Some New Large Sets of Geometric Designs

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Dedicated to Earl Kramer and Dale Mesner

REV. T. P. KIRKMAN's PROBLEM

Resolutions of $t-$designs were studied as early as 1850 by the Rev. T. P. KIRKMAN who proposed the famous **15 SCHOOLGIRLS** problem. Kirkman's problem is equivalent to finding a **resolvable** $2-(15,3,1)$ design with $r=7$, and $b=35$.

A Solution to Kirkman's Problem

$$\alpha = (1\ 2\ 3\ 4\ 5\ 6\ 7)(8\ 9\ \mathrm{a}\ \mathrm{b}\ \mathrm{c}\ \mathrm{d}\ \mathrm{e})(\mathrm{f})$$

1 8 f	2 9 f	3 a f	4 b f	5 c f	6 d f	7 e f
2 3 5	3 4 6	4 5 7	5 6 1	6 7 2	7 1 3	1 2 4
4 a d	5 b e	6 c 8	7 d 9	1 e a	2 8 b	3 9 c
6 9 e	7 a 8	1 b 9	2 c a	3 d b	4 e c	5 8 d
7 b c	1 c d	2 d e	3 e 8	4 8 9	5 9 a	6 a b

- A *simple*, ordinary t-(v, k, λ) *design*, $(X, \mathcal{B})$, is a v-element set X of *points* and a collection $\mathcal{B}$ of k-element subsets of X called *blocks*, such that every t-element subset of X is contained in precisely λ blocks.

We denote by $\binom{X}{k}$ the collection of all k-subsets of X.

- By a *large set* $\mathrm{LS}[N](t, k, v)$ we mean a collection $\mathcal{L} = \{(X, \mathcal{B}_i)\}_{i=1}^{N}$ of t-(v, k, λ) designs where $\mathbb{B} = \{\mathcal{B}_i\}_{i=1}^{N}$ is a partition of $\binom{X}{k}$.

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Kramer's 7 × 7 × 7 Steiner cube.

			147			
	568					
	169				138	
489						
246		125	278		239	
			367			
		345	579			

	289		139	679		459
237	157					
	346					478
		168			124	
358				256		

					356	
			348			
				145	268	379
259		578				
					167	189
	123					
				247	469	

	378				257	
128	359	467				
			249			
179						
134					458	
			689			
156		236				

			126	468		
	245	149				
				589		567
369				238	347	135
		279		178		

						234
				136		269
						127
			456			
	479			357		
					159	
678	148	389				258

			158			
					789	
			235			
	267					
		569				368
457		248		349		146
		137		129		

Let V be an n-dimensional vector space over $GF(q)$. For brevity, by an r-space we mean an r-dimensional subspace of V.

We denote by $\begin{bmatrix} V \\ k \end{bmatrix}$ the collection of all k-subspaces of V.

- A *simple, geometric* t-$[q^n, k, \lambda]$ *design*, $[V, \mathcal{B}]$, is a collection $\mathcal{B}$ of k-spaces (called *blocks*) of an n-space V over $GF(q)$, such that every t-space is contained in precisely λ blocks.
- By a *large set* $\mathrm{LS}[N][t, k, q^n]$ we mean a collection $\mathcal{L} = \{\, [V, \mathcal{B}_i] \,\}_{i=1}^{N}$ of t-$[q^n, k, \lambda]$ geometric designs where $\mathbb{B} = \{\mathcal{B}_i\}_{i=1}^{N}$ is a partition of $\begin{bmatrix} X \\ k \end{bmatrix}$.

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- By a *large set* $\mathrm{LS}[N][t, k, q^n]$ we mean a collection $\mathcal{L} = \{\, [V, \mathcal{B}_i] \,\}_{i=1}^{N}$ of t-$[q^n, k, \lambda]$ geometric designs where $\mathbb{B} = \{\mathcal{B}_i\}_{i=1}^{N}$ is a partition of $\begin{bmatrix} X \\ k \end{bmatrix}$.

- A $1\text{-}[q^n, k, 1]$ design is known as a *$(k\text{-}1)$-spread* in $PG(n\text{-}1, q)$. A large set of $1\text{-}[q^n, k, 1]$ designs is called a *$(k\text{-}1)$-parallelism* of $PG(n\text{-}1, q)$.

  By a parallelism we mean a 1-parallelism, (i.e. $k = 2$).

- Geometric t-designs with $t \geq 2$ have been constructed by S. Thomas, and others, but very few for $t = 2$, only one for $t = 3$, and none for $t > 3$. Moreover, until very recently, the only large sets of geometric t-designs had been for $t = 1$.

- Parallelisms in $PG(n - 1, q)$ have been constructed in a number of papers : Beutelspacher ($n = 2^i$); Baker & Wettl (n even) ; and Penttila & Williams ($n = 4$, $q \equiv 2 \mod 3$).

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- There is vast and rich literature on ordinary t-(v, k, λ) designs that was developed over the past 1.5 centuries. More recently, the theory evolved to include strong recursive constructions, large sets , classification , and a strong connection with other areas of mathematics such as group theory , coding theory , statistics , cryptology , and application areas.

- In 1986 **Luc Teirlinck** proved a remarkable theorem: Non-trivial t-designs without repeated blocks exist for all t. The high road to Luc's construction was by means of the construction of Large Sets of t-designs.

- In a most significant 2014 paper **Peter Keevash** settled the question of existence of Steiner systems $S(t, k, v)$ whenever the necessary conditions are satisfied, under some additional probabilistic conditions.

- In contrast, the analogous questions are far from being answered for geometric $t-$designs. Thus, work on geometric t-designs is still in its infancy.

- However, a number of recent papers, have used geometric t-designs in *network coding and error correction* and are generating new interest in the area. (eg. a paper by R. Kötter and F.R. Kschischang's)

- In a recent preprint A. Fazeli, S. Lovett, and A. Vardy claim to have proved the analog of L. Tierlinck's theorem, to geometric t-designs. If the paper is valid, it will be another remarkable result in geometric t-designs. However, this result would still be purely existential and would not provide an efficient algorithm which could produce t-$[q^n, k, \lambda]$ designs for $t > 3$. The authors provide the reader with the following challenge:

Problem:

Design an efficient algorithm to produce simple, non-trivial $t - [q^n, k, \lambda]$ designs for large t, say $t \geq 4$.

- Let q be a prime power, and for natural numbers j, r, k, n define

$$[j]_q = 1 + q + q^2 + \ldots + q^{j-1} \tag{1}$$

$$[r]_q! = [1]_q! \cdot [2]_q! \cdots [r]_q! \tag{2}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! \cdot [n-k]_q!} \tag{3}$$

It is well known that:

- If V is an n-dimensional vector space over $GF(q)$, and $0 \leq k \leq n$, then

$$\left|\begin{bmatrix} V \\ k \end{bmatrix}\right| = \begin{bmatrix} n \\ k \end{bmatrix}_q$$

It is also easy to show that:

If $[V, \mathcal{B}]$ is a t-$[q^n, k, \lambda]$ design, then it is also an s-$[q^n, k, \lambda_s]$ design for all $s \in \{0, 1, \ldots, t\}$, with

$$\lambda_s = \lambda \begin{bmatrix} n-s \\ t-s \end{bmatrix}_q \Big/ \begin{bmatrix} k-s \\ t-s \end{bmatrix}_q . \tag{4}$$

Thus a set of necessary conditions for a t-$[q^n, k, \lambda]$ to exist is that all the λ_s above are integers for $s \in \{0, 1, \ldots, t\}$.

A necessary condition for the existence of an $LS[N][t, k, q^n]$ to exist is that $b = |\mathcal{B}| = \lambda_0$ divides $|\left[\begin{smallmatrix} V \\ k \end{smallmatrix}\right]| = \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q$, in which case

$$N = \begin{bmatrix} n \\ k \end{bmatrix}_q \Big/ b = \begin{bmatrix} n-t \\ k-t \end{bmatrix}_q \Big/ \lambda. \tag{5}$$

- A group $G \leq GL(V)$ is said to be an *automorphism group* of a t-$[q^n, k, \lambda]$ design $[V, \mathcal{B}]$, if for all $g \in G$, $\mathcal{B}^g = \mathcal{B}$, that is, if $B^g \in \mathcal{B}$ for all $B \in \mathcal{B}$ and $g \in G$. In this case we also say that $[V, \mathcal{B}]$ or $\mathcal{B}$ is G-invariant.

  Thus, if a geometric design $[V, \mathcal{B}]$ is G*-invariant*, $\mathcal{B}$ is the union of G-orbits of $\begin{bmatrix} V \\ k \end{bmatrix}$.

  A group $G \leq GL(V)$ is said to be an *automorphism group* of a large set $\mathcal{L} = [V, \mathbb{B}]$ if $\mathbb{B}^g = \mathbb{B}$ for all $g \in G$, that is, if $\mathcal{B}_i^g \in \mathbb{B}$ for all $\mathcal{B}_i \in \mathbb{B}$ and $g \in G$. Equivalently, we say that $\mathcal{L}$ is G-invariant.

  If the stronger condition $\mathcal{B}_i^g = \mathcal{B}_i$ for all $\mathcal{B}_i \in \mathbb{B}$ and $g \in G$ holds, we say that $\mathcal{L}$ is $[G]$-invariant.

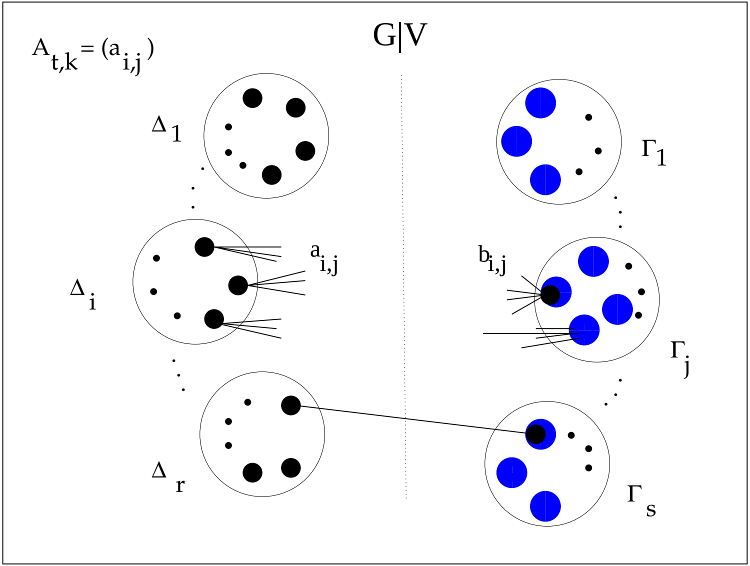
- The **KM** matrices present precise conditions under which G-orbits of $\begin{bmatrix} V \\ k \end{bmatrix}$ can be selected to form a t-$[q^n, k, \lambda]$ design, as well as conditions for the existence of $\mathrm{LS}[t, k, q^n]$.

- Let $G \leq GL(V)$, and $0 < t < k \leq v/2$. Let $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_r$ and $\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_s$ be the orbits of G on $\begin{bmatrix} V \\ t \end{bmatrix}$ and $\begin{bmatrix} V \\ k \end{bmatrix}$ respectively. The *Kramer-Mesner matrix* is defined to be the $r \times s$ matrix $\mathbf{A_{t,k}} = (a_{i,j})$ with:

$$a_{i,j} = |\{K \in \mathcal{K}_j \mid T \subset K\}|, \tag{6}$$

for any fixed $T \in \mathcal{T}_i$

Kramer–Mesner matrices



Now, it can be easily shown that the Kramer-Mesner theorem (1974) is readily addopted to geometric t-designs as follows:

- **Theorem** There exists a G-invariant t-$[q^n, k, \lambda]$ design $[V, \mathcal{B}]$ if and only if there exists a vector $\mathbf{u} \in \{0,1\}^s$ satisfying the equation:

$$\mathbf{A}_{\mathbf{t},\mathbf{k}} \cdot \mathbf{u}^{\mathbf{T}} = \lambda \, \mathbf{J} \tag{7}$$

where $\mathbf{J}$ is the $r \times 1$ column vector of all ones.

The following theorem is an easy generalization of the 1974 K-M theorem, and can easily be adopted to geometric designs.

- **Theorem** [Cusack & S.M., 1999]
  A $[G]$-invariant LS$[N]$-(t, k, v) exists if and only if there exists a matrix $\mathbf{w} \in \{0,1\}^{s\times N}$ with constant row sum 1 satisfying the matrix equation :

$$\mathbf{A_{t,k}} \cdot \mathbf{w} = \lambda \begin{pmatrix} 1 & 1 & \dots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix}_{r\times N} \tag{8}$$

- **Corollary**

  Let $\mathbf{U}$ denote the $m \times s$ matrix whose rows are all vectors $\mathbf{u}$ satisfying $\mathbf{A_{t,k}} \cdot \mathbf{u^T} = \lambda \, \mathbf{J_{r\times 1}}$. Then, there is a $[G]$-invariant **large set** of t-(v,k,λ) designs if and only if there exists a vector $\mathbf{L} \in \{0,1\}^m$ such that

$$\mathbf{L} \cdot \mathbf{U} = \mathbf{J}. \tag{9}$$

- We illustrate the above by a tiny example in the context of ordinary 2-(9,3,1) designs.

- Let $G = \langle\,(1,2,3)(4,5,6)(7,8,9)\,\rangle$, t-$(v,k,\lambda) = $ 2-(9,3,1) Then, the matrix $\mathbf{A_{2,3}}$ is :

Example (i)

- $\mathbf{A_{2,3}} =$

```
111111100000000000000000000000
011000011111000000000000000000
001100010000111100000000000000
010100001000100011100000000000
000011000100010010011000000000
000001100010001001010100000000
000010100001000100101100000000
000000011000100000000011110000
000000000100001000100001101100
000000000001000011000000111010
000000000000101000100000101011 0
000000000000000000001110000111 1
```

____

```
100000000010000010100001000000 1
```

$\leftarrow$ $\mathbf{u_1}$ a solution

- There are exactly 21 solutions $\mathbf{u_i}$ such that $\mathbf{A_{t,k}} \cdot \mathbf{u_i} = \mathbf{J}$:

$\mathbf{L}_1$	$\mathbf{U}$ − **designs**	↓
1	1 0 0 0 0 0 0 0 0 1 0 0 0 0 0 1 0 1 0 0 0 0 1 0 0 0 0 0 0 1	← $\mathbf{u}_1$
0	1 0 0 0 0 0 0 0 0 0 1 0 0 1 0 0 0 0 1 0 0 0 1 0 0 0 0 0 0 1	·
0	1 0 0 0 0 0 0 0 0 0 0 1 0 0 1 0 1 0 0 0 0 0 1 0 0 0 0 0 0 1	·
1	0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 0 1 0 0 0 0 0	← $\mathbf{u}_4$
0	0 1 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 1 0 0 0 0	·
0	0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 0 0 0	·
0	0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 1 0 1 0 0 0 0 0 0	·
0	0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 1 0 0 0 1 0 0 0 0 0	·
1	0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 0 0 0 0 1 0 0 0 0	← $\mathbf{u}_9$
0	0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 1 0 0 0 0	·
1	0 0 0 1 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 1 0 0 1 0 0 0 0 0 0	← $\mathbf{u}_{11}$
0	0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 0 1 0 0 0 0 0	·
0	0 0 0 0 1 0 0 1 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 1 0 0 0	·
1	0 0 0 0 1 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0	← $\mathbf{u}_{14}$
0	0 0 0 0 1 0 0 0 0 0 1 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0	·
0	0 0 0 0 0 1 0 1 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 1 0	·
0	0 0 0 0 0 1 0 0 1 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0	·
1	0 0 0 0 0 1 0 0 0 0 0 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0	← $\mathbf{u}_{18}$
1	0 0 0 0 0 0 1 1 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 1 0 0	← $\mathbf{u}_{19}$
0	0 0 0 0 0 0 1 0 1 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0	·
0	0 0 0 0 0 0 1 0 0 1 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0	·

The Six Large sets LS[7](2,3,9) invariant under $\mathbb{Z}_3$

$\mathbf{L_1}$	**1**	0	0	**1**	0	0	0	0	**1**	0	**1**	0	0	**1**	0	0	0	**1**	**1**	0	0
$\mathbf{L_2}$	1	0	0	0	1	0	1	0	0	0	0	1	0	0	1	1	0	0	0	1	0
$\mathbf{L_3}$	0	1	0	0	1	0	1	0	0	0	0	1	1	0	0	0	1	0	0	0	1
$\mathbf{L_4}$	0	1	0	0	0	1	0	1	0	1	0	0	0	1	0	0	0	1	1	0	0
$\mathbf{L_5}$	0	0	1	1	0	0	0	0	1	0	1	0	1	0	0	0	1	0	0	0	1
$\mathbf{L_6}$	0	0	1	0	0	1	0	1	0	1	0	0	0	0	1	1	0	0	0	1	0

and ... Two Large sets of Large sets ...

1	0	1	0	0	1
0	1	0	1	1	0

Partitioning $A_{t,k}$

- **Theorem**

Let $\mathbf{A} = \mathbf{A_{t,k}}$ be the KM matrix for a group action $G|X$, with $1 \leq t \leq k$. Then, there exists a $[G]-$invariant large set of $t-(v,k,\lambda)$ (or t-$[q^n,k,\lambda]$) designs if and only if there exists a **partition** $\{P_1, \ldots, P_\ell\}$ of the columns of $\mathbf{A}$ such that the sum of column vectors within each part P_i is equal to $\lambda\mathbf{J}$.

$\mathbf{A_{t,k}} =$

P_1	P_2	$\cdots$	P_ℓ

About a year ago, M. Braun, A. Kohnert, Patric Östergård and Alfred Wasserman proved the following:

Theorem

Nontrivial large sets of $t-$designs over finite fields exist for $t \geq 2$.

They accomplished this by showing the existence of a large set of three disjoint $2\text{-}[2^8, 3, 21]$ designs, invariant under a cyclic group of order 255.

Subsequently, Michael Hurley and S.M. confirmed the validity of the above result, and with Bal Khadka constructed 9 new and distinct large sets, different from the original, with the same parameters. We still don't know if our designs are non-isomorphic, and we don't know much about their automorphism groups.

- In what follows we construct 9 distinct large sets $LS[3][2,3,2^8]$ of geometric $2\text{-}[2^8,3,21]$ designs. Each of the $\mathcal{L}_1,\ldots,\mathcal{L}_9$ will be $[G]$-invariant under a fixed Singer subgp $G=\langle x\rangle \leq GL_8(2)$, $|G|=255$.

A **Singer** subgroup G in $\Gamma = GL(8,2)$ is the **centralizer** of a Sylow-17 subgroup, thus, all Singer subgroups are conjugate in Γ, and the **normalizer** of G in Γ is the extension of G by the **Frobenius** automorphism of order 8.

We represent the vectors in $V=\mathbb{F}_2^8$ as the integers in $\mathbb{Z}_{256}$ in their standard base 2 representation. Two-dimensional subspaces of V are elementary abelian 4-groups, and three-dimensional subspaces are elementary abelian groups of order 8. Projectively, 2-spaces are collinear triples, and 3-spaces are Fano planes. More economically, we represent 2-spaces and 3-spaces by their alphabetically minimal bases.

- We begin with the following Singer cycle S in $GL_8(2)$:

$$S = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

  and compute the orbits of $G = \langle S \rangle$ on $\begin{bmatrix} V \\ 2 \end{bmatrix}$ and $\begin{bmatrix} V \\ 3 \end{bmatrix}$.

- There are in all **43** G-orbits on $\begin{bmatrix} V \\ 2 \end{bmatrix}$, with a single short orbit of length 85, all remaining 42 orbits of length 255. There are in all **381** G-orbits on $\begin{bmatrix} V \\ 3 \end{bmatrix}$ all of length 255.

- Thus, the KM matrix $\mathbf{A_{2,3}}$ has shape $\mathbf{43} \times \mathbf{381}$. The entries in $A_{2,3}$ come from the set $\{0, 1, 3\}$. Each of the rows corresponding to the 42 long orbits has a single 3 in it. The short orbit has 21 3's in it. The row sums of $A_{2,3}$ are $\begin{bmatrix} v-t \\ k-t \end{bmatrix} = 63$. Moreover, 360 of the column sums are 7 and 21 of the column sums are 9.

Nine distinct LS[3][2,3,2^8]

- Each of the large sets we seek is comprised of 3 block-disjoint $2-[2^8, 3, \lambda]$ designs with $\lambda = 21$. For each case we proceed as follows:

  (i) We find a single geometric design $2-[2^8, 3, 21]$ design D_1 from $\mathbf{A_{2,3}}$,

  (ii) We delete the 127 orbits used in D_1 from $\mathbf{A_{2,3}}$ to arrive at a 43×254 matrix $\mathbf{A_{2,3}}\!\downarrow$.

  (iii) We attempt to find a second design D_2 among the remaing 254 orbits, using the residual matrix $\mathbf{A_{2,3}}\!\downarrow$.

  (iv) If we find D_1 and D_2, then the remaining 127 orbits will constitute a third 2-$[2^8, 3, 21]$ design D_3, and $\{D_1, D_2, D_3\}$ will be a large set of $[G]$-invariant 2-$[2^8, 3, 21]$ designs.

- We find D_1 and D_2 by using Lattice basis reduction, as for example, discussed in D.L. Kreher and D.R. Stinson's book "*Combinatorial Algorithms*", pages 294-300.
- In each case, the matrix to be reduced is of the form:

$$\left[\begin{array}{c|c} \mathbf{I} & \mathbf{0} \\ & \\ \hline \mathbf{A} & \mathbf{B} \end{array}\right]$$

Where:

(i) $\mathbf{I}$ is the $k \times k$ identity matrix, $k \in \{381, 254\}$,
(ii) $\mathbf{0}$ is the $k \times 1$ zero vector,
(iii) $\mathbf{A}$ is the matrix $\mathbf{A_{2,3}}$ (k=381) when we are looking for design D_1; and $\mathbf{A_{2,3}} \downarrow$ (k=254) when we are looking for D_2.
(iv) In both cases, $\mathbf{B}$ is the 43×1 vector with all entries -21.

Part of encoded $A_{2,3}$

1	2 4	3k0000000040004	65	4 26	k00l0040001000	129	6 80	400g000001hg0g
2	2 8	1lg000100000100	66	4 32	h000h040g00003	130	6 82	404400401004g0
3	2 12	1k5000000011000	67	4 34	gg004g10000140	131	6 88	4000000400s004
4	2 16	111k0000g000040	68	4 40	hg0004g00gg000	132	6 96	40g40100001500
5	2 20	10kh0h000000000	69	4 42	h040g500400000	133	6 98	40g00g0000kk00
6	2 24	1140M0400000000	70	4 48	gc00000h010000	134	6 106	405000g0000k10
7	2 28	10h050000000013	71	4 50	g10ggg50000000	135	6 112	40g014g00g0g00
8	2 32	100g1k00000040g	72	4 56	g540000h000010	136	6 114	41g040g0001100
9	2 36	100541g000000g0	73	4 58	g1001004g01010	137	6 120	40544010040000
10	2 40	1101g05000000g0	74	4 64	ggh00001gg0000	138	6 122	5000g000150g00
11	2 44	110gg04g4000000	75	4 66	g100001g410400	139	6 130	40010014g0g040
12	2 48	104gg004000g400	76	4 72	g04100g00k4000	140	6 136	4001004140004g
13	2 52	1045g0010004000	77	4 74	ggg000001400k0	141	6 138	4k00k004000100
14	2 56	100g400gk00000g	78	4 80	g0h10040410000	142	6 144	44000g0g0h0010
15	2 60	100110001g10040	79	4 82	h04000000gM000	143	6 146	4000g0140g00h0
16	2 64	10144g001g00000	80	4 90	g040441000g010	144	6 154	40400004h400g0
17	2 68	100004001l00100	81	4 96	g00g0000g11g0g	145	6 160	44000104004440
18	2 72	10001h00gh00000	82	4 98	g0g101000044g0	146	6 162	4000000140k04g
19	2 76	1g0000g040k1000	83	4 104	k00l00g0400000	147	6 168	4g00044g001400
20	2 80	104400k01010000	84	4 106	g0500000504400	148	6 170	50100050g000g0
21	2 84	1000001c000g010	85	4 112	g1040045000g00	149	6 176	400000g140005g
		. . .			. . .			. . .
		. . .			. . .			. . .
		. . .			. . .			. . .
249	12 132	4011104000140	312	18 100	4g1000011g0g	375	38 68	k00M04g00
250	12 144	4040k00k01000	313	18 132	40g0044k1000	376	40 68	4g0g54g00
251	12 148	4144g10040000	314	18 160	41k0000g4400	377	42 76	3000M0003
252	12 162	40g0500g04g00	315	18 164	414000000g5g	378	42 78	111140g04
253	12 166	40000100014k3	316	18 196	401g400101g0	379	44 64	M10010g4
254	12 176	400g504400040	317	18 200	4000000415g3	380	44 78	kg0c4000
255	12 182	4050000440110	318	18 206	403g000000gg	381	58 128	c0k10g0

Part 1 : encoded $\{\mathcal{L}_1, \ldots, \mathcal{L}_9\}$

1	222332233	65	311113121	129	122221313	193	233223323	257	111111111	321	123111112
2	111111111	66	211232221	130	333112123	194	233333323	258	132321213	322	311333321
3	211232221	67	233333333	131	223333232	195	122111113	259	111221311	323	211222321
4	133111112	68	111111111	132	211323231	196	223332232	260	132111112	324	223112133
5	132111112	69	232112123	133	132111112	197	133321213	261	233112122	325	123221213
6	311323331	70	322332323	134	323113123	198	311222331	262	232112133	326	233112133
7	333233333	71	323113133	135	323223233	199	311112131	263	111321311	327	233232223
8	332112122	72	322323333	136	123111113	200	322112133	264	211232231	328	211333321
9	133111112	73	111221311	137	311222231	201	322322323	265	311112131	329	232332322
10	211222331	74	122111113	138	133111112	202	211223321	266	211323221	330	223333232
11	311333221	75	311223321	139	211322221	203	332233323	267	223113132	331	111331311
12	133321313	76	233113132	140	111111111	204	123111112	268	223112123	332	211333221
13	311223331	77	322233223	141	332322232	205	322323333	269	211113131	333	233223232
14	122331213	78	223333332	142	232113132	206	133111112	270	322323232	334	122221212
15	211223221	79	232322322	143	332113123	207	333113123	271	322233233	335	111111111
16	333333223	80	111331211	144	111331211	208	211322331	272	111331211	336	323333323
17	222222332	81	111331311	145	323232223	209	223322332	273	123221213	337	111111111
18	232322323	82	211112131	146	323112123	210	332322233	274	323332222	338	232233233
19	111111111	83	222112122	147	111231311	211	211113131	275	122331312	339	211113121
20	223332232	84	222333232	148	123231313	212	123221213	276	222222232	340	223333322
21	233223332	85	211232321	149	133321212	213	111111111	277	232232323	341	133111113
22	222332323	86	332113133	150	132231212	214	233332233	278	223333332	342	133221212
23	322232223	87	111111111	151	323233223	215	311112121	279	332113132	343	233113132
24	223233322	88	133321313	152	322322323	216	332232222	280	233323322	344	311232221
25	322333322	89	111111111	153	211223331	217	132231313	281	332333223	345	233333232
26	211333331	90	332323232	154	223113132	218	332333223	282	111231311	346	223333332
27	132231212	91	332333332	155	223222222	219	132111112	283	332113122	347	332233222
28	323113132	92	233332222	156	333113123	220	223323232	284	223112123	348	111221311
29	111111111	93	123111113	157	111221311	221	211232331	285	211222221	349	222232333
30	311323321	94	122331312	158	322333233	222	323332322	286	333112123	350	323323323
31	222112133	95	222222232	159	223332322	223	222222222	287	111321211	351	323232232
32	132111113	96	132221213	160	133231212	224	332113123	288	223332233	352	111111111

part 2 : encoded $\{\mathcal{L}_1, \ldots, \mathcal{L}_9\}$

33	311113121	97	332222323	161	222332223	225	311112121	289	132111112	353	132331212
34	233113133	98	322232233	162	311333331	226	111111111	290	111221211	354	311112131
35	322223223	99	133111112	163	211232331	227	211332321	291	132221213	355	222233223
36	133221313	100	322112133	164	332232233	228	111111111	292	223113133	356	322223333
37	133111112	101	233112122	165	133111112	229	322112132	293	311112121	357	122111113
38	232113132	102	211113131	166	122111112	230	332333333	294	211113121	358	222112132
39	222233222	103	133221312	167	333333332	231	122221212	295	132111112	359	311332231
40	333112132	104	211322221	168	332112132	232	111321211	296	111111111	360	311323321
41	111221211	105	323223233	169	223113122	233	133231212	297	322322222	361	322222223
42	111231211	106	111321211	170	232323223	234	311112121	298	322323333	362	223112122
43	322233232	107	223112132	171	123111112	235	123331212	299	333232233	363	332223333
44	111111111	108	111221211	172	323223233	236	132231312	300	332332333	364	111111111
45	233322233	109	311323331	173	232233333	237	111321211	301	122221313	365	311112131
46	133321312	110	133331312	174	111231211	238	333113122	302	211332221	366	311113131
47	111111111	111	133231212	175	123111113	239	323112122	303	222222232	367	122321213
48	111221311	112	333233233	176	111221311	240	222223332	304	311332321	368	122111112
49	211333231	113	223113133	177	233332322	241	333233333	305	211232321	369	311233321
50	311112121	114	323322223	178	222222333	242	132231313	306	333323322	370	311222221
51	123111113	115	111111111	179	211322221	243	123221213	307	311223321	371	111321311
52	323322222	116	232233332	180	133111112	244	322113123	308	133111113	372	333323232
53	323332322	117	332323332	181	211233331	245	323113122	309	111111111	373	222223322
54	333112133	118	211232221	182	211322221	246	233112122	310	223223332	374	311332331
55	222222322	119	311112121	183	322223222	247	232332232	311	322323232	375	332333332
56	232322233	120	211233321	184	211323231	248	123111113	312	223112123	376	233222232
57	323223223	121	132231313	185	311112121	249	122221312	313	232332332	377	322333333
58	311113121	122	222112122	186	322322223	250	211222231	314	111111111	378	211223331
59	122321312	123	232322332	187	132231213	251	111221311	315	311112121	379	232233332
60	333222323	124	133321213	188	111231211	252	332322323	316	222332223	380	322333323
61	111231311	125	111221311	189	322322223	253	211233331	317	232232323	381	333322222
62	333223322	126	222112132	190	111321211	254	223113123	318	332333333		
63	311332231	127	133321312	191	133231312	255	133231213	319	111111111		
64	133321313	128	123231213	192	232323333	256	311233231	320	111111111		

What do we know about the 9 large sets ?

(i) They are all distinct. i.e. if $i \neq j$, then $\mathcal{L}_i \neq \mathcal{L}_j$,

(ii) Each $\mathcal{L}_i$ is $[G]$-invariant, thus, $G \leq Aut(\mathcal{L}_i)$ for each i.

(iii) There are 8 conjugacy classes $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_8$, of maximal subgroups in $GL_8(2)$,

(iv) If $N = N_\Gamma(G)$, then $|N| = 8 \cdot 255 = 2040$,

(v) N is not maximal in Γ, (unlike the case when q is odd.)

(vi) If $x \in N - G$, then $\mathcal{L}_i^x \neq \mathcal{L}_j$, for any $i, j \in \{1, \ldots, 9\}$,

(vii) If $G \leq T \leq M \in \mathcal{M}_i$, then $i = 2$, $|M| = 2^{13} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17$ $= 5922201600$.

(viii) At the moment we cannot assert that $G = Aut(\mathcal{L}_i)$, but if this were true, it easily follows that the 9 $\mathcal{L}_i$ are pairwise non-isomorphic.

(i) Find some more $3-[q^n,k,\lambda]$ designs.

(ii) Find an example of a $4-[q^n,k,\lambda]$ design.

(iii) Find large sets $LS[N][t,k,q^n]$ of geometric designs for $t\geq 3$.

(iv) Discover recursive constructions for geometric $t-[q^n,k,\lambda]$ designs.

(v) Settle the questions about the 9 $\mathcal{L}_i$ in this presentation.

(vi) Discover more $t-[q^n,k,1]$ designs, i.e. geometric Steiner systems.

(vii) Establish, or otherwise the validity of the Fazeli, Lovett, Vardy result.

THANK YOU