On {3}-GDDs with 5 groups

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Joint investigation with
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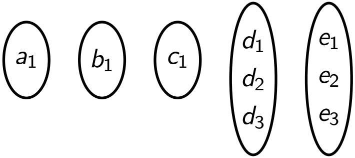
ACA 2015 — MTU August 2015

Definition.

A *group divisible design* ($\{3\}$-GDD) is a decomposition of the complete multipartite graph into triangles called *triples*

- The partite sets are called *groups* or *holes*.
- $\{3\}$-GDD of type $g_1^{t_1} g_2^{t_2} \dots g_\ell^{t_r}$ has t_i groups of size g_i, $i = 1, 2, \dots, \ell$

Ex: a 3-GDD of type $1^3 3^2$ is a decomposition of $K_{1,1,1,3,3}$ into triangles.



$\{F_1, F_2, F_3\}$ is a one-factorization of $K_{3,3}$ on $\{d_1, d_2, d_3\}$ vs $\{e_1, e_2, e_3\}$

$\{a_1, d_1, e_1\}$
$\{a_1, d_2, e_2\}$ $a_1 \to F_1$
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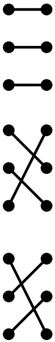

$\{a_1, b_1, c_1\}$ 3-GDD of type 1^3

Existence

Theorem 1. Colbourn 1993 For a 3-GDD of type $g_1 g_2 \cdots g_s$ with $g_1 \geq \cdots \geq g_s \geq 1$, $s \geq 2$, and $v = \sum_{i=1}^{s} g_i$ to exist, necessary conditions include :

1. $\binom{v}{2} \equiv \sum_{i=1}^{s} \binom{g_i}{2} \pmod 3$;
2. $g_i \equiv v \pmod 2$ for $1 \leq i \leq s$;
3. $g_1 \leq \sum_{i=3}^{s} g_i$;
4. whenever $\alpha_i \in \{0, 1\}$ for $1 \leq i \leq s$ and $v_0 = \sum_{i=1}^{s} \alpha_i g_i$,
$$v_0(v - v_0) \leq 2 \left[\binom{v_0}{2} + \binom{v - v_0}{2} - \sum_{i=1}^{s} \binom{g_i}{2} \right]$$
5. $2g_2 g_3 \geq g_1[g_2 + g_3 - \sum_{i=4}^{s} g_i]$; and
6. if $g_1 = \sum_{i=3}^{s} g_i$ then $2g_3 g_4 \geq (g_1 - g_2)[g_3 + g_4 - \sum_{i=5}^{s} g_i]$.

Whats known

The Colbourn conditions are know to be sufficient when

1. (Wilson 1972) $g_1 = \cdots = g_s$;
2. (Colbourn, Hoffman, and Rees 1992) $g_1 = \cdots = g_{s-1}$ or $g_2 = \cdots = g_s$;
3. (Colbourn, Cusack, and Kreher 1995) $1 \leq t \leq s$, $g_1 = \cdots = g_t$, and $g_{t+1} = \cdots = g_s = 1$;
4. (Bryant and Horsley 2006) $g_3 = \cdots = g_s = 1$; and
5. (Colbourn 1993) $\sum_{i=1}^{s} g_i \leq 60$.

Partial results are known when $g_3 = \cdots = g_s = 2$ (Colbourn, M.A. Oravas, and R. Rees 2000)
Surprisingly, in no other cases are necessary and sufficient conditions known for any other class of 3-gdds (of index 1).

Small number of groups

1. No 3-GDD with two groups exists;
2. 3-GDD of type $g_1 g_2 g_3$ exist if and only if $g_1 = g_2 = g_3$;
3. 3-GDD with four groups exist if and only if they have type g^4 or $g^3 u^1$;
4. 3-GDD with five groups are known to exist of all types g^5, $g^4 u^1$, and $g_1 \cdots g_5$ with $\sum_{i=1}^{5} g_i \leq 60$, that satisfy the necessary conditions.
   However many more cases are possible.

A 3-GDD with five groups can have all groups of different sizes.
A 3-GDD of type $17^1 11^1 9^1 7^1 5^1$ exists Colbourn 1993.
The general existence problem for five groups appears to be substantially more complicated than cases with fewer groups.

Type g^3u^2

The necessary conditions disencumber for 3-GDDs of type g^3u^2. They are simply:

1. $u \equiv 0 \pmod 3$, and
2. $g \equiv u \pmod 2$.

We prove that this conditions are also sufficient

Easy result.

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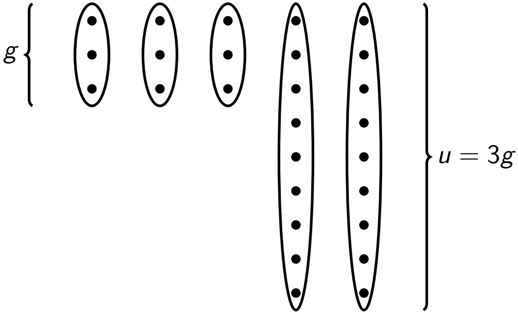

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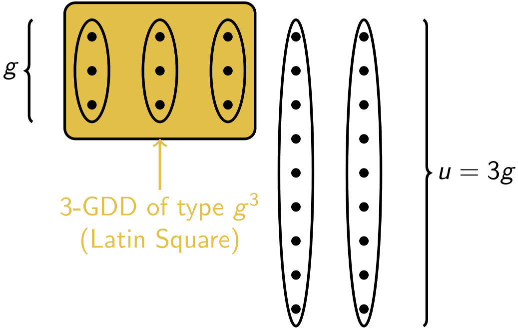

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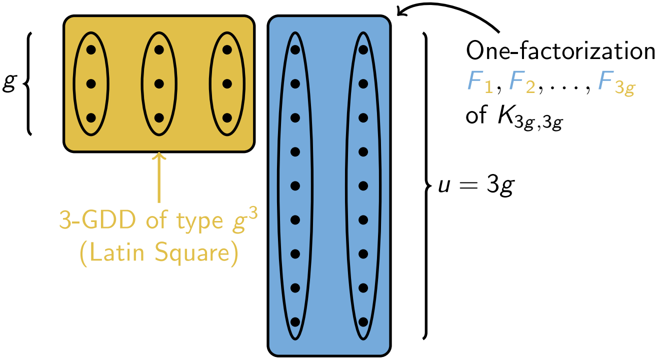

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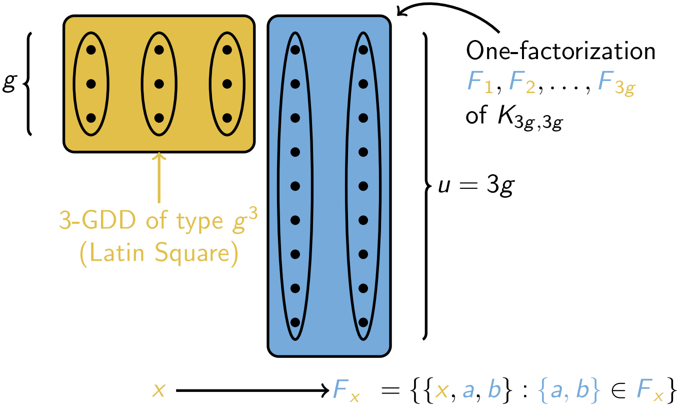
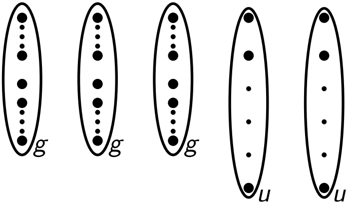


Rolf Rees established the following:

Lemma 3. Let $h \geq 1$ and $0 \leq r \leq 2h$, $(h, r) \neq (1, 2)$ or $(3, 6)$. There exists a $\{2, 3\}$-GDD of type $(2h)^3$ which is resolvable into r parallel classes of blocks of size 3 and $4h - 2r$ parallel classes of blocks of size 2.



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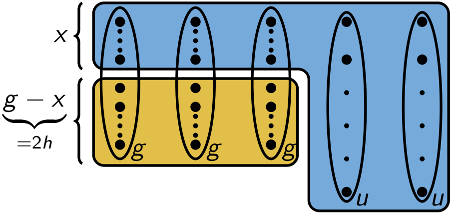

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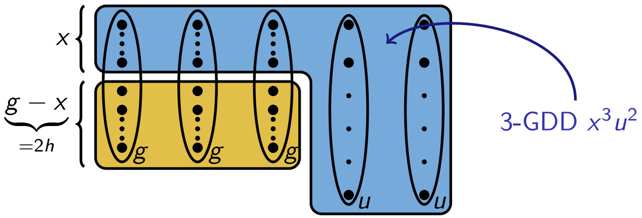

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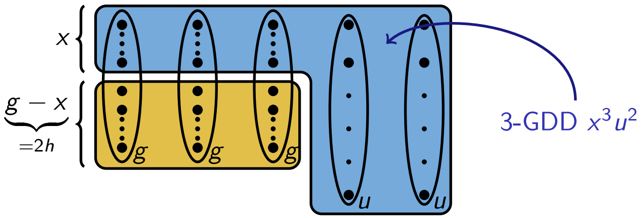

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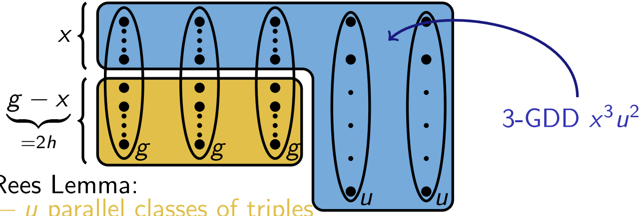

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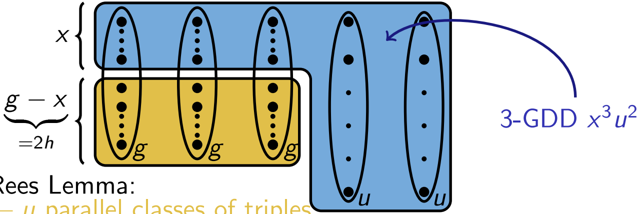

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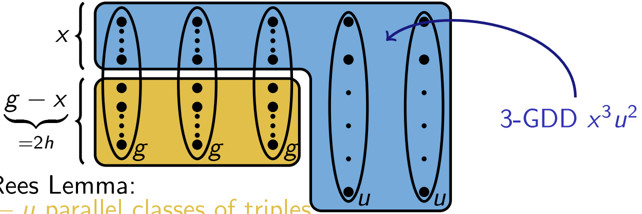

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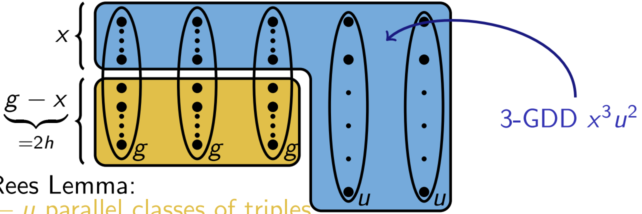

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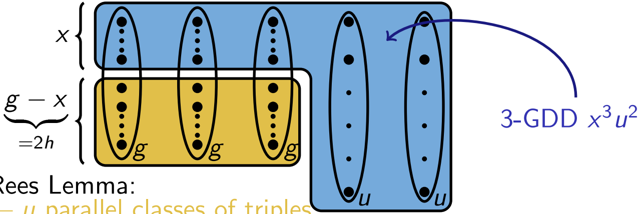

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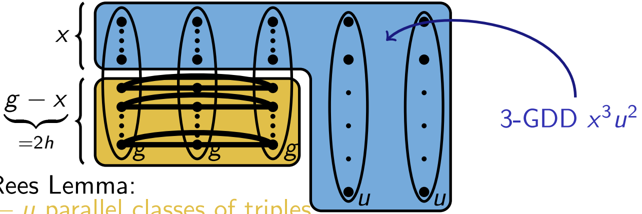

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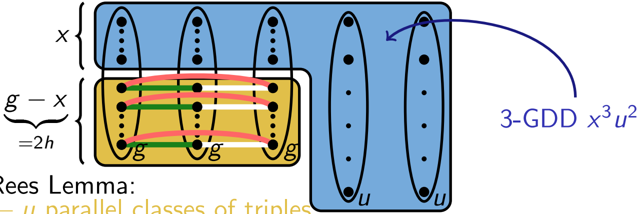

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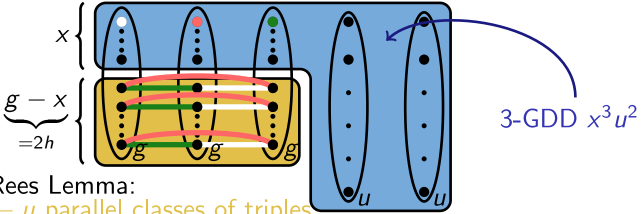

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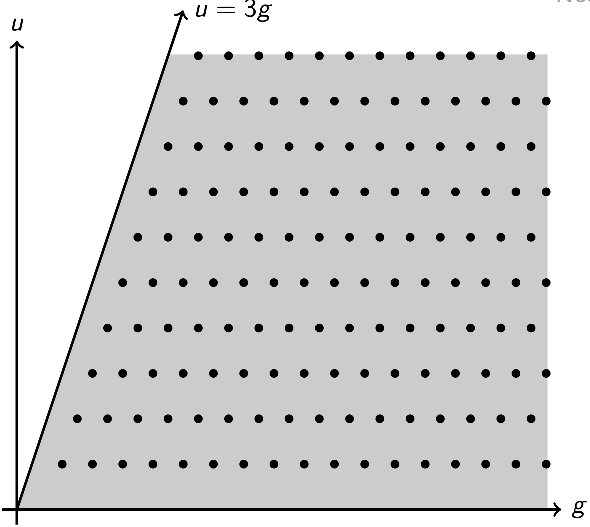
$\square$

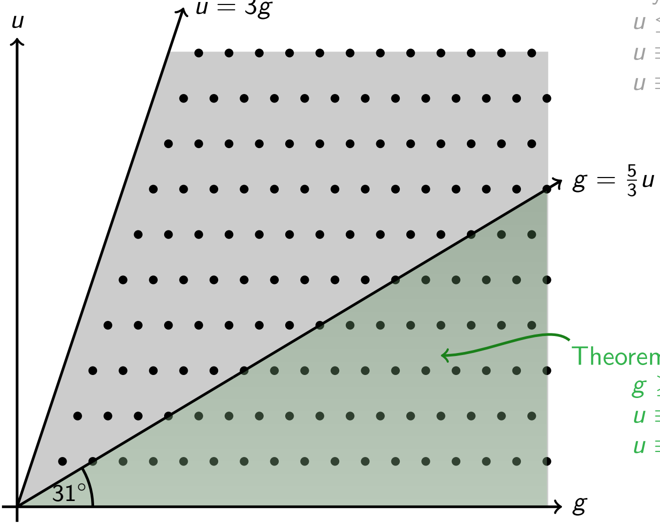
Whats left.



Necessary Conditions.

$u \leq 3g$
$u \equiv g \pmod 2$
$u \equiv 0 \pmod 3$.

Whats left.



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Theorem 7.

$$g \geq \frac{5}{3}u$$
$$u \equiv g \pmod 2$$
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Whats left.

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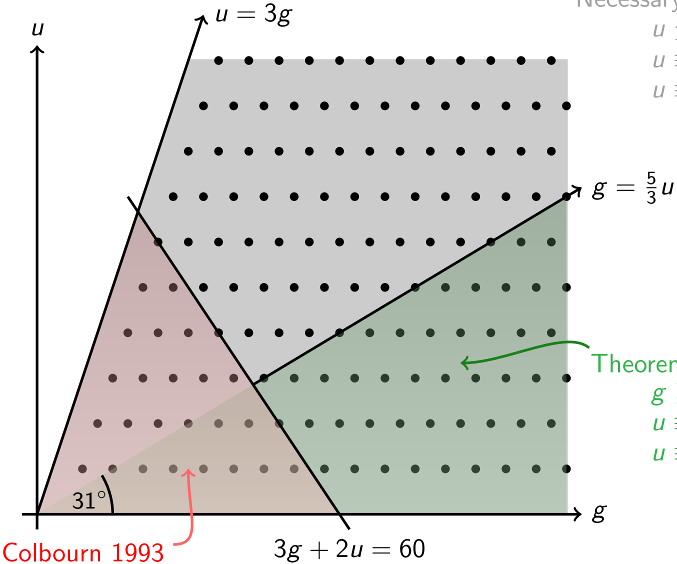

Theorem 7.

$g \geq \frac{5}{3}u$

$u \equiv g \pmod 2$

$u \equiv 0 \pmod 3$.

$3g + 2v \leq 60 \quad u \equiv g \pmod 2 \quad u \equiv 0 \pmod 3$.

Whats left.

Necessary Conditions.

$u \leq 3g$
$u \equiv g \pmod 2$
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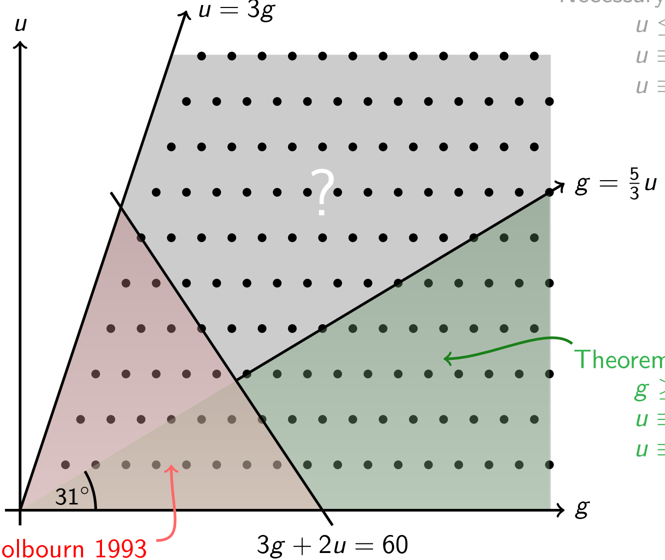

Theorem 7.

$g \geq \frac{5}{3}u$
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Colbourn 1993

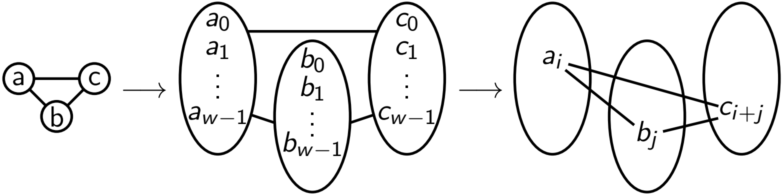
$3g + 2v \leq 60 \quad u \equiv g \pmod 2 \quad u \equiv 0 \pmod 3$.

Giving weight w

If a $\{3\}$-GDD of type $g_1 g_2 g_3 g_4 g_5$ exists

- replace each point by w points
- build on each triple a $\{3\}$-GDD of type w^3 (for example a Latin Square)

to make a $\{3\}$-GDD of type $(wg_1)(wg_2)(wg_3)(wg_4)(wg_5)$.

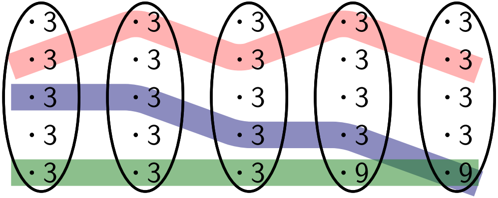


Theorem 7. If w divides $\gcd(g, u)$, and a 3-GDD of type $(g/w)^3(u/w)^2$ exists, then a 3-GDD of type g^3u^2 also exists.

3 Mutually Orthogonal Latin Squares

- 3 MOLS of order k is equivalent to a 5-GDD of type k^5.
- They are known to exist when $k \neq 2, 3, 6, 10$.
- We give different weights to the points in different groups.

Example:



We require:

3-GDD of type 3^5

3-GDD of type $3^4 9^1$

3-GDD of type $3^3 9^2$

All < 60 points and satisfy N.C.

- Therefore a 3-GDD of type $(15)^3(21)^2$ exists.

Final result

(a) various weights on the points of 5-GDD of type k^5s

(b) the 31° Theorem, and

(c) Colbourn's: 3-GDDs on less than 60 points

Theorem 10. A 3-GDD of type g^3u^2 exist if and only if $u \equiv 0 \pmod 3$, $u \equiv g \pmod 3$ except possibly when

$$g^3u^2 \in \left\{ \begin{array}{cccccc} 9^3 21^2, & 10^3 24^2, & 11^3 15^2, & 11^3 21^2, & 11^3 27^2, & 13^3 15^2, \\ 13^3 21^2, & 13^3 27^2, & 13^3 33^2, & 18^3 12^2, & 18^3 24^2, & 18^3 42^2, \\ 18^3 48^2, & 20^3 42^2, & 20^3 48^2, & 20^3 54^2, & 22^3 24^2, & 22^3 30^2, \\ 22^3 36^2, & 22^3 42^2, & 22^3 48^2, & 22^3 54^2, & 22^3 60^2, & 30^3 24^2, \\ 30^3 48^2, & 30^3 72^2, & 30^3 84^2, & 32^3 30^2, & 32^3 42^2, & 32^3 54^2, \\ 32^3 66^2, & 32^3 78^2, & 32^3 90^2, & 34^3 24^2, & 34^3 36^2, & 34^3 48^2, \\ 34^3 60^2, & 34^3 72^2, & 34^3 84^2, & 34^3 96^2 & & \end{array} \right\}$$

Final result

(a) various weights on the points of 5-GDD of type k^5s

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Additional weighing constructions on incomplete group divisible designs found by a Hill climbing algorithm we were able to resolve all of these exceptions.

Final result

(a) various weights on the points of 5-GDD of type k^5s

(b) the 31° Theorem, and

(c) Colbourn's: 3-GDDs on less than 60 points

(d) Hill climbing

(e) Miscelaneous constrcutions of 3-IGDD*s* and 3-GDD*s*.

Theorem 10. A 3-GDD of type g^3u^2 exist if and only if $u \equiv 0 \pmod 3$, $u \equiv g \pmod 3$

Final result

(a) various weights on the points of 5-GDD of type k^5s

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(e) Miscelaneous constrcutions of 3-IGDD*s* and 3-GDD*s*.

Theorem 10. A 3-GDD of type g^3u^2 exist if and only if $u \equiv 0 \pmod 3$, $u \equiv g \pmod 3$.

A general result

An integer partition $g_1 + g_2 + g_3 + g_4 + g_5 = v$ has ratio $(1:3)$ if

$$\frac{\min_i g_i}{\max_i g_i} \geq \frac{1}{3}$$

A 3-GDD of type $g_1 g_2 g_3, g_4, g_5$ has ratio $(1:3)$ if its type is a ratio $(1:3)$ partition.

Lemma 11. If there exits a 5-RDD of type k^5 and a ratio $(1:3)$ 3-GDD of type $\{r_1, r_2, r_3, r_4, r_5\}$, Then for nonnegative integers $A_i < k$, there exists a ratio $(1:3)$ 3-GDD of type $\{g_1, g_2, g_3, g_4, g_5\}$, where $g_i = 9k - 6A_i + r_i$, $i = 1, 2, 3, 4, 5$.

It follows except for a possible small number of exceptions that ratio $(1:3)$ 3-GDDs will exist whenever they satisfy the necessary conditions.