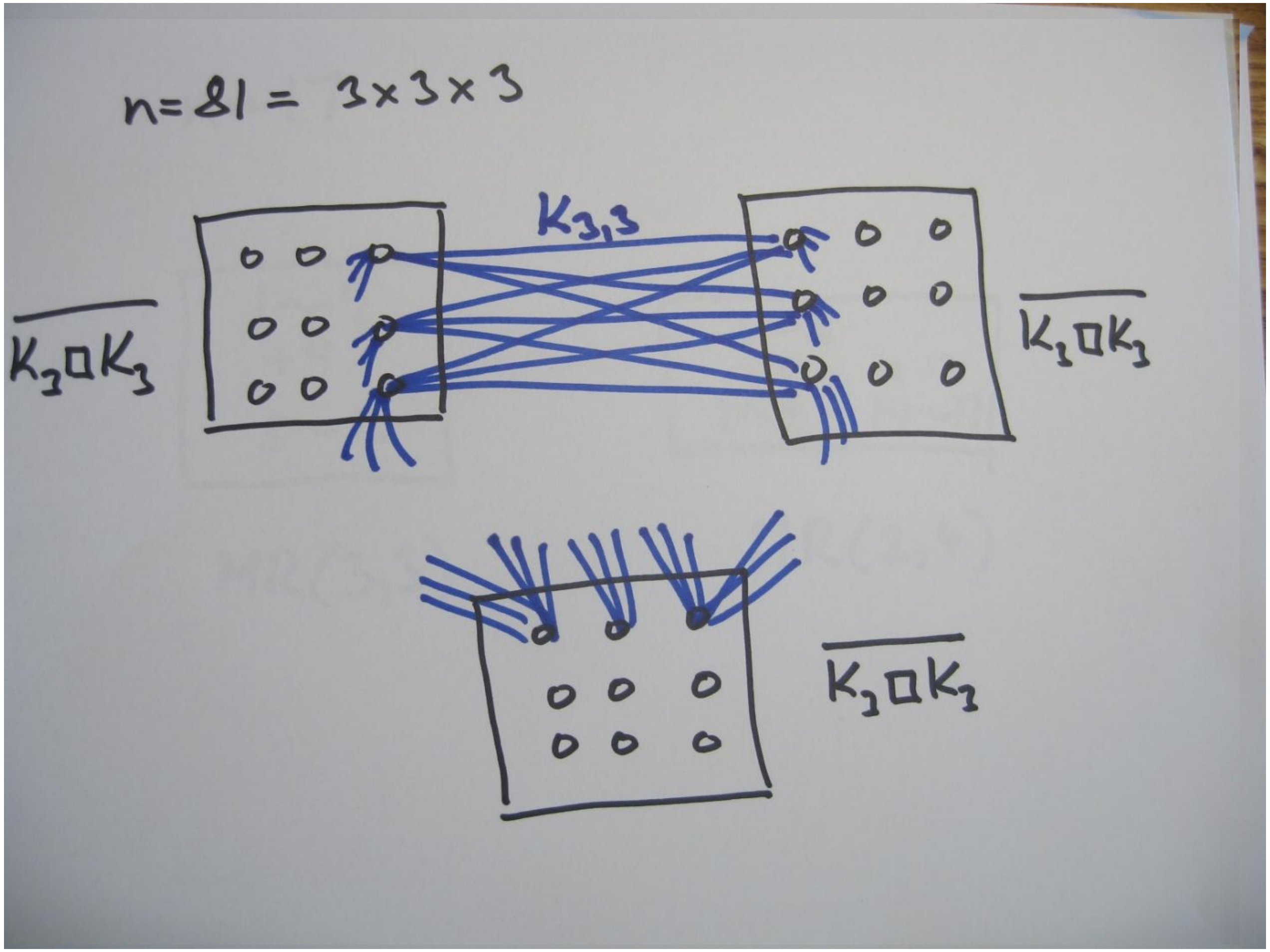
Observation:

For n odd, a HIT(n,r) exists (for some values of r) if $n \equiv 1 \pmod 4$.

Search for the spectrum… for *n* odd



Search for the spectrum… for *n* odd



Thank you!