

Difference systems of sets and code synchronization

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Abstract

Difference systems of sets were introduced by Levenshtein in connection with code synchronization. The paper discusses some constructions of difference systems of sets obtained from cyclic difference sets and finite geometry.

1 Introduction

A *difference system of sets* (DSS) with parameters $(n, \tau_0, \dots, \tau_{q-1}, \rho)$ is a collection of q disjoint subsets $Q_i \subseteq \{1, 2, \dots, n\}$, $|Q_i| = \tau_i$, $0 \leq i \leq q-1$, such that the multi-set

$$\{a - b \pmod{n} \mid a \in Q_i, b \in Q_j, i \neq j\} \quad (1)$$

contains every number i , $1 \leq i \leq n-1$ at least ρ times. A DSS is *perfect* if every number i , $1 \leq i \leq n-1$ is contained exactly ρ times in the multi-set of differences (1). A DSS is *regular* if all subsets Q_i are of the same size: $\tau_0 = \tau_1 = \dots = \tau_{q-1} = m$. We use the notation (n, m, q, ρ) for a regular DSS on n points with q subsets of size m .

Difference Systems of Sets were introduced by V. Levenshtein [6] and were used for the construction of codes that allow for synchronization in the presence of errors. A q -ary code of length n is a subset of the set F_q^n of all vectors of length n over $F_q = \{0, 1, \dots, q-1\}$. If q is a prime power, we often identify F_q with a finite field of order q , in which case i ($0 < i \leq q-1$) stands for the i th power of a primitive element. A *linear* q -ary code (q a prime power), is a linear subspace of F_q^n . If $x = x_1 \cdots x_n$, $y = y_1 \cdots y_n \in F_q^n$, and $0 \leq i \leq n-1$, the i th *joint* of x and y is defined as $T_i(x, y) = x_{i+1} \cdots x_n y_1 \cdots y_i$. In particular, $T_i(x, x)$ is a cyclic shift of x . The *comma-free index* $\rho = \rho(C)$ of a code $C \subseteq F_q^n$ is defined as

$$\rho = \min d(z, T_i(x, y)),$$

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where the minimum is taken over all $x, y, z \in C$ and all $i = 1, \dots, n-1$, and d is the Hamming distance between vectors in F_q^n . The comma-free index $\rho(C)$ allows one to distinguish a code word from a joint of two code words (and hence provides for synchronization of code words) provided that at most $\lfloor \rho(C)/2 \rfloor$ errors have occurred in the given code word [5].

Since the zero vector belongs to any linear code, the comma-free index of a linear code is zero. Levenshtein [6] gave the following construction of comma-free codes of index $\rho > 0$ obtained as cosets of linear codes, that utilizes difference systems of sets. Given a DSS $\{Q_0, \dots, Q_{q-1}\}$ with parameters $(n, \tau_0, \dots, \tau_{q-1}, \rho)$, define a linear q -ary code $C \subseteq F_q^n$ of dimension $n - r$, where

$$r = \sum_{i=0}^{q-1} |Q_i|,$$

whose information positions are indexed by the numbers not contained in any of the sets $Q_0, \dots, Q_{q-1}$, and having all redundancy symbols equal to zero. Replacing in each vector $x \in C$ the positions indexed by Q_i with the symbol i ($0 \leq i \leq q-1$), yields a coset C' of C that has a comma-free index at least ρ .

This application of DSS to code synchronization requires that the redundancy

$$r = r_q(n, \rho) = \sum_{j=0}^{q-1} |Q_j|$$

is as small as possible.

Levenshtein [6] proved the following lower bound on $r_q(n, \rho)$:

Theorem 1.1

$$r_q(n, \rho) \geq \sqrt{\frac{q\rho(n-1)}{q-1}}, \quad (2)$$

with equality if and only if the DSS is perfect and regular.

In [6], Levenshtein found optimal DSS for $q = 2$ and $\rho = 1$ or $\rho = 2$, and proved that for all $n \geq 2$

$$r_2(n, 1) = \lceil \sqrt{2(n-1)} \rceil, \quad r_2(n, 2) = \lceil 2\sqrt{n-1} \rceil.$$

Similar results are not known for $q \geq 3$.

In a recent paper Levenshtein [7] introduced some constructions of imperfect regular DSS obtained as products of cyclic difference sets. In particular, he proved that the existence of a cyclic (v, q, ρ) difference set with $2 \leq q < v$ implies the existence of an DSS with parameters $(n = v^h, m, q, \rho)$ for every $h \geq 2$. A corollary of this result is that for any prime power t and any integer h there exists a regular DSS with $n = (t^2 + t + 1)^h$, $m = \frac{(t+1)^h - 1}{t}$, $q = t + 1$, and $\rho = 1$.

In this paper we describe some direct constructions of perfect and regular, hence optimal difference systems of sets obtained as partitions of cyclic difference sets.

2 DSS as partitions of difference sets

Let $D = \{x_1, x_2, \dots, x_k\}$ be a (v, k, λ) difference set (cf. [1], [2], [9]), that is, a subset of k residues modulo v such that every positive residue modulo v occurs exactly λ times in the multi-set of differences

$$\{x_i - x_j \pmod{v} \mid x_i, x_j \in D, x_i \neq x_j\}.$$

Then the collection of singletons $Q_0 = \{x_1\}, \dots, Q_{k-1} = \{x_k\}$ is a perfect regular DSS with parameters $(n = v, m = 1, q = k, \rho = \lambda)$. Thus, DSS are a generalization of cyclic difference sets. The next lemma generalizes this simple construction by using more general partitions of difference sets.

Lemma 2.1 *Let $D \subseteq \{1, 2, \dots, n\}$, $|D| = k$, be a cyclic (n, k, λ) difference set. Assume that D is partitioned into q disjoint subsets $Q_0, \dots, Q_{q-1}$ that are the base blocks of a cyclic design $\mathcal{D}$ with block sizes $\tau_i = |Q_i|$, $i = 0, \dots, q-1$ such that every two points are contained in at most λ_1 blocks. Then the sets $Q_0, \dots, Q_{q-1}$ form a DSS with parameters $(n, \tau_0, \dots, \tau_{q-1}, \rho = \lambda - \lambda_1)$. The DSS $\{Q_i\}_{i=0}^{q-1}$ is perfect if and only if $\mathcal{D}$ is a pairwise balanced design with every two points occurring together in exactly λ_1 blocks.*

Proof. Since D is a cyclic (n, k, λ) difference set, the multi-set of differences

$$\{a - b \pmod{n} \mid a, b \in D, a \neq b\}$$

contains every $i \in Z_n \setminus \{0\}$ exactly λ times. The assumption that $Q_0, \dots, Q_{q-1}$ are base blocks of a cyclic design with every two points being contained in at most λ_1 blocks is equivalent to the property that the multi-set of differences

$$\{a - b \pmod{n} \mid a, b \in Q_i, a \neq b, i = 0, \dots, q-1\} \quad (3)$$

contains every $i \in Z_n \setminus \{0\}$ at most λ_1 times. It follows that if $a \in Q_i, b \in Q_j$ and $i \neq j$, $0 \leq i, j \leq q-1$, the number $a - b$ appears at least $\rho = \lambda - \lambda_1$ times in the multi-set of differences (1).

The condition that $Q_0, \dots, Q_{q-1}$ are base blocks of a pairwise balanced design with parameter λ_1 is equivalent to the property that the multi-set of differences (3) contains every $i \in Z_n \setminus \{0\}$ exactly λ_1 times, in which case if $a \in Q_i, b \in Q_j$ and $i \neq j$, $0 \leq i, j \leq q-1$, the number $a - b$ appears exactly $\lambda - \lambda_1$ times in the multi-set of differences (1), hence the collection $\{Q_i\}_{i=0}^{q-1}$ is perfect DSS of index $\rho = \lambda - \lambda_1$. $\square$

The following theorem gives infinitely many perfect and regular DSS obtained by partitioning the trivial cyclic $(n, n-1, n-2)$ difference set $D = \{1, 2, \dots, n-1\}$, where n is an arbitrary prime number.

Theorem 2.2 *Let $n = mq + 1$ be a prime, and let α be a primitive element of the finite field of order n , $GF(n)$. The collection of sets*

$$Q_0 = \{\alpha^q, \alpha^{2q}, \dots, \alpha^{mq} = 1\}, Q_1 = \alpha Q_0, \dots, Q_{q-1} = \alpha^{q-1} Q_0$$

is a perfect regular $(n, m, q, \rho = n - m - 1)$ DSS.

Proof. Note that Q_0 is a subgroup of the multiplicative group of $GF(n)$, and the sets $\alpha^i Q_0$, $0 \leq i \leq q-1$ are pairwise disjoint cosets of Q_0 that partition $GF(n)^* = GF(n) \setminus \{0\}$. The group $GA(n)$ of affine transformations

$$y = ax + b, \quad a, b \in GF(n), \quad a \neq 0$$

acts 2-transitively on $GF(n)$. Note that Q_0 is a subgroup of $GA(n)$ of order m . Consequently, the orbit of Q_0 under $GA(n)$ is a 2 -(n, m, λ) design $\mathcal{D}$ with

$$b = \frac{|GA(n)|}{|Q_1|} = nq$$

blocks, whence $\lambda = m - 1$.

The cyclic group $Z_n \leq GA(n)$ partitions the nq blocks of $\mathcal{D}$ into q orbits of length n . The multiplicative group $GF(n)^*$ is a subgroup of $GA(n)$ of order $n - 1$ that fixes only one element (namely 0) of $GF(n)$. Consequently, $GF(n)^*$ fixes exactly one block in each of the q orbits of blocks of $\mathcal{D}$ under Z_n , and these fixed blocks are $Q_0, \dots, Q_{q-1}$. It follows that $\mathcal{D}$ can be viewed as a cyclic design with base blocks $Q_0, \dots, Q_{q-1}$. Since

$$\bigcup_{i=0}^{q-1} Q_i = GF(n)^*,$$

and $GF(n)^*$ is a cyclic $(n, n - 1, n - 2)$ difference set, the statement of the theorem follows from Lemma 2.1. $\square$

The DSS described in Theorem 2.2 has redundancy $r_q(n, \rho) = n - 1$. However, the following example suggests that it is sometimes possible to obtain a DSS with a smaller value of $r_q(n, \rho)$ being a sub-collection of the DSS described in Theorem 2.2.

Example 2.3 Let $n = 19$, $q = 6$, $m = 3$. The DSS from Theorem 2.2 has $\rho = 15$, and the six sets Q_i of size 3 are

$$\{1, 7, 11\}, \{2, 14, 3\}, \{4, 9, 6\}, \{5, 16, 17\}, \{8, 18, 12\}, \{10, 13, 15\}.$$

The two sets $\{1, 7, 11\}$, $\{2, 14, 3\}$ form a perfect DSS with $q = 2$, $\rho = 1$, and $r = 6$.

It is an interesting open problem to find an infinite class of such examples.

The following theorem gives perfect regular DSS's obtained as partitions of difference sets of quadratic-residue (QR) type.

Theorem 2.4 *For every prime $n = 2mq + 1 \equiv 3 \pmod{4}$ there exists a perfect regular DSS with parameters $(n, m, q, \rho = (n - 2m - 1)/4)$.*

Proof. Let α be a primitive element of $GF(n)$. The set of quadratic residues

$$D = \{\alpha^{2i} \mid 1 \leq i \leq (n-1)/2\}$$

is a cyclic difference set with parameters

$$v = n, \quad k = (n-1)/2, \quad \lambda = (n-1)/4.$$

Note that D is a multiplicative cyclic group of order $(n-1)/2 = mq$. Let $D_m = \{\alpha^{2iq} \mid 1 \leq i \leq m\}$ be the cyclic subgroup of D of order m . We define $Q_0, Q_1, \dots, Q_{q-1}$ to be the cosets of D_m in D :

$$Q_0 = D_m, \quad Q_1 = D_m \alpha^2, \dots, Q_{q-1} = D_m \alpha^{2(q-1)}.$$

We claim that $Q_0, \dots, Q_{q-1}$ are the base blocks of a cyclic $2-(n, m, (m-1)/2)$ design. To see that, consider the group G of transformations $\phi: GF(n) \rightarrow GF(n)$ of the form

$$\phi(x) = a^2x + b \pmod{n}; \quad a, b \in GF(n), \quad a \neq 0.$$

The group G is of order $n(n-1)/2$ and contains the cyclic group Z_n and the multiplicative group D as subgroups. G acts transitively on the 2-subsets of $GF(n)$. Consequently, the orbit of any m -subset of $GF(n)$ under G is a cyclic $2-(n, m, \lambda)$ design for some λ . In particular, the orbit D_m^G of D_m is a $2-(n, m, \lambda)$ design with total number of blocks equal to

$$\frac{|G|}{|D_m|} = qn = \frac{n(n-1)}{m(m-1)}\lambda,$$

whence $\lambda = (m-1)/2$. Now the theorem follows from Lemma 2.1. □

Example 2.5 Let $n = 31 = 2 \cdot 5 \cdot 3 + 1$. We take $m = 5$, $q = 3$, and $\alpha = 3$ as a primitive element modulo 31. The set D_5 defined as in Theorem 2.4 for $m = 5$ consists of the elements

$$3^6 \equiv 16, \quad 3^{12} \equiv 8, \quad 3^{18} \equiv 4, \quad 3^{24} \equiv 2, \quad 3^{30} \equiv 1.$$

The sets

$$Q_0 = D_5 = \{16, 8, 4, 2, 1\}, \quad Q_1 = D_5 3^2 = \{20, 10, 5, 18, 9\}, \quad Q_2 = D_5 3^4 = \{25, 28, 14, 7, 19\}$$

are base blocks of a cyclic $2-(31, 5, 2)$ design, and their union $Q_0 \cup Q_1 \cup Q_2$ is the set of all nonzero quadratic residues modulo 31. Consequently, the collection Q_0, Q_1, Q_2 is a perfect regular DSS with parameters $n = 31$, $m = 5$, $q = 3$, $\rho = 5$.

3 Difference systems of sets from finite geometry

Perfect DSS with reasonably small redundancy $r_q(n, \rho)$ can be obtained from difference sets related to finite geometry.

Let H be a hyperplane in the $2s$ -dimensional projective space $PG(2s, p)$ over $GF(p)$. The $(p^{2s} - 1)/(p - 1)$ points of H form a cyclic difference set with parameters

$$v = \frac{p^{2s+1} - 1}{p - 1}, \quad k = \frac{p^{2s} - 1}{p - 1}, \quad \lambda = \frac{p^{2s-1} - 1}{p - 1}$$

in a cyclic group acting regularly on the points of $PG(2s, p)$, known in design theory and geometry as the Singer difference set. It is known [4] that the points of H can be partitioned into disjoint lines $Q_0, Q_1, \dots, Q_{q-1}$, where

$$q = \frac{p^{2s} - 1}{p^2 - 1} = p^{2s-2} + \dots + p^2 + 1.$$

On the other hand, the collection of all lines in $PG(2s, p)$ is a cyclic $2-(\frac{p^{2s+1}-1}{p-1}, p+1, 1)$ design $\mathcal{D}$. If the partition

$$H = Q_0 \cup Q_1 \cup \dots \cup Q_{q-1}$$

is chosen so that $Q_0, \dots, Q_{q-1}$ are base blocks of $\mathcal{D}$, then by Lemma 2.1 the collection $Q_0, Q_1, \dots, Q_{q-1}$ is a perfect regular DSS with parameters

$$n = \frac{p^{2s+1} - 1}{p - 1}, \quad m = p + 1, \quad q = \frac{p^{2s} - 1}{p^2 - 1}, \quad \rho = \frac{p^{2s-1} - p}{p - 1}.$$

Hyperplane partitions with the above property were studied by Fuji-Hara, Jimbo and Vanstone in a different context in [3], who showed that such partitions exist in $PG(2s, 2)$ for $s \leq 5$, and in $PG(2s, 3)$ for $s \leq 3$.

Example 3.1 Let $p = 2$, $s = 2$. We consider $1, \alpha, \alpha^2, \dots, \alpha^{30}$ as points of $PG(4, 2)$, where α is a primitive element of $GF(2^5)$ defined by the polynomial $x^5 + x^3 + 1$. The following set of 15 points

$$H = \{\alpha, \alpha^2, \alpha^4, \alpha^8, \alpha^{16}, \alpha^3, \alpha^6, \alpha^{12}, \alpha^{24}, \alpha^{17}, \alpha^{29}, \alpha^{27}, \alpha^{23}, \alpha^{15}, \alpha^{30}\}$$

is a hyperplane in $PG(4, 2)$, and hence a $(31, 15, 7)$ difference set in the multiplicative group of $GF(2^5)$. The following partition of H ,

$$H = \{\alpha, \alpha^3, \alpha^{29}\} \cup \{\alpha^2, \alpha^6, \alpha^{27}\} \cup \{\alpha^4, \alpha^{12}, \alpha^{23}\} \cup \{\alpha^8, \alpha^{24}, \alpha^{15}\} \cup \{\alpha^{16}, \alpha^{17}, \alpha^{30}\}$$

has the property that each of the five 3-subsets is a projective line, and these five lines are the base blocks of a cyclic $2-(31, 3, 1)$ design under the multiplicative group of $GF(2^5)$. Thus, these five 3-subsets define a perfect DSS with parameters $(n = 31, m = 3, q = 5, \rho = 6)$.

Example 3.2 The following set of 63 residues modulo 127 is a Singer difference set $(127, 63, 31)$, i.e., a hyperplane in $PG(6, 2)$, partitioned into 21 disjoint triples being lines in $PG(6, 2)$ that form the base of a cyclic $2-(127, 3, 1)$ design:

1 96 66; 2 65 5; 4 3 10; 8 6 20; 16 12 40; 32 24 80; 64 48 33; 9 101 83; 18 75 39; 36 23 78; 72 46 29; 17 92 58; 34 57 116; 68 114 105; 11 113 91; 22 99 55; 44 71 110; 88 15 93; 49 30 59; 98 60 118; 69 120 109.

The hyperplane, as well as the partition are fixed by the multiplier $i \rightarrow 2i \pmod{127}$.

Example 3.3 The following set of 40 residues modulo 121 is a Singer difference set $(121, 40, 13)$, i.e., a hyperplane in $PG(4, 3)$, partitioned into ten disjoint lines in $PG(4, 3)$ that are base blocks of a cyclic 2 -(121, 4, 1) design, that is, of the collection of all lines in $PG(4, 3)$:

1 11 34 115; 3 33 102 103; 4 63 75 117; 7 13 89 108; 9 64 67 99; 12 68 104 109; 21 25 39 82; 27 55 71 80; 36 70 83 85; 44 81 92 119.

Note 3.4 The above examples share the property that the hyperplane partition is fixed by a multiplier. It is an interesting open problem to find a general explicit construction, or to prove that such partitions always exist.

The above construction cannot be extended directly to a projective space of odd dimension since the points of a hyperplane cannot be partitioned into disjoint lines. However, the following example shows that it may be possible to use partitions of the points of a hyperplane into disjoint subsets of difference sizes that yield a perfect DSS. Another possibility is to study partitions of a hyperplane into subspaces of dimension greater than one.

Example 3.5 Let $n = 15$. The set

$$D = \{0, 1, 2, 4, 8, 5, 10\}$$

is a cyclic $(15, 7, 3)$ difference set in Z_{15} isomorphic to the Singer difference set in $PG(3, 2)$. The two sets $\{5, 10\}$ and $\{1, 2, 4, 8\}$ are base blocks of a cyclic pairwise balanced 2 -(15, $\{2, 4\}$, 1) design. Thus, the partition

$$D = \{0\} \cup \{5, 10\} \cup \{1, 2, 4, 8\}$$

yields a perfect DSS via Lemma 2.1 with parameters

$$n = 15, \tau_0 = 1, \tau_1 = 2, \tau_2 = 4, q = 3, \rho = 2.$$

Note that $r_q(n, \rho) = 7 = \lceil \sqrt{42} \rceil$. Thus, this DSS is optimal with respect to the bound (2).

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