Symmetric Functions and Quasisymmetric Functions

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Outline

Symmetric Functions

Definition

- R: commutative ring with identity
- $\mathbf{x} = (x_1, x_2, \cdots)$: set of indeterminates
- n: nonnegative integer

A homogeneous symmetric function of degree n is a formal power series $f(\mathbf{x}) = \sum_\alpha c_\alpha \mathbf{x}^\alpha$ where

- α ranges over all weak compositions $\alpha = (\alpha_1, \alpha_2, \cdots)$ of n,
- $c_\alpha \in R$,
- $\mathbf{x}^\alpha$ stands for the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \cdots$,
- $f(x_{w(1)}, x_{w(2)}, \cdots) = f(x_1, x_2, \cdots)$ for every permutation w of the positive integers.

Symmetric Functions

Definition

Let Λ_R^n be the set of all homogeneous symmetric functions of degree n.

$$\Lambda_R = \Lambda_R^0 \oplus \Lambda_R^1 \oplus \cdots$$

is a commutative, unital, graded R-algebra.

Bases for $\Lambda_{\mathbb{Q}}^n$

- Monomial symmetric functions $\{m_\lambda : \lambda \vdash n\}$
- Elementary symmetric functions $\{e_\lambda : \lambda \vdash n\}$
- Complete homogeneous symmetric functions $\{h_\lambda : \lambda \vdash n\}$
- Power sum symmetric functions $\{p_\lambda : \lambda \vdash n\}$
- Schur functions $\{s_\lambda : \lambda \vdash n\}$

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Symmetric Functions Over Integers

$\Lambda_{\mathbb{Z}} = \mathrm{Sym} = \mathbb{Z}[e_1, e_2, \cdots] \subset \mathbb{Z}[x_1, x_2, \cdots]$

- $e_1 = x_1 + x_2 + \cdots$
- $e_2 = x_1 x_2 + x_1 x_3 + x_2 x_3 + \cdots$
- $e_n = \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n}$

Sym as a Hopf algebra

- $\triangle : \mathrm{Sym} \to \mathrm{Sym} \otimes \mathrm{Sym}, \quad e_n \mapsto \sum_{i+j=n} e_i \otimes e_j$
- $\epsilon : \mathrm{Sym} \to \mathbb{Z}, \quad e_n \mapsto 0, \ n \geq 1$

Connection to Representation Theory

- (Geissinger 1977) $\mathrm{Sym} \cong \bigoplus_{n=0}^{\infty} \mathcal{K}_0(\mathbb{C}[S_n]\text{-mod})$

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Duality of Sym

Bilinear Form on Sym

- Define $\langle \cdot\ ,\ \cdot \rangle : \mathrm{Sym} \times \mathrm{Sym} \to \mathbb{Z}$ by $\langle m_\lambda, h_\mu \rangle = \delta_{\lambda,\mu}$ for $\lambda, \mu \in \mathcal{P}$.
- $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}$

Bilinear Form on $\mathrm{Sym} \otimes \mathrm{Sym}$

- Define $(\cdot\ ,\ \cdot) : \mathrm{Sym} \otimes \mathrm{Sym}\ \times\ \mathrm{Sym} \otimes \mathrm{Sym} \to \mathbb{Z}$ by $(x \otimes y, x' \otimes y') = \langle x, x' \rangle \langle y, y' \rangle$.
- $(x \otimes y, \triangle(z)) = \langle \triangledown(x \otimes y), z \rangle$

$\mathrm{Sym} \cong \mathrm{Sym}^*$

- $\mathrm{Sym}^* = \bigoplus_{n \in \mathbb{N}} (\Lambda^n_{\mathbb{Z}})^*$: graded dual of Sym
- $\Phi : \mathrm{Sym} \cong \mathrm{Sym}^*$ by $\Phi(x)(y) = \langle x, y \rangle$.

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Noncommutative Symmetric Functions

Definition

$\mathrm{NSym} = \mathbb{Z}\langle \mathbf{h}_1, \mathbf{h}_2, \cdots \rangle$: free algebra

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Quasisymmetric Functions

Definition (Gessel 1984)

$\mathrm{QSym} \subset \mathbb{Z}[[x_1, x_2, \cdots]]$ consisting of shift invariant formal power series of bounded degree, i.e., $f \in \mathrm{QSym}$ if and only if

$$\text{coeff of } x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} \text{ in } f = \text{coeff of } x_{i_1}^{n_1} x_{i_2}^{n_2} \cdots x_{i_k}^{n_k} \text{ in } f$$

for all $0 < i_1 < i_2 < \cdots < i_k$ and $n_1, n_2, \cdots, n_k \in \mathbb{N}$.

Example

- $\sum_{i<j} x_i^2 x_j$ quasisymmetric, not symmetric.
- $\sum_{i<j} x_i x_j^5$ quasisymmetric, not symmetric.

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Additive Basis for QSym

- $M_\alpha = \sum_{i_1 < \cdots < i_k} x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k}$, where $\alpha \in \mathrm{Comp}(n)$.

QSym as a Hopf algebra

- Multiplication: overlapping shuffles
- Comultiplication: cut

Duality of NSym and QSym

- Define $\langle \cdot\, ,\, \cdot \rangle : \mathrm{NSym} \times \mathrm{QSym} \to \mathbb{Z}$ by $\langle \mathbf{h}_\alpha, M_\beta \rangle = \delta_{\alpha,\beta}$.
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- $(\triangle(\mathbf{h}_\alpha), M_\beta \otimes M_\gamma) = \langle \mathbf{h}_\alpha, \triangledown(M_\beta \otimes M_\gamma) \rangle$
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Polynomial Freeness of QSym

Ditters Conjecture 1972

The algebra QSym is a free commutative algebra over the integers.

Hazewinkel 2001, 2002

- Ditters Conjecture is proved.
- An explicit free commutative polynomial basis is constructed.

QSym is free over Sym

- $E = \{e_n(\alpha) \mid \alpha \in eLYN, n \in \mathbb{N}\}$: free polynomial basis for QSym.
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The Heisenberg Double

Definition (Dual Pair)

(H^+, H^-) is a dual pair of Hopf algebras if

- $H^\pm$ are graded connected Hopf algebras,
- we have a perfect Hopf pairing $\langle \cdot\, , \,\cdot \rangle : H^- \times H^+ \to R$.

Via this pairing, identify $H^\pm$ with the grade dual of $H^\mp$.

Definition (Heisenberg Double)

The Heisenberg double of H^+ is the algebra $\mathfrak{h} = \mathfrak{h}(H^+, H^-)$ given by

- $\mathfrak{h} = H^+ \otimes H^-$ as R-modules.
  We write $a\sharp x$ for $a \otimes x$, viewed as an element of $\mathfrak{h}$.
- Multiplication is given by:
  $(a\sharp x)(b\sharp y) = \sum_{(x)} a\, {}^R x^*_{(1)}(b)\sharp x_{(2)} y = \sum_{(x),(b)} \langle x_{(1)}, b_{(2)} \rangle a b_{(1)} \sharp x_{(2)} y.$

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Fock Space Representation

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The algebra $\mathfrak{h}$ has a natural representation on H^+ given by

$$(a\sharp x)(b) = a\,{}^R x^*(b), \quad a, b \in H^+, x \in H^-.$$

Stone-von Neumann Type Theorem (Savage, Yacobi 2015)

- The representation $\mathcal{F}$ is faithful.
- If R is a field, then $\mathcal{F}$ is irreducible.
- Any representation of $\mathfrak{h}$ generated by a lowest weight vacuum vector is isomorphic to $\mathcal{F}$.

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Example

$$\begin{array}{ccc} & & \mathrm{Sym} \\ & & \cap \\ \mathrm{NSym} & \longleftrightarrow & \mathrm{QSym} \\ \downarrow & & \\ \mathrm{Sym} & & \end{array}$$

Heisenberg Algebra $\mathfrak{h} = \mathfrak{h}(\mathrm{Sym}, \mathrm{Sym})$

- $p_1, p_2, \cdots$: the power sums in $H^+ = \mathrm{Sym}$.
- $p_1^*, p_2^*, \cdots$: the power sums in $H^- = \mathrm{Sym}$.
- $p_m p_n = p_n p_m, \quad p_m^* p_n^* = p_n^* p_m^*, \quad p_m^* p_n = p_n p_m^* + m\delta_{m,n}$.

Quasi-Heisenberg Algebra $\mathfrak{q} = \mathfrak{h}(\mathrm{QSym}, \mathrm{NSym})$

- Fock space representation: natural action on QSym.
- $\mathfrak{q}_{\mathrm{proj}}$: subalgebra generated by $\mathrm{Sym} \subset \mathrm{QSym}$ and NSym.

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Categorification

Goal

To categorify Heisenberg doubles and their Fock space representations.

What is categorification?

Suppose M is a module for a ring R.
We would like to find an abelian category $\mathcal{M}$ such that

$$\mathcal{K}_0(\mathcal{M}) \xrightarrow[\cong]{\phi} M \qquad \text{(as } \mathbb{Z}\text{-modules)},$$

where $\mathcal{K}_0(\mathcal{M})$ is the Grothendieck group of $\mathcal{M}$.

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For each $r \in R$ (or, for those r in a fixed generating set), we want an exact endofunctor F_r of $\mathcal{M}$ such that we have a commutative diagram:

$$\begin{array}{ccc} \mathcal{K}_0(\mathcal{M}) & \xrightarrow{[F_r]} & \mathcal{K}_0(\mathcal{M}) \\ \phi \downarrow & & \phi \downarrow \\ M & \xrightarrow{r} & M \end{array}$$

Here $[F_r]$ denotes the map induced by F_r on $\mathcal{K}_0(\mathcal{M})$.

We would also like isomorphisms of functions lifting the relations of R. For example, suppose we have a relation in R: $rs = 2sr + 3$. Then we would like isomorphisms of functors $F_r \circ F_s \cong (F_s \circ F_r)^{\oplus 2} \oplus \mathrm{Id}^{\oplus 3}$.

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Categorification

Fruits of Categorification

- Classes of objects (simple, indecomposable projective) give distinguished bases with positivity and integrality properties.
- Uncovers hidden structure in the algebra and its representation.
- Provides tools for studying the category $\mathcal{M}$.
- Applications to topology and physics.

Example

- (Lusztig) Categorification of quantum groups yields canonical bases with positivity and integrality properties.

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Categorification of the Heisenberg Double

Goal

- Find categories whose Grothendieck groups are isomorphic to $\mathfrak{h}$ as $\mathbb{Z}$-modules,
- Find functors lifting the action of $\mathfrak{h}$ on Fock space,
- Find isomorphisms of functors lifting the defining relations of $\mathfrak{h}$.

Module Categories

- $A = \oplus_{n \in \mathbb{N}} A_n$: a tower of algebras.
- A_n-mod: category of f.g. left A_n-modules.
- A_n-pmod: category of f.g. projective left A_n-modules.
- $G_0(A_n)$: Grothendieck group of A_n-mod.
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Categorification of the Heisenberg Double

Theorem (Bergeron, Li 2009)

Let $\mathcal{G}(A) = \oplus_{n\in\mathbb{N}} G_0(A_n)$ and $\mathcal{K}(A) = \oplus_{n\in\mathbb{N}} K_0(A_n)$. Then $(\mathcal{G}(A), \mathcal{K}(A))$ is a dual pair of Hopf algebras.

Definition (Heisenberg double associated to a tower)

To a tower of algebras A, we associate the Heisenberg double $\mathfrak{h}(A) := \mathfrak{h}(\mathcal{G}(A), \mathcal{K}(A))$ and its Fock space $\mathcal{F}(A) = \mathcal{G}(A)$.

Theorem (Savage, Yacobi 2015)

The functors Ind_M and Res_P for $M \in A$-mod and $P \in A$-pmod categorify the Fock space representation $\mathcal{F}(A)$ of $\mathfrak{h}(A)$.

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Definition (Heisenberg double associated to a tower)

To a tower of algebras A, we associate the Heisenberg double $\mathfrak{h}(A) := \mathfrak{h}(\mathcal{G}(A), \mathcal{K}(A))$ and its Fock space $\mathcal{F}(A) = \mathcal{G}(A)$.

Theorem (Savage, Yacobi 2015)

The functors Ind_M and Res_P for $M \in A$-mod and $P \in A$-pmod categorify the Fock space representation $\mathcal{F}(A)$ of $\mathfrak{h}(A)$.

Categorification of the Heisenberg Double

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Application: QSym is free over Sym

Tower of 0-Hecke algebras

- $A = \bigoplus_{n \in \mathbb{N}} H_n(0)$
- $\mathcal{G}(A) = \mathrm{QSym}$, $\mathcal{K}(A) = \mathrm{NSym}$
- $\mathfrak{q} = \mathfrak{h}(\mathrm{QSym}, \mathrm{NSym})$: quasi-Heisenberg algebra
- $\mathfrak{q}_{\mathrm{proj}}$: subalgebra generated by $\mathrm{Sym} \subset \mathrm{QSym}$ and NSym.

Theorem (Savage, Yacobi 2015)

Any representation of $\mathfrak{q}_{\mathrm{proj}}$ generated by a lowest weight vacuum vector is isomorphic to Sym.

Theorem (Hazewinkel 2001, Savage, Yacobi 2015)

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Further Applications

Towers of Superalgebras

- 0-Hecke-Clifford algebras (Li 2015)
- The ring of peak quasisymmetric functions is free over the subring of symmetric functions spanned by Schur's Q-functions.
- Other towers of (super)algebras (ongoing work)

Thank you for your attention!

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Thank you for your attention!